

COHOMOLOGICAL STABILITY AND HARMONIC ANALYSIS OF DISCRETE DYNAMICAL SYSTEMS OF COMPACT CONNECTED LIE GROUPS

¹M.E. Egwe and ²N.O. Okeke

¹Department of Mathematics, University of Ibadan, Ibadan, Nigeria
²Physical and Mathematical Sciences, Dominican University, Ibadan

Abstract

The study of cohomological equations on compact Lie groups has in recent times undergone much development, revealing deep connections with Harmonic Analysis of discrete dynamical systems and ergodicity. Here, we give a necessary and sufficient conditions for cohomological stability and Harmonic Analysis of discrete dynamical systems of compact connected Lie groups.

Keywords: Cohomological equations, compact Lie groups, Cohomological stability, Dynamical system, ergodicity
Mathematics Subject Classification (2010): 37-XX, 22Fxx, 22Exx, 22E40

Motivation

Given the cohomological equation $f - f \circ \gamma = g$, we study its relationship with harmonic analysis of orbits of discrete dynamical system (G, γ) of compact connected Lie groups. The orbits (Γ) of discrete dynamical systems of compact connected Lie groups reveal a structure isomorphic to maximal torus of the compact Lie groups. Using \mathbb{R}^n as a typical example, we employ Fourier analysis to obtain a representation of the effective group of transformation G/Γ on the space of translation invariant smooth functions on the orbit of interest, and the theory of distributions on a Schwartz space to have a representation of the group on the dual space. The diffeomorphism defining the discrete dynamical system was found to be C^∞ -cohomologically stable whenever it is defined by a Diophantine vector $a \in G$, or in the case of a Liouville vector, whenever the Lebesgue measure μ defined on the orbit of translation of the Liouville vector $a \in G$ is ergodic. When this is the case, the solution space of the cohomological equation $f - f \circ \gamma = g$ is the space of distribution on \mathbb{T}^n . Since this is a Schwartz space, all the γ -invariant distribution have emergent Fourier series expansion.

1. Introduction

Many problems which usually arise due to the consideration of certain forms of rigidity and stability of physical bodies are modeled using dynamical systems. A discrete dynamical system is a couple (M, γ) , where M is a manifold and γ a diffeomorphism of M . The dynamics is usually given by the diffeomorphism on the manifold M . The most basic cohomological equation which usually arises is a first order linear difference equation of the form:

$$f - f \circ \gamma = g, \text{ where } f, g \in C^\infty(M) \quad (1.1)$$

This is derived as follows.

First, we require a structure on the Lie group G in order to measure the effect of γ . Objective measurement requires that the structure be invariant on G or be preserved by the action of any vector $a \in G$. In other words, the structure will need to be invariant under change of coordinates. So, we assume invariant Haar measure on the compact Lie group G . Invariant integration also follows on the invariant measures [1].

Secondly, we assume G is a compact smooth oriented Lie group and γ a diffeomorphism; our assumption implies the existence of a measure that is absolutely continuous with respect to the measure induced on G by Ω , a measurable space defined on G which is compatible with its topology.

Thirdly, following from the measurable space Ω defined on G , there exists a positive function $\rho: G \rightarrow \mathbb{R}$ such that $\rho\Omega$ is invariant or fixed under γ which is considered a change of coordinates in G . Following [2] and [3]. The cohomological equation is derived by considering the pullback

$$\gamma^*(\rho\Omega) = \rho \circ \gamma(\Omega) = \rho\Omega,$$

and recalling that the Jacobian Jh of any smooth function $h: G \rightarrow G$ is defined via its pullback (which is the differential of the smooth function) as

$$h^*\Omega = (Jh)\Omega,$$

the equation then becomes

Correspondence Author: Egwe M.E., Email: Murphy.egwe@ui.edu.ng, Tel: +2348036792310

