Transactions of the Nigerian Association of Mathematical Physics
***Volume 12, (July – Sept., 2020 Issue*), pp11 – 16**
© Trans. of NAMP

SOME SHORT NOTES ON THE TOPOLOGY OF THE SEMINORM

Sunday Oluyemi

Former ARMTI Quarters, Off Basin Road, Basin Area, Ilorin, Kwara State.

Abstract

We prove that the topology of a seminorm is a vector topology, and then describe two local bases of neighbourhoods of zero, one from General Topology (GT) and another from Topological Vector Spaces (TVS). Some seminorm topologies' equalities and inequalities are also established.

***Keywords*:** Topology, vector space, seminorm, topolo- gical vector spaces, neighbourhood of zero, local base of neighbour- hoods.

1 LANGUAGE AND NOTATION If somebody describes the study of *Locally Convex Spaces* in Topological Vector Spaces as *the study of the topology of the seminorm*, it is very unlikely that *heresy charges* are consequently preferred against him/her. And so, therefore, a fortiori, it may not be out of place to point out in detail some elementary facts, here called *Short Notes*, about the topology of the seminorm, that the literature is subconscious of but not fully conscious of.

I GT *and* **TVS** The *Abstract* succinctly describes the task of this paper. A minimum of background is, of course, needed. And so, we assume the reader is familiar with the beginnings of :

(i) General Topology (**GT**)

and

(ii) Topological Vector Spaces **(TVS).**

II *Language and Notation* Our language and notation shall be pretty standard as found, for example, in the classics [1], [2], [3] and [4]. We denote by $\mathbb{R} = (\mathbb{R}, +, \cdot, 0, 1)$ the *real number field* and by $\mathfrak{C} = (\mathfrak{C}, +, \cdot, 0, 1)$ the *complex number field*. By $\mathbb{K} = (\mathbb{K}, +, \cdot, 0, 1)$ we denote either $\mathbb{R}$ or $\mathfrak{C}$. By /// we denote the end or absence of a proof.

III $\mathscr{N}_x(\tau)$ Let X be a non-empty set and τ a topology on X. The pair (X,τ) is called a *topological space*. The elements of X are called the *elements* of (X,τ) and subsets of X are called *sets* of (X,τ). The reader, of course, knows what are called the *open sets* of $(X,\tau) \equiv$ the sets that comprise the family τ. Let $x \in X$, $\varnothing \neq G \in \tau$, and $x \in G \subseteq U \subseteq X$. The set U is called a *neighbourhood of x*, or for emphasis, a τ-*neighbourhood* of x. The family of all the neighbourhoods of x is denoted $\mathscr{N}_x(\tau)$ and called the (τ-) *neighbourhood system of x*, or, *at x*. A subfamily, $\mathcal{R}_x(\tau)$, say, of $\mathscr{N}_x(\tau)$ such that for every $W \in \mathscr{N}_x(\tau)$ there exists $U \in \mathcal{R}_x(\tau)$ with $U \subseteq W$, is called a *local base*, or, for emphasis, a τ-*local base*, of *neighbourhoods of x*, or *at x*.

IV *Topological Vector Space* Let $(V, +, \theta)$ be an additive Abelian group with an external multiplication by the elements of $\mathbb{K}$,

$\mathbb{K}\mathrm{x}(V, +, \theta) \to (V, +, \theta)$

$(\lambda,\nu) \mapsto \lambda\nu$

satisfying : For $\alpha, \beta \in \mathbb{K}$ and $\upsilon, \omega \in V$,

(a) $1\nu = \nu$

(b) $\alpha(\nu + \omega) = \alpha\nu + \alpha\omega$

(c) $(\alpha\beta)\nu = \alpha(\beta\nu) = \beta(\alpha\nu)$

(d) $(\alpha + \beta)\nu = \nu + \beta\nu$.

