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THE PSEUDOMETRIC TOPOLOGY OF A SEMINORM IS A VECTOR TOPOLOGY

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Abstract

We show that the pseudometric topology of a seminorm is a vector topology. The proof here is a General Topological proof based on net convergence in a topological space.

Keywords: net convergence, seminorm, pseudometric topology, vector topology

1 LANGUAGE AND NOTATION Our language, notation and citations shall be pretty standard, as found, for example, in [1] and [2]. The *pseudometric topology* τ_d of the pseudometric space (X, d) holds a commanding position in General Topology. And you are not very likely to be admonished for *heresy* if you define General Topology as the *study of the pseudometric topology*. J.D. Pryce, in his book [3], ob- served that the Euclidean n-space ($\mathbb{R}^n$, $\| \, \| n$) *holds a commanding position in Analysis.* But the topology $\tau_{\| \, \| n}$ of ($\mathbb{R}^n$, $\| \, \| n$) is a pseudometric topology. Undiluted *General Topology* **(GT)** constitutes about seventy-five percent (75%) of the theory of *Topological Vector Spaces* **(TVS)**. And so, it should not be surprising, therefore, that the *pseudometric topology of the seminorm* also holds a commanding position in **TVS** theory. At the peak of the subject of **TVS** are the theory of *Locally Convex Spaces* and *Duality Theory*. And one may not be very incorrect to define these subjects as the *study* of the *pseudometric topology of the seminorm*. For an instance, the popular strong topology $\beta(X, Y)$ of the duality $< X, Y >$ is generated by the collection of all the lower semicontinuous seminorms of the duality[4].

To embark on the study of **TVS** without first taking the pains to wade through *the oppressive detail* of ***General Topology*** $\equiv$ Open set, Closed set, Closure, Net and Filter convergence, the pseudometric topology, the product topology, Continuity, Compactness, Connectedness Completion, Uniform Space, Topological Group, is only good for geniuses. I repeat: *More than 75% of the subject of **TVS** is undiluted, naked, unadulterated General Topology from Open Set to Topological Group, e.g., a topological group is a uniform space, and a topological vector space is a topological group. A dominant concept in Locally Convex Space Theory is Equicontinuity. And, Equicontinuity is a Uniform Space concept. Sterling K. Berberian, in a private communication, once scolded me "……." but that* (*Uniform Space theory*) *is part of General Education…….".* ***Observation*** A motivation for the study of *Uniform Spaces* and a key example are furnished by the pseudometric topology.

In what follows, the reader is assumed familiar with the beginnings of General Topology. By $\mathbb{R}$ we denote the *real numbers*, $\mathfrak{C}$ denotes the *complex numbers* and $\mathbb{N}$ denotes the *positive integers* 1, 2, …….

Let $X \neq \varnothing$. A subfamily of 2^X, τ, say, is called a *topology on* X, and the pair (X, τ) called a *topological space*, if

TOP 1 $\varnothing, X \in \tau$,

TOP 2 τ is closed under finite intersection, and

TOP 3 τ is closed under arbitrary union.

If $G \in \tau$, G is called an *open set* of τ or of (X, τ).

If (X, τ) is a topological space, $x_0 \in X$, $G \in \tau$, $\varnothing \neq A \subseteq X$ and $x_0 \in G \subseteq A$, then A is called a *neighbourhood* x_0 .

Let $X \neq \varnothing$, $\varnothing \neq A \subseteq X$, $(I, \leq)$ a *directed set,* and $(x_i)_{i\in(I, \leq)}$ a *net* in X based on the directed set $(I, \leq)$. The net $(x_i)_{i\in(I, \leq)}$ is said to be *eventually in* A if there exists $\delta \in I$ such that $x_i \in A$ for all $i \in I$, $i \geq \delta$. And we write
$x_i \in A$ *eventually*.

Let (X, τ) be a *topological space* and $x_0 \in X$. If the net $(x_i)_{i\in(I, \leq)}$ in X is eventually in every neighbourhood of x_0, we say that $(x_i)_{i\in(I, \leq)}$ *converges to* x_0, and we may write $x_i \xrightarrow{\tau} x_0$

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The words *function*, *map* and *mapping* are synonymous. We indic- ate by /// the end or absence of a proof.

2 THE PSEUDOMETRIC TOPOLOGY Let $X \neq \varnothing$. A function
$d : X\text{x}X \to \mathbb{R}$,
such that, for $x, y, z \in X$,
PSDM 1 $d(x, y) \geq 0$ and $d(x, x) = 0$ (*Positivity of the pseudometric*),
PSDM 2 $d(x, y) = d(y, x)$(Symmetry of the pseudometric) and
PSDM 3 $d(x, z) \leq d(x, y) + d(y, z)$ (*The Triangle Inequality*)
is called a *pseudometric* on X, and the pair (X, d) then also called a *pseudometric space*. A pseudometric d on X with the additional property
METC $d(x, y) = 0$ if and only if $x = y$,
is called a *metric on X*, and the pair (X, d) then also called a *metric space*.