Transactions of the Nigerian Association of Mathematical Physics Volume 12, (July – Sept., 2020), 21 –28

$$\begin{aligned}
 (\gamma \circ h)^* \rho \Omega &= h^* \circ \gamma^* (\rho \Omega) = h^* (\gamma^* (\rho \Omega)) \\
 &= (\rho \circ \gamma) (Jh) \Omega = \rho \Omega \Rightarrow Jh = \frac{\rho}{\rho \circ \gamma}
 \end{aligned}$$

Hence, we obtain a multiplicative version of the cohomological equation. Taking log of both sides gives the standard form of the cohomological equation

$$\log |Jh| = \log \rho - \log \rho \circ \gamma \Rightarrow f - f \circ \gamma = g. \tag{1.2}$$

From this derivation, it is clear that the necessary condition for an invariant measure is that $Jh(x)Jh(\gamma(x)) \dots Jh(\gamma^{n-1} x) = 1$. In the cohomological equation above, $g: M \rightarrow \mathbb{C}$ is the given or known function and $f: M \rightarrow \mathbb{C}$ is the unknown function or the solution of the equation. In other words, given $g \in C^\infty(M)$, is there $f \in C^\infty(M)$ such that the equation holds? The solution of the equation will be such $f \in C^\infty(M)$ that satisfies the equation $f - f \circ \gamma = g$, such that $\int g dx = 0$ which is the necessary condition. That is, the solutions of the equation are smooth functions on M such that g defines a zero distribution in $\mathcal{D}'(M)$. Geometrically, the main quest is to discover the geometric objects invariant under the action of γ . Our analysis is geared towards specifying the space of distributions on G which are invariant under the translation effect of γ . Let us give some preliminaries which we shall need in the sequel.

2. Preliminaries

2.1 Structure of the Orbit

Let G be a compact Lie group and $a \in G$. The topological features of the quotient group $G/\langle a \rangle$ is determined by the structure of the subgroup $\langle a \rangle$ generated by continuous translations by a . So, the effective transformation group is the quotient group $G/\langle a \rangle$. Usually, as in the case of \mathbb{R}^n , the action of a discrete subgroup is equivalent to \mathbb{Z}^n -action on \mathbb{R}^n [4]. Hence, $\langle a \rangle$ is the stabilizer subgroup of $f \in C^\infty(G)$ satisfying the cohomological equation. The effective transformation group is then $G/\langle a \rangle \cong \mathbb{R}^n/\mathbb{Z}^n \cong \mathbb{T}^n$. The functions defined on the quotient space are the γ -invariant functions on \mathbb{R}^n . We consider in details the general structure of the discrete subgroup $\Gamma = \langle a \rangle$ for any compact connected abstract Lie group G .

It is generally established that a compact connected Lie group is isomorphic to \mathbb{R}^n and the closed subgroup $K \subset G$ is isomorphic to a maximal torus \mathbb{T}^n [5]. Further, it can be shown that the closure of Γ is isomorphic to an extension of a torus[6].

2.2 Maximal Torus and Centre of G

The closure of the connected component generated by $a \in G$ is a *maximal torus* of G [7]. The maximal torus of a Lie group is not generally unique, but it is a maximal subgroup contained in G which is isomorphic to $\mathbb{T}^k = \mathbb{S}^1 \times \mathbb{S}^1 \times \dots \times \mathbb{S}^1$ (k -times). The connected components of G form disjoint subsets of G , only the subset containing the identity element e is a subgroup of G . Further, every discrete normal subgroup of a path-connected matrix Lie group is contained in the centre of the group.

Remark 2.1. We note the following:

- 1.) The subgroup K which is the closure of Γ is finite if Γ is closed in G , which implies that $K = \Gamma$, and since Γ is discrete, we have $G/K \cong \mathbb{T}^n$.
- 2.) The subgroup K is infinite and an extension of a torus \mathbb{T}^n by a finite group F if the subgroup Γ is open in G . This implies that $K = \bar{\Gamma}$. Since F is a trivial group in this case, K then is a connected maximal subgroup of G , which implies $K \cong \mathbb{T}^n$.