We say that $(V, +, \theta)$ is a *vector space over* $\mathbb{K}$; we simply write $(V, +, \theta)_{(\mathbb{K},+,\cdot,0,1)}$ or $(V, +, \theta)_{\mathbb{K}}$ or simply $V_{\mathbb{K}}$ for a vector space over $\mathbb{K}$. The addi- tive identity, θ, of $(V, +, \theta)$ is called the *zero* of the vector space $(V, +, \theta)_{(\mathbb{K},+,\cdot,0,1)} = (V, +, \theta)_{\mathbb{K}} = V_{\mathbb{K}}$. An element ν of V is called an *element* of $V_{\mathbb{K}}$ and we write $\nu \in V_{\mathbb{K}}$. Similarly, a subset U of V is called a *set of* $V_{\mathbb{K}}$. Let $\varnothing \neq A, B$ be sets of $V_{\mathbb{K}}$ and $\lambda \in \mathbb{K}$. We define $A + B = \{a + b : a \in A, b \in B\}$ and $\lambda A = \{\lambda a : a \in A\}$.

V Let $(V, +, \theta)_{\mathbb{K}} = V_{\mathbb{K}}$ be a vector space. A positive function $p : V_{\mathbb{K}} \to \mathbb{R}$ such that, for $\lambda \in \mathbb{K}$ and $\nu, w \in V_{\mathbb{K}}$, we have $p(\lambda\nu) = |\lambda|p(\nu)$ and $p(\nu + w) \leq p(\nu) + p(w)$, is called a *seminorm* on $V_{\mathbb{K}}$. Clearly, $p(\theta) = 0$. If $p(\nu) \neq 0$ for $\nu \neq \theta$, p is called a *norm*. The function

$dp : V_{\mathbb{K}}\mathrm{x}V_{\mathbb{K}} \to V_{\mathbb{K}}$

$(\nu, \omega) \mapsto p(\nu - \omega)$

Correspondence Author: Sunday O., Email: soluyemi19@yahoo.com, Tel: +2348160865176

is a pseudometric on $V_{\mathbb{K}}$. The topology of this pseudometric, denoted τ_p in this paper, is called the *topology of the seminorm p*. If p is a norm, τ_p is, of course, called a *norm topology.*

Well-known is that $\mathbb{R}_{\mathbb{R}}$, $\mathfrak{C}_{\mathfrak{C}}$, and so, $\mathbb{K}_{\mathbb{K}}$ are vector spaces, since any field is a vector space over itself. Well-known is that $\mathbb{R}$ has its *usual topology,* which is a norm topology; we denote it $\tau_{\mathbb{R}}$ in this paper. Similarly, the Euclidean topology of $\mathfrak{C} = \mathbb{R}^2$ is also called $\mathfrak{C}$'s usual topology; we denote it here by $\tau_{\mathfrak{C}}$. Hence, since $\mathbb{K} = \mathbb{R}$ or $\mathfrak{C}$, $\mathbb{K}$'s topology shall be denoted $\tau_{\mathbb{K}}$, it is a norm topology.

VI Let $(V, +, \theta)_{\mathbb{K}} = V_{\mathbb{K}}$ be a vector space, and τ a topology on $V_{\mathbb{K}}$. We call τ a *vector topology*, and the topological space $(V_{\mathbb{K}}, \tau)$ called a *topological vector space* if *addition*

Add: $(V_{\mathbb{K}} x V_{\mathbb{K}}, \tau x \tau) \to (V_{\mathbb{K}}, \tau)$

$(\nu, \omega) \mapsto \nu + \omega$

and *scalar multiplication*

Sc: $(\mathbb{K} x V_{\mathbb{K}}, \tau_{\mathbb{K}} x \tau) \to (V_{\mathbb{K}}, \tau)$

$(\lambda, \nu) \mapsto \lambda\nu$

are continuous functions.

VII By $\mathbb{N}$ we denote the *natural numbers* 1, 2, …… We indicate by /// the end or absence of a proof,

VIII Let X be a non-empty set and suppose that τ_1 and τ_2 are topologies on X. If $\tau_1 \subseteq \tau_2$, we say that τ_1 is *coarser* than τ_2 and that τ_2 is finer than τ_1, and we indicate this by writing $\tau_1 \le \tau_2$. If Φ is a collection of topologies on X, clearly, the topology 2^X ($\equiv$ the family of all the subsets of X) is finer than each $\tau \in \Phi$. There exists a coarsest of all the topo- logies on X that are finer than each $\tau \in \Phi$. This coarsest topology is called the *supremum* of Φ and denoted $\vee\Phi$.