Let (X, d) be a pseudometric space, $x_0 \in X$, $r \in \mathbb{R}$, $r > 0$. The set
$B_d(x_0, r) \equiv \{x \in X : d(x, x_0) < r\}$
is called a *ball in* (X, d) *of radius r centered on* x_0. Clearly, $x_0 \in B_d(x_0, r)$.
THEOREM 1 *The Open Ball Theorem* Let (X, d) be a pseudo- metric space, $x_0 \in X$, $r > 0$, $B_d(x_0, r)$ a ball in (X, d) and $a \in B_d(x_0, r)$. Then, there exists $r' > 0$ such that $B_d(a, r') \subseteq B_d(x_0, r)$. ///
Define
$\tau_d \equiv \{\varnothing, X\} \cup \{\varnothing \neq G \subseteq X : G \text{ is a union of balls}\}$.
THEOREM 2 τ_d is a topology on X. ///
The topology τ_d is called the *pseudometric topology* of d or of (X, d), and, of course, (X, τ_d) is called a *pseudometric topological space*. If d is a metric, of course, τ_d is called the *metric topology* of d or of (X, d), and (X, τ_d) called a *metric topological space*.

Example 3 The *absolute value metric*
$d_{|\,|} : \mathbb{R}\text{x}\mathbb{R} \to \mathbb{R}$
$(x, y) \mapsto |x - y|$
on $\mathbb{R}$, yields the metric topology $\tau_{d_{|\,|}}$ on $\mathbb{R}$ called, in this paper, the *topology* of $\mathbb{R}$. Clearly if $a \in \mathbb{R}$, and $r > 0$, the ball of radius r centered on a is
$B_{d_{|\,|}}(a, r) \equiv \{x \in \mathbb{R} : d_{|\,|}(x, a) < r\}$
$= \{x \in \mathbb{R} : |x - a| < r\}$
$= (a - r, a + r)$,
the finite open interval with extremities $a - r$ and $a + r$. See

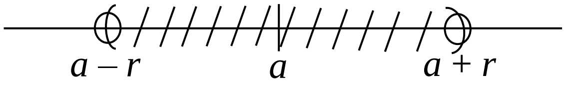

Fig. 1

Figure 1 above. So, the open sets of the topology of $\mathbb{R}$ are $\varnothing$, $\mathbb{R}$ and unions of finite open intervals. The topology of $\mathbb{R}$, $\tau_{d_{|\,|}}$ is denoted $\tau_{\mathbb{R}}$ in this paper. So, $\tau_{d_{|\,|}} = \tau_{\mathbb{R}}$.
FACT 4 Let (X, τ_d) be a pseudometric topological space, $x_0 \in X$ and $\varnothing \neq A \subseteq X$. Then, A is a τ_d –neighbourhood of x_0 if and only if there exists a ball $B_d(x_0, r)$, $r > 0$, such that $B_d(x_0, r) \subseteq A$.
Proof Immediate from the definition of τ_d and the definition of a neighbourhood, noting that the ball $B_d(x_0, r)$ is itself a neighbourhood of x_0. ///

THEOREM 5 Let (X, τ_d) be a pseudometric topological space, $x_0 \in X$ and $(x_i)_{i\in(I, \leq)}$ a net in X. Then, $(x_i)_{i\in(I, \leq)}$ τ_d -converges to x_0
$\Leftrightarrow^1$
Whenever given $r > 0$, $(x_i)_{i\in(I, \leq)}$ is eventually in $B_d(x_0, r)$
$\Leftrightarrow^2$
For every $r > 0$, there exists $i_0 \in I$ such that $d(x_i, x_0) < r$ for all $i \in I$, $i \geq i_0$.
That is,
$|d(x_i, x_0) - 0| < r$ for all $i \in I$, $i \geq i_0$.

⇔[3]

The net $(d(x_i, x_0))_{i\in(I,\le)}$ $\tau_{\mathbb{R}}$-converges to 0.
Proof Immediate from definitions. ///

COROLLARY 6 Let $x_0 \in \mathbb{R}$. The net $(x_i)_{i\in(I,\le)}$ in $\mathbb{R}$ $\tau_{\mathbb{R}}$-converges to x_0

⇔

Whenever given $r > 0$, there exists $i_0 \in I$ such that
$|x_i - x_0| < r$ for all $i \in I$, $i \ge i_0$.

⇔

$$x_i - x_0 \xrightarrow{\tau_R} 0$$

(i.e., the net $(x_i - x_0)_{i\in(I\le)}$ $\tau_{\mathbb{R}}$-converges to 0. ///

THEOREM 7 Let $(x_i)_{i\in(I\le)}$ and $(y_i)_{i\in(I\le)}$ be nets in $\mathbb{R}$ based on the same directed set $(I, \le)$. Suppose $x, y, \alpha \in \mathbb{R}$ and that

$$x_i \xrightarrow{\tau_R} x$$

and

$$y_i \xrightarrow{\tau_R} y.$$

Then,

(i) $x_i + y_i \xrightarrow{\tau_R} x + y$
and

(ii) $\alpha x_i \xrightarrow{\tau_R} \alpha x$.