These two correspond to the nature of the vector $a \in G$ which defines γ . For according to [8] the structure of the orbit of a transformation contains much information about the original element $a \in G$. Hence, the following definition is pertinent.

Definition 2.2. The vector $a \in \mathbb{R}^n$ (or G in general) can be of two possible kinds:

- i) a is said to be *Diophantine* if there exist real numbers C and τ such that $|1 - e^{2i\pi\langle m,a \rangle}| \geq \frac{C}{|m_\tau|^\tau}$ for any nonzero $m \in \mathbb{Z}^n$, where $\langle m, a \rangle$ is the scalar product of the two vectors.
- ii) a is said to be *Liouville* if there exists $C > 0$ such that for any $\tau > 0$, there exists $m_\tau \in \mathbb{Z}^n$ satisfying $|1 - e^{2i\pi\langle m_\tau, a \rangle}| \leq \frac{C}{|m_\tau|^\tau}$.

When the vector $a \in \mathbb{R}^n$ is Diophantine it generates a subgroup Γ which is finite; and when it is Liouville the subgroup Γ is open and infinite. In both cases the orbit of the action is clearly the n -dimensional torus [9], [10]. Thus, in both cases, the translative action of $\gamma: G \rightarrow G$ on $C^\infty(G)$, defines a map $K \rightarrow C^\infty(G)$. But to measure the regularity of this translation or map, a topology is needed on $C^\infty(G)$. Using the C^∞ -topology on the space of smooth functions on G defined in [11], we proceed to examine the orbits of the dynamical systems.

3. Analysis of Orbits of the Dynamical System

We will now consider the two cases of the order of Γ and their translative action on $C^\infty(G)$ following [11]. The notions of covering spaces and maps are crucial for the analysis of the orbit space of the diffeomorphism.

3.1 Case I: Γ is finite

In this case $G(B, K, \pi)$ form a principal bundle, where the principal fibration $\Gamma \hookrightarrow G \xrightarrow{\pi} B = G/K$ is a covering with group Γ over the compact manifold B as given by [11]. We have an exact sequence:

$$e \rightarrow K \xrightarrow{i} G \xrightarrow{\pi} B \rightarrow e. \tag{3.1}$$

Hence, π induces a map $\pi_*: C^\infty(G) \rightarrow C^\infty(B)$ defined as $\pi_*(f) = \sum_{\sigma \in \Gamma} f \circ \sigma$, where σ is a diffeomorphism defined by action of element $a \in \Gamma$ on $C^\infty(B)$.

Since π_* is defined for all $f \in C^\infty(G)$ that are invariant on Γ , it is defined on the quotient group $B = G/K$ and hence, it is periodic on G . That is, any $f \in C^\infty(B)$ is just $f \in C^\infty(G)$ satisfying $f = f \circ \gamma$. Hence, π_* maps functions of the form $f - f \circ \gamma = g$ to 0 distributions in $C^\infty(B)$. In other words, the kernel of π_* contains the subspace \mathcal{C} of all smooth functions of the form $f - f \circ \gamma = g$. Thus we have the exact sequence:

$$0 \rightarrow \mathcal{C} \xrightarrow{i} C^\infty(G) \xrightarrow{\pi_*} C^\infty(B) \rightarrow 0 \tag{3.2}$$

Remark 3.1. It follows, therefore, that $\mathcal{C} = \ker \pi_*$. From the exactness of the sequence we conclude that $C^\infty(B)$ is the space of smooth functions/distributions satisfying $\pi_*(g) = \sum_{a \in \Gamma} g \circ \gamma = 0$. For since π_* is the push-forward of smooth functions $f \in C^\infty(G)$ that satisfy $f = f \circ \gamma$, it defines γ -invariant distributions on the quotient group $B = G/K \cong \mathbb{T}^n$. Therefore, $C^\infty(B)$ is K -invariant space of distributions on the torus. The exactness of the sequence implies