Let $X \neq \varnothing$, $(\tau_i)_{i \in I}$ a family of topologies on X and suppose $x_0 \in X$. Let $\mathcal{R}_x{}^0(\tau_i)$ be a τ_i – local base of neighbourhoods of x_0, $i \in I$. We have from (**GT**)

FACT 1 Let $X \neq \varnothing$, $(\tau_i)_{i \in I}$ a family of topologies on X, $x_0 \in X$, and for $i \in I$, $R_{x_0}(\tau_i)$ is a τ_i –local base of neighbourhoods of x_0. Then, a $\underset{i \in I}{\vee} \tau_i$ -local base of neighbourhoods of x_0 is given by the family

$\Re_{x_0}\left(\underset{i \in I}{\vee} \tau_i\right) = \{ B_{i_1} \cap B_{i_2} \cap \ldots\ldots\cap B_{i_n} : n \in \mathbb{N},\ B_{i_k} \in \Re_{x_0}(\tau_{i_k})$ for $k = 1, 2, \ldots.., n\}$[| The indices $i_1, i_2, \ldots\ldots., i_n$ needn't be distinct. |]. ///

Let $X \neq \varnothing$ and $d : X x X \to \mathbb{R}$ be a pseudometric on X. Let $x_0 \in X$. By $B_d(x_0, r)(r > 0)$ we denote the *ball of radius r centred on* x_0. By definition, the topology, τ_d, of the pseudometric d comprises of the empty set $\varnothing$ and unions of balls. And the family

$\Re_{x_0}(\tau_d) = \{B_d(x_0, r) : r > 0\}$ ….(Δ*)

is a τ_d –local base of neighbourhoods of x_0.

Let D be a collection of pseudometrics on the non-empty set X, and $(\tau_d)_{d \in D}$ the family of the pseudometric topologies of the pseudometrics of D. Clearly, we have from FACT 1 and (Δ*).

FACT 2 Let D be a collection of pseudometrics on the non-empty set X. Then, a $\underset{d \in D}{\vee} \tau_d$-local base of neighbourhoods of $x_0 \in X$ is the family

$\{ B_{d_1}(x_0, r_1) \cap B_{d_2}(x_0, r_2) \cap \ldots\ldots\cap B_{d_n}(x_0, r_n) : n \in \mathbb{N}, r_1, r_2, \ldots., r_n > 0, \quad d_1, d_2, \ldots\ldots, d_n \in D\}$. ///

Let $(V, +, \theta)_{\mathbb{K}} = V_{\mathbb{K}}$ be a vector space and p a seminorm on $V_{\mathbb{K}}$. Form the pseudometric of p :

$dp : V_{\mathbb{K}} x V_{\mathbb{K}} \to \mathbb{R}$

$(\nu, \omega) \mapsto p(\nu - \omega)$ ….(∇)

From (Δ*), a τ_p-local base of neighbourhoods of θ is

$\mathcal{R}_\theta(\tau_p) = \{B_{dp}(\theta, r) : r > 0\}$

whereby (∇),

$B_{dp}(\theta, r) = \{\nu \in V_{\mathbb{K}} : p(\nu - \theta) < r\}$

$= \{\nu \in V_{\mathbb{K}} : p(\nu) < r\}$.

That is,

$\mathcal{R}_\theta(\tau_p) = \{B_{r\theta}(\tau_p) : r > 0\}$

where

$B_{r\theta}(\tau_p) = B_{dp}(\theta, r)$

$= \{\nu \in V_{\mathbb{K}} : p(\nu) < r\}$.

That is,

$\mathcal{R}_\theta(\tau_p) = \{B_{r\theta}(\tau_p) : r > 0\}$ ….($\Delta\Delta^*$)

Now suppose P is a collection of seminorms on the vector space $(V, +, \theta)_{\mathbb{K}} = V_{\mathbb{K}}$. We define $\tau_P = \underset{p \in P}{\vee} \tau_p$. Clearly, from FACT 2 and ($\Delta\Delta^*$), we have