Proof (i): Let $r > 0$. By hypotheses and COROLLARY 6, there exist $i_0', i_0'' \in I$ such that

$$\left.\begin{aligned} &i \ge i_0' \Rightarrow |x_i - x_0| < \frac{r}{2} \\ \text{and}\quad &i \ge i_0'' \Rightarrow |y_i - y| < \frac{r}{2} \end{aligned}\right\} \qquad \ldots\ldots\ldots(*)$$

Since $(I, \le)$ is a directed set, there exists $i_0 \in I$ such that $i_0 \ge i_0'$ and $i_0 \ge i_0''$. Hence,

$$i \ge i_0 \Rightarrow |x_i - x_0| < \frac{r}{2},\ |y_i - y| < \frac{r}{2}$$

By arguments from *Elementary Real Analysis*, therefore,

$$i \ge i_0 \Rightarrow |x_i + y_i - (x + y)| = |x_i - x + y_i - y| \le |x_i - x| + |y_i - y|$$
$$< \frac{r}{2} + \frac{r}{2} = r.$$

(ii): If $\alpha = 0$ we have nothing to show since *a* constant net converges to the lone value of its terms. So, suppose $\alpha \ne 0$ and $r > 0$. By hypothesis, there exists $i_0 \in I$ such that

$$i \ge i_0 \Rightarrow |x_i - x| < \frac{r}{|\alpha|}$$

Hence,

$$i \ge i_0 \Rightarrow |\alpha x_i - \alpha x| = |\alpha||x_i - x| < |\alpha| \cdot \frac{r}{|\alpha|} = r \ ///$$

If $z \in \mathfrak{C}$, then $z = a + ib$, where $a, b \in \mathbb{R}$ and i is the *imaginary unit* ($i^2 = -1$). The *modulus*, $|z|$, of z is defined as $\sqrt{a^2 + b^2}$. That, is

$$|z| = \sqrt{a^2 + b^2} .$$

The modulus shares many of the properties of the absolute value on $\mathbb{R}$. Incidentally, they have same notation, | |. From textbooks of beginning *Complex Function Theory,* confirm that

FACT 8 The function

$$d_{|\ |} : \mathfrak{C} \text{x} \mathfrak{C} \to \mathbb{R},$$
$$(z_1, z_2) \mapsto |z_1 - z_2|$$

is a metric on $\mathfrak{C}$. ///

We denote $\tau_{d|\,|}$ by $\tau_{\mathfrak{C}}$ and call it the *topology of* $\mathfrak{C}$, in this paper.

FACT 9 Let $z_0 \in \mathfrak{C}$. Then, $(z_i)_{i\in(I,\,\leq)}$ $\tau_{\mathfrak{C}}$ -converges to z_0
$\Leftrightarrow$
Whenever given $r > 0$, there exists $i_0 \in I$ such that
$|z_i - z_0| < r$ for all $i \in I$, $i \geq i_0$
$\Leftrightarrow$

$z_i - z_0 \xrightarrow{\tau_{\mathfrak{C}}} 0$.

(i.e., the net $(z_i - z_0)_{i\in(I,\,\leq)}$ $\tau_{\mathfrak{C}}$-converges to 0). ///

FACT 10 Let $(x_i)_{i\in(I,\,\leq)}$ and $(y_i)_{i\in(I,\,\leq)}$ be nets in $\mathfrak{C}$ based on the same directed set $(I, \leq)$. Suppose $x, y, \alpha \in \mathfrak{C}$ and that

$x_i \xrightarrow{\tau_{\mathfrak{C}}} x$
and

$y_i \xrightarrow{\tau_{\mathfrak{C}}} y$.
Then,

(i) $x_i + y_i \xrightarrow{\tau_{\mathfrak{C}}} x + y$
and

$\alpha x_i \xrightarrow{\tau_{\mathfrak{C}}} \alpha x$. ///

K=ℝ or ℭ

Denote by K either of ℝ and ℭ. And we can restate COROLLARY 6 and FACT 9 as

THEOREM 11 Let $x_0 \in$ K. The net $(x_i)_{i\in(I,\,\leq)}$ in K τ_K-converges to $x_0 \Leftrightarrow^1$
Whenever given $r > 0$, there exists $i_0 \in I$ such that
$|x_i - x_0| < r$ for all $i \in I$, $i \geq i_0$
$\Leftrightarrow^2$

$x_i - x_0 \xrightarrow{\tau_K} 0$.
(i.e., the net $(x_i - x_0)_{i\in(I,\,\leq)}$ τ_K –converges to 0.)
Similarly, we can restate THEOREM 7 and FACT 10 as

THEOREM 12 Let $(x_i)_{i\in(I,\,\leq)}$ and $(y_i)_{i\in(I,\,\leq)}$ be nets in K based on the same directed set $(I, \leq)$. Suppose $x, y, \alpha \in$ K and that