$$C^\infty(G)/\mathcal{C} \cong C^\infty(B) \tag{3.3}$$

3.2 Case II: Γ is infinite

Next we consider the case where Γ is infinite, in which case, Γ is open and strictly contained in its closure $K \cong \mathbb{T}^n$. Hence, the action of K on G defines a principal bundle and this is given as

$$\mathbb{T}^n \curvearrowright G \xrightarrow{\pi} B = G/\mathbb{T}^n$$

Where G is a smooth manifold and \mathbb{T}^n a Lie group. So (G, \mathbb{T}^n, π) form a principal bundle, where the free right action of \mathbb{T}^n on G given as $(x, t) \mapsto xt$, forms orbits which coincide with the fibres of the surjective submersion $\pi: G \rightarrow B$, and B is covered by the domain of the local section $\sigma: U \rightarrow G, U \subset B$, such that $\pi \circ \sigma|_U = Id_U$ and $\cup U = B$.

By definition, orbits of the free right action \mathbb{T}^n are equivalent classes on G , where $x, y \in G$ are in the same class if there exists $t \in \mathbb{T}^n$ such that $xt = y$. That the orbits of the action coincide with the fibres of π implies $\pi^{-1}(U), U \subset B$, are the orbits of the free action of \mathbb{T}^n on G . Hence, if $\{U_1, \dots, U_p\}$ is a cover of B by open sets which are all diffeomorphic to \mathbb{R}^d , where d is the dimension of B , then for each U_i there exists a diffeomorphism $\Psi: \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{T}^n$ such that the following diagram commutes.

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\Psi} & U \times \mathbb{T}^n \\ \pi \downarrow & & \downarrow p_1 \\ U & \xrightarrow{Id_U} & U \end{array}$$

(p_1 is the projection on the first factor.) The diffeomorphism Ψ is constructed as follows [11]. Since σ is a local section of π , inverse of Ψ is defined to coincide with image of the cross section σ in G . Thus, $\Psi^{-1}: U \times \mathbb{T}^n \rightarrow \pi^{-1}(U)$ defined by $\Psi^{-1}(u, x) = x \cdot \sigma(u)$ is well defined since we have identified the closure of Γ to be a torus \mathbb{T}^n -the connected component of identity in G . So (\cdot) is the group multiplication in \mathbb{T}^n .

Further, the continuous function defined by the action of $a \in G$ on $U \subset B$ leaves \mathbb{T}^n invariant since it is the orbit space; therefore, we have:

$$u \in U \xrightarrow{a} a(u) = (a_1(u), \dots, a_n(u)) \in \mathbb{T}^n.$$

Now because B is totally disconnected, the action is like a change of coordinates in B , hence $a_1(u), \dots, a_n(u)$ are linearly independent over \mathbb{Q} . We see then that the action of Γ on the open set $\pi^{-1}(U)$ is equivalent (via Ψ) to the action:

$$(u, x) \in U \times \mathbb{T}^n \mapsto (u, x + a(u)) \in U \times \mathbb{T}^n.$$

Again, since B is a quotient group/space of G by $K = \mathbb{T}^n, u \in U \subset B$ defines or parameterizes a coset which is orbit space of action of a , so the vector a has no effect on it, which implies that $a(u)$ is independent of u .

The map Ψ is a trivialization of π (or G) over U and the left and right actions on G commute. Consequently, the translations on G are action of K generated by the vector $a \in G$. The conclusion then is that the translation in each torus $F_u = \pi^{-1}(u) = \mathbb{T}^n$ is by the vector a . So each torus F_u is the orbit space of the action of the vector a independent of $u \in B$ on which it is fibred. Hence, in both cases the orbit is the n -dimensional torus \mathbb{T}^n . We now consider space of distributions on \mathbb{T}^n or γ -invariant distributions on G .

4. γ -Invariant Distribution on \mathbb{R}^n

It is shown that the Haar measures on \mathbb{T}^n are the only (modulo multiplication by a constant) γ -invariant distributions on \mathbb{T}^n , and the diffeomorphism γ is cohomologically C^∞ -stable if and only if the vector $a \in \mathbb{R}^n$ is Diophantine [3].