FACT 3 Let P be a collection of seminorms on the vector space $(V, +, \theta)_{\mathbb{K}} = V_{\mathbb{K}}$. Then, a τ_P-local base of neighbourhoods of θ, $\mathcal{R}_\theta(\tau_P)$, is given by $\mathcal{R}_\theta(\tau_P) = \{ B_{r_1\theta}(\tau_{p_1}) \cap B_{r_2\theta}(\tau_{P_2}) \cap \ldots\ldots \cap B_{r_n\theta}(\tau_{P_n}) : n \in \mathbb{N}, r_1, r_2, \ldots, r_n > 0, p_k \in P \text{ for } k = 1, 2, \ldots, n\}$. ///

IX Definition 1 Let $(V, +, \theta)_{\mathbb{K}} = V_{\mathbb{K}}$ be a vector space and suppose p is a seminorm on $V_{\mathbb{K}}$. We define
$p(\leq 1) \equiv \{\nu \in V_{\mathbb{K}} : p(\nu) \leq 1\}$
and, define
$\overline{B}_{r\theta} \equiv \{\nu \in V_{\mathbb{K}} : p(\nu) \leq r\}$.

Clearly, $p(\leq 1) = \overline{B}_{1\theta}$.

X Let $(V, +, \theta)_{\mathbb{K}} = V_{\mathbb{K}}$ be a vector space and $\varnothing \neq A \subseteq V_{\mathbb{K}}$. If for every $x \in V_{\mathbb{K}}$ there exists $\varepsilon(x) > 0$ such that
$\lambda x \in A$ for all $\lambda \in \mathbb{K}, |\lambda| \leq \varepsilon(x)$
the set A is said to be *absorbing*.
If $\lambda A \subseteq A$ for all $\lambda \in \mathbb{K}, |\lambda| \leq 1$, the set A is said to be *balanced*.

Example 1 Let $(V, +, \theta)_{\mathbb{K}} = V_{\mathbb{K}}$ be a vector space, p a seminorm on $V_{\mathbb{K}}$ and $r > 0$. One checks easily, from definitions, that $\overline{B}_{r\theta}$ is absorbing and balanced.

Observation 2 Let $(V, +, \theta)_{\mathbb{K}} = V_{\mathbb{K}}$ be a vector space, and suppose the set A of $V_{\mathbb{K}}$ is absorbing/balanced. Then, $\theta \in A$.

XI *Theorems from* ***TVS*** In the next section we shall employ some results of **TVS**, possibly without citation.

FACT 1 Let τ and τ' be two vector topologies on the vector space $(V, +, \theta)_{\mathbb{K}} = V_{\mathbb{K}}$. Then,

(i) $\mathcal{N}_\theta(\tau) \subseteq \mathcal{N}_\theta(\tau') \Leftrightarrow \tau \leq \tau'$
and
(ii) $\mathcal{N}_\theta(\tau) = \mathcal{N}_\theta(\tau') \Leftrightarrow \tau = \tau'$. ///

FACT 2 Let $(V, +, \theta)_{\mathbb{K}} = V_{\mathbb{K}}$ be a vector space, and τ, τ' two vector topologies on $V_{\mathbb{K}}$. If $\mathcal{N}_\theta(\tau)$ includes all the members of a τ'-local base of neighbourhoods of θ, then $\mathcal{N}_\theta(\tau') \subseteq \mathcal{N}_\theta(\tau)$ and $\tau' \leq \tau$. ///

FACT 3 Let $((V, +, \theta)_{(\mathbb{K},+,\cdot,0,1)}, \tau) = (V_{\mathbb{K}}, \tau)$ be a topological vector space, and suppose

(i) $U \in \mathcal{N}_\theta(\tau)$,
and
(ii) $\lambda \in \mathbb{K}, \lambda \neq 0$.

Then, $\lambda U \in \mathcal{N}_\theta(\tau)$, ///

FACT 4 Let $V_{\mathbb{K}}$ be a vector space, and $P = \{ p_1, p_2, \ldots\ldots, p_n\}, n \in \mathbb{N}, n \geq 2$, seminorms on $V_{\mathbb{K}}$. Define $P_{\max}$ by

$P_{\max}(x) = \max\limits_{1 \leq i \leq n} p_i(x), x \in V_{\mathbb{K}}$.

Then,

$P_{\max}(\leq 1) = \bigcap\limits_{1 \leq i \leq n} p_i(\leq 1)$.