$x_i \xrightarrow{\tau_K} x$
and

$y_i \xrightarrow{\tau_K} y$.
Then,

(i) $x_i + y_i \xrightarrow{\tau_K} x + y$
and

(ii) $\alpha x_i \xrightarrow{\tau_K} \alpha x$ ///
One deduces that

THEOREM 13 If $(t_i)_{i\in(I,\,\leq)}$ and $(s_i)_{i\in(I,\,\leq)}$ are nets in K, based on the same directed set $(I, \leq)$ and

$t_i \xrightarrow{\tau_K} 0$ and $s_i \xrightarrow{\tau_K} 0$.
Then,

$t_i + s_i \xrightarrow{\tau_K} 0$. ///
Call a net $(t_i)_{i\in(I,\,\leq)}$ in K an *eventually bounded net* if there exists $M > 0$ such that $|t_i| \leq M$ eventually. Immediate is that if a net $(t_i)_{i\in(I,\,\leq)}$ in K is eventually bounded, so is the net $(|t_i|)_{i\in(I,\,\leq)}$ in ℝ.

THEOREM 14(i) If the net $(t_i)_{i\in(I,\,\leq)}$ in K τ_K-converges, then, it is eventually bounded.
(ii) Suppose $(t_i)_{i\in(I,\,\leq)}$ and $(s_i)_{i\in(I,\,\leq)}$ are nets in K based on the same directed set $(I, \leq)$. Suppose
(α) $(t_i)_{i\in(I,\,\leq)}$ is eventually bounded
and

(β) $(s_i)_{i\in(I,\,\leq)}$ is *null* (i.e., $s_i \xrightarrow{\tau_K} 0$).

Then, $(t_i s_i)_{i\in(I,\leq)}$ is null. ///

3 THE PRODUCT TOPOLOGY Let $\Delta \neq \varnothing$ and $(X_\alpha, \tau_\alpha)_{\alpha\in\Delta}$ an indexed family of topological spaces indexed by Δ. With $\prod_{\alpha\in\Delta} X_\alpha$ we de- note the *Cartesian Product* of the indexed family $(X_\alpha)_{\alpha\in\Delta}$ of non-empty sets X_α, $\alpha \in \Delta$. For $\beta \in \Delta$, X_β is called the β*th factor set* of $\prod_{\alpha\in\Delta} X_\alpha$ and the map

$$p_\beta : \prod_{\alpha\in\Delta} X_\alpha \to X_\beta$$

$$x = (x(\alpha))_{\alpha\in\Delta} \mapsto x(\beta)$$

is called the *projection of* $\prod_{\alpha\in\Delta} X_\alpha$ onto the βth factor set X_β. Define

$$\mathscr{L} = \{ p_\alpha^{-1}(G_\alpha) : \varnothing \neq G_\alpha \in \tau_\alpha, \alpha \in \Delta\}.$$

The topology on $\prod_{\alpha\in\Delta} X_\alpha$ generated by $\mathscr{L}$ is called the *product topology* of the family $(X_\alpha, \tau_d)_{\alpha\in\Delta}$, denoted $\prod_{\alpha\in\Delta} \tau_\alpha$. And the pair $(\prod_{\alpha\in\Delta} X_\alpha, \prod_{\alpha\in\Delta} \tau_\alpha)$ is called a *product topological space* or simply called a *product space*.

If $\Delta = \{1, 2\}$, so that we have $(X_\alpha, \tau_\alpha)_{\alpha\in\Delta}$ as the two-terms family $((X_1, \tau_1), (X_2, \tau_2))$, the product set $\prod_{\alpha\in\Delta} X_\alpha$ may be written $X_1 \text{x}\, X_2$ and the product topology $\prod_{\alpha\in\Delta} X_\alpha$ written $\tau_1 \text{x} \tau_2$, and the product space $(\prod_{\alpha\in\Delta} X_\alpha, \prod_{\alpha\in\Delta} \tau_\alpha)$ written $(X_1 \text{x}\, X_2, \tau_1 \text{x} \tau_2)$.

A main theorem on net convergence in the product space is

THEOREM 1 (GT) *Net Convergence in the Product Topological Space* Let $(\prod_{\alpha\in\Delta} X_\alpha, \prod_{\alpha\in\Delta} \tau_\alpha)$ be a product topological space, $(x_i)_{i\in(I,\leq)}$ a net in $\prod_{\alpha\in\Delta} X_\alpha$ and $x = (x(\alpha))_{\alpha\in\Delta} \in \prod_{\alpha\in\Delta} X_\alpha$. Then, the net $(x_i)_{i\in(I,\leq)}$ $\prod_{\alpha\in\Delta} \tau_\alpha$-converges to x if and only if its αth coordinate net, $(x_i(\alpha))_{i\in(I,\leq)}$, τ_α-converges to $x(\alpha)$ for each $\alpha \in \Delta$. That is,

$$x_i \to x \text{ in } (\prod_{\alpha\in\Delta} X_\alpha, \prod_{\alpha\in\Delta} \tau_\alpha)$$

$$\Leftrightarrow$$

$x_i(\alpha) \to x(\alpha)$ in (X_α, τ_α) for each $\alpha \in \Delta$. ///

Partial Maps

Let $\varnothing \neq X, X', X^*$, and suppose $x_0 \in X$, $x_0' \in X'$ and $f : X \text{ x } X' \to X^*$ a map. Associated with this map are the maps

$$f_{x_0} : X' \to X^*$$

$$x' \mapsto f(x_0, x')$$

and

$$f_{x_0'} : X \to X^*$$

$$x \mapsto f(x, x_0')$$

called its *partial maps*.