Rudin [1] showed that each $f \in C^\infty(G)$ is related to a unique number m_f , which can be interpreted as the Haar measure on the space of linear representation of the orbit \mathbb{T}^n on the vector space $C^\infty(\mathbb{R}^n)$. Being uniformly approximated by convex combinations of either left translates or right translates of f , m satisfies $m(L_a f) = m_f = m(R_a f)$ for every $a \in \mathbb{T}^n$. Hence, periodic functions form the invariant set of the representation.

By Riesz Theorem, there is a bijective map, $\psi: m(f) \mapsto \int_{\mathbb{T}^n} f dm$. Let $Q = \{m: m \text{ is a Radon measure and } m(\mathbb{T}^n) = 1\}$ as in [12].

So a Haar probability measure on \mathbb{T}^n is a point in the set Q . The set Q is invariant under \mathfrak{F} where

$$\mathfrak{F} = \{R_a | a \in \mathbb{T}^n\} \cup \{L_a | a \in \mathbb{T}^n\}$$

is the family of left and right translates of $C^\infty(\mathbb{R}^n)$.

Let $\mathcal{D}'(\mathbb{R}^n)$ denote the space of distributions on \mathbb{R}^n , Riesz Representation Theorem for Radon measures on \mathbb{R}^n implies $\psi(Q) \subset \mathcal{D}'(\mathbb{R}^n)$. So we consider Q as defining a subset of distributions $Q' \subset \mathcal{D}'(\mathbb{R}^n)$, where Q' is the image of Q under ψ in the space of linear functionals on $C^\infty(\mathbb{R}^n)$.

Definition 4.1. Let $T: C^\infty(\mathbb{R}^n) \rightarrow \mathbb{C}$ such that $T(f) = \langle T, f \rangle$. Then T is said to be γ -invariant if for any function $f \in C^\infty(G)$, we have $\langle T, f \circ \gamma \rangle = \langle T, f \rangle$.

Remark 4.2. T is continuous since $T \in Q'$ is finite and so bounded, for $\psi(m) = \psi(m_f) = \int_{\mathbb{T}^n} f dm < \infty, \forall m \in Q, f \in C^\infty(\mathbb{R}^n)$. So $T(f) = \langle T, f \rangle = \langle T, f \circ \gamma \rangle$ is a continuous linear function from $C^\infty(\mathbb{R}^n) \rightarrow \mathbb{C}$, vanishing on ℓ -the subspace generated by the functions of the form $g = f - f \circ \gamma$ in $C^\infty(\mathbb{R}^n)$, where $\int_{\mathbb{T}^n} g dm = 0$. It follows then that the γ -invariant distributions are those defined on the quotient group $\mathbb{T}^n \cong \mathbb{R}^n / \langle a \rangle$. In other words, the continuous linear functional $\langle T, f \rangle$ is defined on $C^\infty(\mathbb{T}^n)$. So we identify the space Q' with $\mathcal{D}'_\gamma(\mathbb{R}^n)$ -the space of γ invariant distributions on \mathbb{R}^n which can be identified with the topological dual of the quotient space $C^\infty(\mathbb{R}^n) / \ell$. To give a topology to the dual space, we make use of Fourier transform of smooth function on \mathbb{R}^n and on the torus as in [11].

5. Topology of the Dual Space $\mathcal{D}'(\mathbb{T}^n)$

To determine the regularity of these distributions on \mathbb{T}^n we have to equip the dual space with a topology using Fourier transformation. It follows then that any complex-valued function on \mathbb{T}^n that is integrable admits a Fourier series expansion

$$f(x) = \sum_{m \in \mathbb{Z}^n} f_m \theta_m(x) \tag{5.1}$$

Where f_m are Fourier coefficients of the expansion given by

$$f_m = \int_{\mathbb{T}^n} f(x) e^{-2i\pi \langle m, x \rangle} dx.$$

If f is square integrable, with coefficients f_m satisfying $\sum_{m \in \mathbb{Z}^n} |f_m|^2 < +\infty$. So we have

$$f(x) = \sum_{m \in \mathbb{Z}^n} f_m \theta_m(x)$$

$$\Rightarrow \|f\|^2 = \left| \sum_{m \in \mathbb{Z}^n} f_m \theta_m(x) \right|^2 \leq \sum_{m \in \mathbb{Z}^n} |f_m|^2 < +\infty.$$