Proof A simple set inclusion Proof. Let $x \in P_{\max}(\leq 1)$. Then, $x \in p_i(\leq 1)$ for all $i \in \{1, 2, \ldots, n\}$ and so, $x \in \bigcap\limits_{1 \leq i \leq n} p_i(\leq 1)$. We have thus shown that $P_{\max}(\leq 1) \subseteq \bigcap\limits_{1 \leq i \leq n} p_i(\leq 1)$. The proof of the reverse inclusion $\supseteq$ is similar. ///

2 SHORT NOTES

Note 1 *The Pseudometric topology of a semimorm is a vector topology* [5].

Note 2 $rp(\leq 1) = \overline{B}_{r\theta}$. Let $(V, +, \theta)_{\mathbb{K}} = V_{\mathbb{K}}$ be a vector space, p a semi- norm on $V_{\mathbb{K}}$ and $r > 0$. We want to prove that

$rp(\leq 1) = \overline{B}_{r\theta}$ …..(ρ^*)

We go by *set-inclusion* proof. Clearly, either side of (ρ^*) is not empty as θ belongs to them. Clearly $x \in rp(\leq 1) \Leftrightarrow (1/r)x \in p(\leq 1) \Leftrightarrow p((1/r)x) \leq 1$ which by the absolute homogeneity of the seminorm,

$\Leftrightarrow (1/r)p(x) \leq 1 \Leftrightarrow p(x) \leq r \Leftrightarrow x \in \overline{B}_{r\theta}$

Note 3 *Two local bases for the topology of the seminorm* By *Note* 1, the topology τ_p of a seminorm p on a vector space $(V, +, \theta)_{\mathrm{K}} = V_{\mathrm{K}}$ is a vector topology. And, for a vector topology τ on V_{K}, for almost all considerations, it suffices to restrict attention to $\mathcal{N}_\theta(\tau)$. This, we do. By a *filterbase* on V_{K} is meant a family $\mathcal{A}$ of non-empty sets of V_{K} such that if $A, B \in \mathcal{A}$, there exists $C \in \mathcal{A}$ and $C \subseteq A \cap B$.
We have

THEOREM 1 [3, Section 2.3 THEOREM 1] Let $(V, +, \theta)_{\mathrm{K}}$ be a vector space, and $\mathcal{R}_\theta$ a filterbase of balanced, absorbing sets satisfying : If $W \in \mathcal{R}_\theta$, then there exists $U \in \mathcal{R}_\theta$, such that $U + U \subseteq W$. Then, there exists a *unique* topology τ making (V_{K}, τ) a topological vector space and $\mathcal{R}_\theta$ a τ-local base of neighbourhoods of θ. /// Let $(V, +, \theta)_{\mathrm{K}} = V_{\mathrm{K}}$ be a vector space, p a seminorm on V_{K} and $r > 0$. By Example 1 of 1. **X**, $\overline{B}_{r\theta}$ is absorbing and balanced. Define

$\mathcal{R}_\theta \equiv \{ \overline{B}_{r\theta} : r > 0\}$

One sees easily that $\mathcal{R}_\theta$ is a filterbase of absorbing, balanced sets. For if $\overline{B}_{r_1\theta}$, $\overline{B}_{r_2\theta} \in \mathcal{R}_\theta$, and $r = \min\{r_1, r_2\}$, then $\overline{B}_{r\theta} \in \mathcal{R}_\theta$ and $\overline{B}_{r\theta} \subseteq \overline{B}_{r_1\theta} \cap \overline{B}_{r_2\theta}$. One sees easily also that for $r > 0$. $\overline{B}_{(r/2)_\theta} + \overline{B}_{(r/2)_\theta} \subseteq \overline{B}_{r\theta}$. So, therefore, by THEOREM 1 above, $\mathcal{R}_\theta$ is a τ^*-local base of neighbourhoods of θ, for a *unique* vector topology τ^*, say, on $(V, +, \theta)_{\mathrm{K}}$.
Thus, we have that with p a seminorm on a vector space $(V, +, \theta)_{\mathrm{K}} = V_{\mathrm{K}}$ there exist two vector topologies on V_{K}, τ^* described above for which $\mathcal{R}_\theta$ is a local base of neighbourhoods of θ, and the pseudometric topology, τ_p, of p, for which $\mathcal{R}_\theta(\tau_p)$ is a local base of neighbourhoods of θ. From 1. VIII