4 CONTINUITY Let (X, τ) and (X', τ') be topological spaces. Sup- pose $x_0 \in X$ and $f : X \to X'$ a map.

Cont.a.Defn 1 [1, p. 52] The function f is said to be *continuous at* x_0 if whenever given a neighbourhood V of $f(x_0)$ in (X', τ'), $f^{-1}(V)$ is a neighbourhood of x_0 in (X, τ).

THEOREM 2 Let (X, τ) and (X', τ') be topological spaces. Sup- pose $x_0 \in X$ and $f : X \to X'$ a map. The following are equiva- lent.

(1) f is continuous at x_0.

(2) Whenever given a neighbourhood V of $f(x_0)$ in (X', τ'), there exists a neighbourhood U of x_0 in (X, τ) such that $f(U) \subseteq V$.

(3) For every $G \in \tau'$ such that $f(x_0) \in G$, there exists $H \in \tau$ such that $x_0 \in H$ and $H \subseteq f^{-1}(G)$. ///

Notation 3 With language and notation still as in the preceding,
(1) By
" $f : (X, \tau) \to (X', \tau')$ is continuous at $\overset{\Delta}{x_0}$ "
shall be meant that $f : X \to X'$ is continuous at x_0.
(2) To emphasize that the continuity of $f : X \to X'$ at x_0 is with respect to the topologies τ and τ', we may write
$f : X \to X'$ is (τ, τ')-continuous at x_0.
=Again, with language and notation as in all the preceding, if f is continuous at EVERY point of X, then f is called a *continuous function* and simply said to be *continuous*. And we may write
(i) $f : (X, \tau) \to (X', \tau')$ is continuous,
or
(ii) $f : X \to X'$ is (τ, τ')-continuous.

THEOREM 4 *Continuity and Net convergence* Let (X, τ) and (X', τ') be topological spaces, $x_0 \in X$ and $f : X \to X'$ a map. The following are equivalent.
(1) f is (τ, τ')-continuous at x_0 .
(2) Whenever given a net $(x_i)_{i\in(I, \leq)}$ in X τ-converging to x_0, the net $(f(x_i))_{i\in(I, \leq)}$ τ'-converges to $f(x_0)$. ///

THEOREM 5 Let (X, τ_d) and (X', τ_{d}') be pseudometric topolo- gical spaces, and $x_0 \in X$. The map $f : (X, \tau_d) \to (X', \tau_{d'})$ is continuous at x_0 if and only if whenever given $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, x_0) > 0$ such that
$x \in X$, and $d(x, x_0) < \delta \Rightarrow d'(f(x), f(x_0)) < \varepsilon$.

Proof Deducible form THEOREM 2. ///

COROLLARY 6 Let (X, τ_d) be a pseudometric topological space
and $f : (X, \tau_d) \to (\mathbb{R}, \tau_{\mathbb{R}})$ a map. Let $x_0 \in X$. Then, f is continuous at x_0 if and only if whenever given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, x_0) > 0$ such that
$x \in X$ and $d(x, x_0) < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$. ///
COROLLARY 7 Let (X, τ_d) be a pseudometric topological space, and suppose $x_0 \in X$. Then, the map
$f : (X, \tau_d) \to (\mathbb{R}, \tau_{\mathbb{R}})$
$x \mapsto d(x, x_0)$
is continuous. ///
Recall from the preceding Section 3 that if $\varnothing \neq X, X', X^*$, $x_0 \in X$, $x_0' \in X'$ and $f : X\text{x}\, X' \to X^*$ is a map, then the maps
$f_{x_0} : X' \to X^*$
$x' \mapsto f(x_0, x')$
and
$f_{x_0'} : X \to X^*$
$x \mapsto f(x, x_0')$
are called the partial maps of f.
THEOREM 8 Let (X, τ), (X', τ') and (X^*, τ^*) be topological spaces. Let $x_0 \in X$, $x_0' \in X'$, and the map $f : (X\text{x}\, X', \tau\text{x}\tau') \to (X^*, \tau^*)$ continuous. Then, the partial maps
$f_{x_0} : (X', \tau') \to (X^*, \tau^*)$
$x' \mapsto f(x_0, x')$
and
$f_{x_0'} : (X, \tau) \to (X^*, \tau^*)$
$x \mapsto f(x, x_0')$
are continuous maps.