To any f defined on \mathbb{T}^n there is a corresponding invariant distribution via the Haar measure on \mathbb{T}^n defined on \mathbb{T}^n which is a \mathbb{Z}^n -periodic distribution on \mathbb{R}^n , which can be written likewise as

$$T = \sum_{m \in \mathbb{Z}^n} T_m \theta_m(x). \tag{5.2}$$

Hence, the distribution T also has a Fourier series expansion, where $\{T_m\}$ is a family of complex numbers indexed by $m \in \mathbb{Z}^n$. But distributions have Fourier series expansion only in a Schwartz space \mathcal{S} -the space of rapidly decreasing test functions. It is easily seen that our space is such a space, hence, assures the Fourier transformations of our invariant distributions on \mathbb{T}^n . So using the tempered distributions which behave well with respect to Fourier transformation, we topologize the dual space $\mathcal{D}'(\mathbb{T}^n)$ [13].

Remark 5.1. Because the topology $\mathcal{D}(\mathbb{T}^n)$ is stronger than the topology on $\mathcal{S}(\mathbb{R}^n)$, continuity of more linear functionals is easily achieved. So any linear functional continuous on $\mathcal{S}(G)$ is continuous on $\mathcal{D}(G)$. Subsequently, Fourier transformation is continuous on \mathcal{D} since it is continuous on the Schwartz space \mathcal{S} [13].

Every locally integrable function f on \mathbb{R}^n defines a regular distribution in \mathcal{D}' . For f to define a tempered distribution in \mathcal{S}' , it must satisfy a growth condition, that is, it must not grow faster than some power of x as $|x| \rightarrow \infty$. This ensures the definition of the integral $\int f \phi$. Therefore, \mathcal{S}' is the space of distributions of polynomial growth as $|x| \rightarrow \infty$.

Distributions are continuous linear functionals on the Hilbert space $C^\infty(G)$, therefore they satisfy $|T(\phi)| \leq M \|\phi\|_{\mathcal{H}}, \phi \in \mathcal{H}$

for some positive constant M . Thus, we see that any distribution T on \mathbb{T}^n which is \mathbb{Z}^n -periodic on \mathbb{R}^n with period $m \in \mathbb{Z}^n$ can be expanded in Fourier series as $T = \sum_{m \in \mathbb{Z}^n} T_m \theta_m$, where $\{T_m\}$ is a family of complex numbers with polynomial growth. The implication is there exists $r \in \mathbb{N}$, a constant $C > 0$ such that $|T_m| \leq C|m|^r$ for any $m \in \mathbb{Z}^n$. The inequality holds because the Fourier transformation \mathcal{F} is continuous and bijective on \mathcal{S}' .

Remark 5.2. The space $C^\infty(\mathbb{T}^n)$ which is contained in $L^2(\mathbb{T}^n)$ is a Sobolev space which allows us to consider distributions as functions instead of measures. Hence, we will focus on the Sobolev space of L^2 -functions on \mathbb{T}^n .

In Sobolev spaces the Fourier transform of derivatives of a function is a multiplication by a polynomial of the Fourier transform of the function; that is $(\mathcal{F}(D^\alpha f) = \xi^\alpha \mathcal{F}(f))$. So distributions in Sobolev spaces (which are derivatives) are tempered. That is, $f \in \mathcal{S}'(\mathbb{T}^n)$ implies $(1 + |\xi|^2)^{(1/2)r} f \in L^2(\mathbb{T}^n)$. We will therefore consider distributions in \mathbb{T}^n as sums of Fourier transforms of functions of $L^2(\mathbb{T}^n)$ [14].