$\mathcal{R}_\theta(\tau_p) = \{B_{r\theta}(\tau_p) : r > 0\}$.
And,

$\mathcal{R}_\theta = \{ \overline{B}_{r\theta} : r > 0\}$.
Also,

$B_{r_\theta}(\tau_p) = \{\nu \in V_{\mathrm{K}} : p(\nu) < r\}$
and

$\overline{B}_{r\theta} = \{\nu \in V_{\mathrm{K}} : p(\nu) \leq r\}$
One sees easily that, for $r > 0$,

$B_{r_\theta}(\tau_p) \subseteq \overline{B}_{r\theta}$ …..(ρ^1)
and

$\overline{B}_{(r/2)\theta} \subseteq B_{r_\theta}(\tau_p)$ …..(ρ^2)

Clearly, by the definition of *neighbourhood*, and the definition of *local base of neighbourhoods*, it follows from (ρ^1) and (ρ^2), respectively, that
$\mathcal{N}_\theta(\tau^*) \subseteq \mathcal{N}_\theta(\tau_p)$
and $\mathcal{N}_\theta(\tau_p) \subseteq \mathcal{N}_\theta(\tau^*)$.
And hence, $\mathcal{N}_\theta(\tau_p) = \mathcal{N}_\theta(\tau^*)$. And so, therefore, $\tau_p = \tau^*$.
Thus, we have

THEOREM 2 Let $(V, +, \theta)_{\mathrm{K}} = V_{\mathrm{K}}$ be a vector space, and p a semi- norm on V_{K}. Then,

$\mathcal{R}_\theta(\tau_p) = \{B_{r\theta}(\tau_p) : r > 0\}$
where $B_{r\theta}(\tau_p) = \{\nu \in V_{\mathrm{K}} : p(\nu) < r\}$
and
$\mathcal{R}_\theta = \{ \overline{B}_{r\theta} : r > 0\}$
where $\overline{B}_{r\theta} = \{\nu \in V_{\mathrm{K}} : p(\nu) \leq r\}$,
are local bases of neighbourhood of θ for the topology τ_p of the seminorm p. ///
Clearly, the above theorem and Note 2 give

COROLLARY 3 Let $(V, +, \theta)_{\mathrm{K}} = V_{\mathrm{K}}$ be a vector space, and p a seminorm on V_{K}. Then, a τ_p-local base of neighbourhoods of θ is the family

$\mathcal{R}_\theta = \{rp(\leq 1) : r > 0\}$.
In particular, $p(\leq 1) \in \mathcal{N}_\theta(\tau_p)$. ///
We have from the above **COROLLARY 3** and FACT 1 of . 1.VII,

COROLLARY 4 Let $(V, +, \theta)_{\mathrm{K}} = V_{\mathrm{K}}$ be a vector space, and P a collection of seminorms on V_{K}. Then, a $\tau_P(= \bigvee_{p \in P} \tau_p)$-local base of neighbourhoods of θ is the family

$\mathcal{R}_\theta(\bigvee_{p\in P} \tau_p) = \{r_1p_1(\leq 1)\cap r_2p_2(\leq 1)\cap\ldots\ldots\cap r_np_n(\leq 1) : n \in\mathbb{N}, r_1, r_2, \ldots\ldots., r_n > 0, p_1, p_2, \ldots., p_n \in P \}$. ///

Note 4 αp, $\tau_p = \tau_{\alpha p}$, $\alpha > 0$. Let $V_\mathbb{K}$ be a vector space, p a seminorm on $V_\mathbb{K}$, and $\alpha > 0$. We show first that αp is also a seminorm on $V_\mathbb{K}$.

FACT 1 Let $V_\mathbb{K}$, be a vector space, p a seminorm on $V_\mathbb{K}$. and $\alpha > 0$. Then, αp is also a seminorm.

Proof The *positivity* of αp is clear. Let $x \in V_\mathbb{K}$ and $\lambda \in \mathbb{K}$. Then

$$(\alpha p)(\lambda x) = \alpha(p(\lambda x)) = \alpha(|\lambda|p(x))$$
$$= |\lambda|(\alpha p(x)) = |\lambda|(\alpha p)(x).$$

That is,

$(\alpha p)(\lambda x) = |\lambda|(\alpha p))(x)$,

confirming the *absolute homogeneity* of αp.