Proof Let $z' \in X'$ and $(z_i')_{i\in(I,\,\le)}$ a net in X' τ'-converging to z'. By 3.1, the net $((x_0, z_i'))_{i\in(I,\,\le)}$ τ x τ'-converges to (x_0, z'). By the continuity of f, therefore, $(f(x_0, z_i'))_{i\in(I,\,\le)} = ((f_{x_0}(z_i'))_{i\in(I,\,\le)}$ τ^*-converges to $f(x_0, z') = f_{x_0}(z')$. Similarly, for f_{x_0}'. ///

5 VECTOR SPACE Let $(V, +, \theta)$ be an additive Abelian group and $(F, +, \cdot, 0, 1)$ a field. $(V, +, \theta)$ is called a *vector space over the field* $(F, +, \cdot, 0, 1)$ if there exists an *external multiplication*
$Ext : F\text{x}V \to V$
$(\alpha,\nu) \mapsto \alpha\nu$
of the elements of V by the elements of F satisfying : For $\nu, \omega \in V$ and $\alpha, \beta \in F$, we have
VSP1 $1\nu = \nu$
VSP2 $(\alpha + \beta)\nu = \alpha\nu + \beta\nu$
VSP3 $\alpha(\nu + \omega) = \alpha\nu + \alpha\omega$
VSP4 $(\alpha\beta)\nu = \alpha(\beta\nu)$
By the notation $(V, +, \theta)_{(F,\, +,\, \cdot,\, 0,\, 1)}$, at times shortened to V_F, we shall mean in this paper that the additive Abelian group $(V, +, \theta)$ is a vector space over the field $(F, +, \cdot, 0, 1)$.

According to Sterling K. Berberian in [5, p.47], the *real field* $(\mathbb{R}, +, \cdot, 0, 1)$ and the *complex field* $(\mathfrak{C}, +, \cdot, 0, 1)$ are the *fields of primary importance in Analysis*. And so, therefore, a fortiori, in the theory of Topological Vector Spaces **(TVS)**, only vector spaces $(V, +, \theta)_{(\mathbb{R},\, +,\, \cdot,\, 0,\, 1)}$, and $(V, +, \theta)_{(\mathfrak{C},\, +,\, \cdot,\, 0,\, 1)}$ are entertained.

By K we have since been denoting either of $\mathbb{R}$ and $\mathfrak{C}$, and so we consider **only** vector spaces V_{K}. Of course, any field $(F, +, \cdot, 0, 1)$ is a vector space over itself and so F_F is a vector space by our notation. In particular, $\mathbb{R}_{\mathbb{R}}$, $\mathfrak{C}_{\mathfrak{C}}$, and K_{K} are vector spaces.
Let V_{K} be a vector space. By *points of* V_{K} are meant the elements of the ground set V. That is, by $x \in V_{\mathrm{K}}$ is simply meant $x \in V$. Similarly, by a *net* $(x_i)_{i\in(I,\,\le)}$ *in* V_{K} is simply meant that $(x_i)_{i\in(I,\,\le)}$ is a net in V. Of course, by a *set* of V_{K} is simply meant a subset of V. Let V_{K} and W_{K} be vector spaces. A map $\ell : V_{\mathrm{K}} \to W_{\mathrm{K}}$ is called a *linear map*, and said to be *linear*, if for $\nu_1, \nu_2 \in V_{\mathrm{K}}$ and $\alpha \in \mathrm{K}$,
(i) $\ell(\nu_1 + \nu_2) = \ell(\nu_1) + \ell(\nu_2)$
and
(ii) $\ell(\alpha\nu) = \alpha\ell(\nu)$.
Immediate from (i) and (ii) is that, for $\nu_1, \nu_2 \in V_{\mathrm{K}}$, $\alpha, \beta \in \mathrm{K}$,
(iii) $\ell(\alpha\nu_1 + \beta\nu_2) = \alpha\ell(\nu_1) + \beta\ell(\nu_2)$.

FACT 1 Let $(V, +, \theta)_{\mathrm{K}}$ and $(V', +, \theta')_{\mathrm{K}}$ be vector spaces, and $\ell : V_{\mathrm{K}} \to V'_{\mathrm{K}}$ a linear map. Then, $\ell(\theta) = \theta'$. ///
Let V_{K} be a vector space. A linear map $\ell : V_{\mathrm{K}} \to \mathrm{K}_{\mathrm{K}}$ is called a *linear functional*. From the preceding FACT 1 follows that
FACT 2 If $(V, +, \theta)_{\mathrm{K}}$ is a vector space and $\ell : (V, +, \theta)_{\mathrm{K}} \to (\mathrm{K}, +, \cdot, 0, 1)_{\mathrm{K}}$ is a linear functional, then, $\ell(\theta) = 0$. ///

6 SEMINORM Let $(V, +, \theta)_{(\mathrm{K},+,\cdot,\,0,\,1)} = V_{\mathrm{K}}$ be a vector space. A map
$p : V_{\mathrm{K}} \to \mathbb{R}$
such that, for $x, y \in V_{\mathrm{K}}$ and $t \in \mathrm{K}$,
(i) $p(x) \ge 0$,
(ii) $p(tx) = |t|\,p(x)$,
and
(iii) $p(x + y) \le p(x) + p(y)$
is called a *seminorm* on V_{K}. Clearly, by (ii) and the fact that $0\theta = \theta$,
(iv) $p(\theta) = 0$
If
(v) $p(x) = 0 \Rightarrow x = \theta$,
then, p is called a *norm* on V_{K}.
Let p be a seminorm on the vector space V_{K}. Define
$d_p ; V_{\mathrm{K}}\text{x}V_{\mathrm{K}} \to \mathbb{R}$
$(x, y) \mapsto p(x - y)$