A Sobolev space of functions on a compact $W^\tau(M)$ is invariant under coordinate transformations, so possesses the property of invariance under translation. Hence, the Fourier coefficients of functions on \mathbb{T}^n and their distributions are also γ -invariant Lebesgue square integrable functions on \mathbb{R}^n , that is, they are γ -invariant in $L^2(\mathbb{R}^n)$. [15].

Proposition 5.3. Let $W^{1,r}$ be the space of functions f on \mathbb{T}^n whose Fourier coefficients (f_m) , $m \in \mathbb{Z}^n$ satisfy the condition $\sum_{m \in \mathbb{Z}^n} |m|^r |f_m| < +\infty$ and $W^{2,r}$ the Sobolev space of functions f whose Fourier coefficients (f_m) , $m \in \mathbb{Z}^n$ satisfy the condition $\sum_{m \in \mathbb{Z}^n} |m|^r |f_m|^2 < +\infty$. It is easily seen that $L^2(\mathbb{T}^n)$ contains all the Sobolev spaces of any order, with $C^\infty(\mathbb{T}^n)$ being the limit of inclusion.

Notice that $W^{i,r}$ are subspaces of distributions $\phi \in \mathcal{D}'(\mathbb{T}^n)$ such that $\partial^\alpha \phi \in L^2(\mathbb{T}^n) \forall |\alpha| \leq r, i = 1, 2$ according to [16]. The inclusions given as:

$$\mathcal{D}'(\mathbb{T}^n) \supset L^2(\mathbb{T}^n) = W^{2,0} \supset W^{2,r} \supset W^{2,r+1} \supset \dots \supset W^{1,0} \dots \supset C^\infty(\mathbb{T}^n)$$

Since $L^2(\mathbb{T}^n)$ is a Hilbert space, one can work with either the space of functions as in [5], or with the dual space as in [11]. The inner product of two functions and the norm of any function can be expressed in both since they have equivalent topology. We now establish the following assertions.

Proposition 5.4. Let $T = \sum_{m \in \mathbb{Z}^n} T_m \theta_m$ to be a series (where T_m are complex numbers). Then the following assertions are equivalent:

- i) T is a regular distribution, that is, T is a C^∞ -function.
- ii) For any $r \in \mathbb{N}$, the series $\sum_{m \in \mathbb{Z}^n} |m|^{2r} |T_m|^2$ converges.
- iii) For any $r \in \mathbb{N}$, the series $\sum_{m \in \mathbb{Z}^n} |m|^r |T_m|$ converges.

The injections $j_{1,r}: W^{1,r+1} \hookrightarrow W^{1,r}$ and $j_{2,r}: W^{2,r+1} \hookrightarrow W^{2,r}$ are compact operators.

Proof. i) \implies ii)

Assume T is a regular distribution, then by definition, there exists a Lebesgue integrable function f such that for $\phi \in \mathcal{D}(\mathbb{T}^n)$, we have

$$(f, \phi) = \int_{\mathbb{T}^n} f(x) \phi(x) dx$$

So, $T \in C^\infty$ since (f, ϕ) has derivative of all order. Hence, $C^\infty(\mathbb{T}^n) \subset L^2(\mathbb{T}^n) \implies T \in L^2(\mathbb{T}^n)$. That is to say that the distribution T is square integrable on \mathbb{T}^n . So, T has a Fourier series expansion $T = \sum_{m \in \mathbb{Z}^n} T_m \theta_m$, where $\{T_m\}$ is a family of complex number with polynomial growth. So the Fourier coefficients $(T_m)_{m \in \mathbb{Z}^n}$ of T satisfy the convergence condition $\sum_{m \in \mathbb{Z}^n} |m|^{2r} |T_m|^2 < \infty$.

ii) \implies iii)