Now, let $x, y \in V_\mathbb{K}$. then,

$(\alpha p)(x + y) = \alpha(p(x + y)) \leq \alpha(p(x) + p(y))$

$= \alpha p(x) + \alpha p(y) = (\alpha p)(x) + (\alpha p)(y)$.

That is,

$(\alpha p)(x + y) \qquad \leq (\alpha p)(x) + (\alpha p)(y)$. ///

Next, we prove

FACT 2 Let $(V, +, \theta)_\mathbb{K} = V_\mathbb{K}$ be a vector space, p a seminorm on $\vee_\mathbb{K}$, and $\alpha > 0$. Then, $\tau_p = \tau_{\alpha p}$.

Proof $\boldsymbol{\tau_p \leq \tau_{\alpha p}}$**:** Let $x \in (\alpha p)(\leq 1)$. Then, $(\alpha p)(x) \leq 1$. That is, $\alpha\cdot p(x) \leq 1$. So, $p(x) \leq 1/\alpha$, and therefore,

$x \in \overline{B}_{(1/\alpha)\theta}$ …..(η^1)

By Note 2,

$\overline{B}_{(1/\alpha)\theta} = (1/\alpha)p(\leq 1)$ ……(η^2)

and so, by (η^1) and (η^2), $x \in (1/\alpha)p(\leq 1)$ and thus, we have shown that

$(\alpha p)(\leq 1) \subseteq (1/\alpha)p(\leq 1)$.

By COROLLARY 3 of Note 3, therefore,

$(1/\alpha)p(\leq 1) \in \mathring{\mathcal{N}}_\theta(\tau_{\alpha p})$.

And from this follow by FACT 3 of 1. **XI** that

$p(\leq 1) = \alpha\cdot(1/\alpha)p(\leq 1) \in \mathring{\mathcal{N}}_\theta(\tau_{\alpha p})$.

Again, from this follows by same FACT 3 of 1. **XI** that, for $r > 0$,

$rp(\leq 1) \in \mathring{\mathcal{N}}_\theta(\tau_{\alpha p})$.

From this follows by COROLLARY 3 of Note 3 and FACT 2 of 1. **XI** that

$\mathring{\mathcal{N}}_\theta(\tau_p) \subseteq \mathring{\mathcal{N}}_\theta(\tau_{\alpha p})$.

By FACT 1(i) of !.XI, therefore,

$\tau_p \leq \tau_{\alpha p}$ ….(1)

$\boldsymbol{\tau_{\alpha p} \leq \tau_p}$: Let $x \in p(\leq 1)$, and so, $p(x) \leq 1$. Therefore, $\alpha\cdot p(x) \leq \alpha$.

Hence, $(\alpha p)(x) \leq \alpha$, and so, $(1/\alpha)(\alpha p)(x) \leq 1$, from which follows, since αp is a seminorm, that $(\alpha p)((1/\alpha)x) \leq 1$. And so, $(1/\alpha)x \in (\alpha p)(\leq 1)$. Hence,

$x \in \alpha[(\alpha p)(\leq 1)]$.

Thus, we have shown that

$p(\leq 1) \subseteq \alpha[(\alpha p)(\leq 1)]$.

By now familiar arguments, we have shown that

$\mathring{\mathcal{N}}_\theta(\tau_{\alpha p}) \subseteq \mathring{\mathcal{N}}_\theta(\tau_p)$,

And so, therefore,

$\tau_{\alpha p} \leq \tau_p$ …..(2)

Clearly, from (1) and (2), we have $\tau_p = \tau_{\alpha p}$. ///

Note 5
$$\left.\begin{array}{l} p \leq q \\ \text{or} \qquad \Rightarrow \tau_p \leq \tau_q \\ p \leq \alpha q \end{array}\right\}$$

Let p, q be seminorms on a vector space $V_\mathbb{K}$. By $p \leq q$ we shall of course mean that $p(x) \leq q(x)$ for all $x \in V_\mathbb{K}$. We have

FACT 1 Let $(V, +, \theta)_\mathbb{K} = V_\mathbb{K}$ be a vector space, p, q xeminorms on $V_\mathbb{K}$, and $\alpha > 0$. Then

$$\left.\begin{array}{l} p \leq q \\ \text{or} \qquad \Rightarrow \tau_p \leq \tau_q. \\ p \leq \alpha q \end{array}\right\}$$

Proof Assume $p \leq q$, and so $p(x) \leq q(x)$ for all $x \in V_\mathbb{K}$. From this fol- lows that

$q(\leq 1) \subseteq p(\leq 1)$.