FACT 1 (i) d_p is a pseudometric on V_{K}, and
(ii) if p is a norm, then d_p is a metric. ///
The pseudometric/metric topology, τ_{d_p}, of the pseudometric/metric d_p, is called the *seminorm/norm topology of the seminorm/norm p*.
We here in this paper, briefly write τ_p for τ_{d_p}.
Example 2(i) The absolute value function

$$p : \mathbb{R}_{\mathbb{R}} \to \mathbb{R}$$

$$x \mapsto |\,x\,|$$

is a norm on $\mathbb{R}_{\mathbb{R}}$. Clearly, here $\tau_p = \tau_{d\,|\,|} = \tau_{\mathbb{R}}$.
(ii) The modulus function

$$p : \mathfrak{C}_{\mathfrak{C}} \to \mathbb{R}$$

$$z \mapsto |\,z\,|$$

is a norm on $\mathfrak{C}_{\mathfrak{C}}$. Also, here

$$\tau_p = \tau_{d\,|\,|} = \tau_{\mathfrak{C}}.$$

Example 3 Let $\ell : V_{\mathrm{K}} \to \mathrm{K}_{\mathrm{K}}$ be a linear functional on the vector space V_{K}. Then,

$$p : V_{\mathrm{K}} \to \mathbb{R}$$

$$\nu \mapsto |\ell(\nu)|$$

is a seminorm on V_{K}.
From 2.5 $\Leftrightarrow^3$ now follows that
THEOREM 4 If p is a seminorm on the vector space V_{K}, $x \in V_{\mathrm{K}}$, and $(x_i)_{i\in(I,\,\leq)}$ a net in V_{K}, then,

$$x_i \xrightarrow{\tau_p} x \Leftrightarrow p(x_i - x) \xrightarrow{\tau_R} 0. \ ///$$

We now prove

THEOREM 5 Let
(i) V_{K} be a vector space,
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(ii) p a seminorm on V_{K},
(iii) $(t_i)_{i\in(I,\,\leq)}$ a net in K τ_{K}-converging to t,
(iv) $(x_i)_{i\in(I,\,\leq)}$ a net in V_{K} τ_p-converging to x.
Then, the net $(t_ix_i)_{i\in(I,\,\leq)}$ τ_p-converges to tx.

Proof By THEOREM 4, it suffices to show that

$$p(t_ix_i - tx) \xrightarrow{\tau_R} 0.$$

And this we proceed to do. Clearly,

$$p(t_ix_i - tx) = p(t_ix_i - t_i\,x + t_i\,x - tx)$$
$$= p(t_i\,(x_i - x) + (t_i - t)x)$$
$$\leq p(t_i\,(x_i - x)) + p((t_i - t)x)$$
$$= |\,t_i\,|\,p(x_i - x) + |t_i - t\,|\,p(x).$$

That is,

$$p(t_i\,x_i - tx) \leq |\,t_i\,|p(x_i - x) + |t_i - t\,|p(x) \qquad \ldots..(\Delta)$$

By THEOREM 4,

$$p(x_i - x) \xrightarrow{\tau_R} 0 \qquad \ldots..(\nabla)$$

By 2.14(i), since $t_i \xrightarrow{\tau_R} t$, the net $(t_i)_{i\in(I,\,\leq)}$ in K is eventually bounded, and so, the net $(|t_i\,|)_{i\in(I,\,\leq)}$ in $\mathbb{R}$ is also eventually bounded. Hence, it follows from 2.14(ii) and (∇) that

$$|\,t_i\,|\,p(x_i - x) \xrightarrow{\tau_R} 0 \qquad \ldots.(\rho^1)$$

Clearly,

$$|t_i - t|\,p(x) \xrightarrow{\tau_R} 0 \qquad \ldots.(\rho^2)$$

By 2.13, therefore, (ρ^1) and $((\rho^2)$ give

$| t_i | p(x_i - x) + |t_i - t | p(x) \xrightarrow{\tau_R} 0$ ….(ΔΔ)

Clearly (Δ) and (ΔΔ) give

$p(t_i x_i - t x) \xrightarrow{\tau_R} 0.$ ///

THEOREM 6 Let V_K be a vector space, $x, y \in V_K$ and p a semi- norm on V_K. Suppose $(x_i)_{i\in(I, \leq)}$ and $(y_i)_{i\in(I, \leq)}$ are nets in V_K, based on same directed set $(I, \leq)$, such that

$x_i \xrightarrow{\tau_p} x$ and $y_i \xrightarrow{\tau_p} y$ ……(Σ^1)

Then,

$x_i + y_i \xrightarrow{\tau_p} x + y$ ……(Σ^2)

Proof To prove (Σ^2), it suffices by THEOREM 4 to show that

$p((x_i + y_i) - (x + y)) \xrightarrow{\tau_R} 0$ ……(Σ^3)

And this, we do. Again by THEOREM 4 and hypotheses (Σ^1)

$p(x_i - x) \xrightarrow{\tau_R} 0$ …….(Σ^4)

and

$p(y_i - y) \xrightarrow{\tau_R} 0$ …….(Σ^5)

By 2.13, therefore, (Σ^4) and (Σ^5) give

$p(x_i - x) + p(y_i - y) \xrightarrow{\tau_R} 0$ …….(Σ^6)

Clearly,

$p((x_i + y_i) - (x + y)) = p((x_i - x) + (y_i - y))$

$\leq p(x_i - x) + p(y_i - y)$.

That is,

$p((x_i + y_i) - (x + y)) \leq p(x_i - x) + p(y_i - y)$ ……(Σ^7)

Clearly from (Σ^6) and (Σ^7) is that

$p((x_i + y_i) - (x + y)) \xrightarrow{\tau_R} 0$

which is (Σ^3). ///

7 TOPOLOGICAL VECTOR SPACE Let V_K be a vector space and τ a topology on V_K. The topology τ is said to be *compatible* with the vector space structure of V_K provided *addition*

$Ad : (V_K \text{x } V_K, \tau \text{x} \tau) \to (V_K, \tau)$

$(\nu,\omega) \mapsto \nu+\omega$

and *scalar multiplication*

$Sc : (K_K \text{x} V_K , \tau_K \text{x} \tau) \to (V_K, \tau)$

$(\alpha,\nu) \mapsto \alpha\nu$

are continuous mappings. If this is so, we call τ a *vector topology*, and the topological space (V_K, τ) called a *topological vector space*.

THEOREM 1 If p is a seminorm on the vector space V_K, then the seminorm topology, τ_p , of p on V_K, is a vector topology, and (V_K, τ_p) is a topological vector space.

Proof We have to show that addition

$Ad : (V_K \text{ x } V_K, \tau_p \text{x} \tau_p) \to (V_K, \tau_p)$

$(\nu, \omega) \mapsto \nu + \omega$

and scalar multiplication

$Sc : (K_K \text{x} V_K , \tau_K \text{x} \tau_p) \to (V_K, \tau_p)$

$(\alpha, \nu) \mapsto \alpha\nu$

are continuous mappings. For *Ad*, let $(\nu, \omega) \in V_K \text{ x } V_K$ and $((\nu_i, \omega_i))_{i\in(I, \leq)}$ a net in $V_K \text{ x } V_K$.

Suppose

$(\nu_i\,,\omega_i) \xrightarrow{\tau_p \text{ x } \tau_p} (\nu, \omega)$ ….(δ^1)
Then, by 3.1, therefore,

$\nu_i \xrightarrow{\tau_p} \nu$

and $\omega_i \xrightarrow{\tau_p} \omega$.
By 6.6, therefore,

$Ad((\nu_i, \omega_i)) = \nu_i + \omega_i \xrightarrow{\tau_p} \nu + \omega$ ….(δ^2)
Clearly, by 4.4, (δ^1) and (δ^2), *Ad* is a continuous mapping. Similarly, *Sc* is a continuous mapping. ///

One also deduces from 2.12 that
THEOREM 2 τ_K is a vector topology on K_K, and so (K_K,, τ_K) is a topological vector space. ///
We conclude this paper with two applications of the *Continuity of the partial maps*.

THEOREM 3 *Continuity of Translation* Let (V_K, τ) be a topolo- gical vector space, and suppose $a \in V_K$. Then, *translation*
$T_a : (V_K, \tau) \to (V_K, \tau)$
$\omega \mapsto a + \omega$
is continuous (Indeed, a *homeomorphism*).
Proof By hypothesis, addition
$Ad\ :\ (V_K x V_K, \tau x \tau) \to (V_K, \tau)$
$(\nu, \omega) \mapsto \nu + \omega$
is continuous. By 4.8, therefore,
$Ad_a : (V_K, \tau) \to (V_K, \tau)$
$\omega \mapsto a + \omega$
is continuous. Clearly, $T_a = Ad_a$. ///
Let (V_K, τ) be a topological vector space and $\alpha \in K$. From the continuity of the scalar multiplication
$Sc\ :\ (K_K x^{\vee_K}, \tau^K x \tau) \to (^{\vee_K}, \tau)$
$(t, \nu) \mapsto t\nu$
and 4.8, therefore, follows that
$Sc_\alpha : (V^K, \tau) \to (^{\vee_K}, \tau)$
$\nu \mapsto \alpha\nu$
is continuous. If we take $\alpha = -1$, then we have
THEOREM 4 *Continuity of the Additive Inversion* Let (V^K, τ) be a topological vector space. The additive inversion
$h : (V^K, \tau) \to (V^K, \tau)$
$\nu \mapsto -\nu$
is continuous. ///

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