And, by now familiar argument, it follows that $\tau_p \leq \tau_q$. Assume $p \leq \alpha q$ and employ Note 4. ///

Note 6 $\tau_P = \tau_{P\max}$. Let $P = \{p_1, p_2, …., p_n\}$, $n \in \mathbb{N}$, $n \geq 2$, be a finite family of seminorms on the vector space $(V, +, \theta)_K = V_K$. Then, $\tau_P = \tau_{P_{\max}}$.

Proof $\boldsymbol{\tau_P \leq \tau_{P_{\max}}}$ **:** Clearly, $p_k \leq P_{\max}$ for each $k \in \{1,2, ….., n\}$, and so by Note 5

$\tau_{p_k} \leq \tau_{p_{\max}}$ for each $k \in \{1,2, ….., n\}$.By the definition of the *supremum*, therefore,

$\tau_P = \bigvee_{1 \leq k \leq n} \tau_{p_k} \leq \tau_{P_{\max}}$.

That is,

$$\tau_P \leq \tau_{p_{\max}} \qquad …...(\xi^1)$$

By FACT 4 of 1.**XI**,

$$P_{\max}(\leq 1) = \bigcap_{1\leq k \leq n} p_k (\leq 1) \qquad …...(\omega^1)$$

Observe that by COROLLARY 4 of Note 3,

$$\bigcap_{1\leq k \leq n} p_k (\leq 1) \in \mathcal{N}_\theta(\tau_P) \qquad …..(\omega^2)$$

and that by COROLLARY 3 of same Note 3,

$$P_{\max}(\leq 1) \in \mathcal{N}_\theta(\tau_{P_{\max}}) \qquad …..(\omega^3)$$

Again from COROLLARY 3 of Note 3, (ω^1), (ω^2) and (ω^3) give

$\mathcal{N}_\theta(\tau_{P_{\max}}) \subseteq \mathcal{N}_\theta(\tau_P)$

And, hence,

$$\tau_{P_{\max}} \leq \tau_P \qquad …...(\xi^2)$$

By((ξ^1) and (ξ^2), we have

$\tau_{P_{\max}} = \tau_P$. ///

Note 7 $P = \{p_1, p_2, …., p_n\}$, $q = \sum_{i=1}^{n} p_i$

Let $P = \{p_1, p_2, …., p_n\}$, $q = \sum_{k=1}^{n} P_k$, $\tau_{P_{\max}} = \tau_q$ Let $\{p_1, p_2, ….,p_n\}$, $n \in \mathbb{N}$, $n \geq 2$, be a finite collection of seminorms on the vector space V_K. Let $q = \sum_{i=1}^{n} p_i$. First, we have trivially.

FACT 1 q is aseminorm on V_K. ///

Next, we have

FACT 2 $\tau_{P_{\max}} = \tau_q$.

Proof Clearly, for $x \in V_K$,

$P_{\max}(x) \leq \sum_{i=1}^{n} p_i (x) \leq n \cdot P_{\max}(x)$.

So, $p_{\max} \leq q \leq n \cdot P_{\max}$. By Note 5, therefore,

$\tau_{P_{\max}} \leq \tau_q \leq \tau_{P_{\max}}$. ///

REFERENCES

[1] Albert Wilansky, *Topology for Analysis*, Gin &Company, Waltham, Massachusetts, 1970.

[2] Albert Wilansky, Modern Methods in Topological *Vector Spaces*, McGraw-Hill International Book Company, New York 1978.

[3] John Horvath, *Topological Vector Spaces and Distributions*, Volume 1, Addision-Wesley Publishing Company, Reading, Massachusetts, 1966.

[4] Sterling K. Berberia, *Lectures in Functional Analysis and Operator Theory*, GTM 15, Springer-Verlag, New York, Heidelberg, Berlin, 1974.

[5] Sunday Oluyemi, The pseudometric topology of a seminorm is a vector topology - To appear.