Assuming that for $r \in \mathbb{N}$, the series $\sum_{m \in \mathbb{Z}^n} |m|^{2r} |T_m|^2 < \infty$ converges. Then, since $\sum_{m \in \mathbb{Z}^n} |m|^r |T_m| \leq \sum_{m \in \mathbb{Z}^n} |m|^{2r} |T_m|^2$, by dominated convergence, $\sum_{m \in \mathbb{Z}^n} |m|^r |T_m|$ converges.

iii) \implies i)

Assuming the series $\sum_{m \in \mathbb{Z}^n} |m|^r |T_m|$ converges for every $r \in \mathbb{N}$. Then, since $\sum_{m \in \mathbb{Z}^n} |m|^r |T_m| = \sum_{m \in \mathbb{Z}^n} |\mathcal{F}(\partial^\alpha T)| < \infty$, there exists a function $f \in W^{1,r}$ which has $\{T_m\}$ as its Fourier coefficients. This implies that f is Lebesgue integrable on \mathbb{T}^n since its Fourier coefficients $\sum_{m \in \mathbb{Z}^n} |T_m|$ is bounded, $f \in L^1(\mathbb{T}^n)$. Hence, f defines a regular distribution T_f on $\mathbb{T}^n \implies T \in C^\infty(\mathbb{T}^n)$. So we conclude that

$$\bigcap_{r \in \mathbb{N}} W^{1,r} = \bigcap_{r \in \mathbb{N}} W^{2,r} = C^\infty(\mathbb{T}^n)$$

It follows therefore, that the injections $j_{1,r}: W^{1,r+1} \hookrightarrow W^{1,r}$ and $j_{2,r}: W^{2,r+1} \hookrightarrow W^{2,r}$ are identity operators, hence, are compact operators. ■

We conclude then that $C^\infty(\mathbb{T}^n)$ and its dual space $\mathcal{D}'(\mathbb{T}^n)$ are Hilbert spaces; every function or distribution has convergence Fourier series expansion and invariant under left and right translations.

5.1 Topology of the Fibre Space

For the second case when Γ is infinite, which resulted in fibration of G , the topology is modified as in [11] to help determine the regularity of these functions. Recalling that U_1, \dots, U_p are the open cover of $B = G/\mathbb{T}^n$, diffeomorphic to \mathbb{R}^d , where d is the dimension of B . Let u and Ψ be as defined above, Ψ is a trivializing diffeomorphism.

Let $f: G \rightarrow \mathbb{C}$ be a measurable function. The restriction of f to $V = \pi^{-1}(u)$ via Ψ is a function $f_u: U \times \mathbb{T}^n \rightarrow \mathbb{C}$. This definition immediately makes the result above useful, for it places the coordinates $(u, x) = (u_1, \dots, u_d, x_1, \dots, x_n)$ within the domain of the function f_u . We assume square integrability of f , which makes it a distribution on \mathbb{T}^n , but modified to reflect the fibre structure on G .

Hence, for fixed $u \in B$ and multi-indices $r \in \mathbb{N}^d$ and $s \in \mathbb{N}^n$, the distribution $\frac{\partial^{|r|+|s|} f}{\partial u^r \partial x^s}$ on the fibre of $u, F_u = \pi^{-1}(u) = \mathbb{T}^n$, admits a Fourier expansion. We have explained that this is so because $L^2(\mathbb{T}^n)$ is a Sobolev space. So, smooth functions on \mathbb{T}^n are also distributions with their derivatives, and all have Fourier expansions which are distributions. It follows therefore, that

$$\frac{\partial^{|r|+|s|} f(u, x)}{\partial u^r \partial x^s} = \sum_{m \in \mathbb{Z}^n} \frac{\partial^{|r|} f_m(u)}{\partial u^r} (2i\pi)^{|s|} m^s \theta_m, \text{ where } \theta_m = e^{2i\pi(m, x)}. \quad (5.3)$$

The norm of each function and the inner product on the space are defined with respect to this structure. Thus, every $f \in L^2(G)$ restricted to V and integrated over \mathbb{T}^n defines an integral operator from $B \rightarrow \mathbb{C}$, given as

$$I(f)(u) = \int_{\mathbb{T}^n} f(u, x_1, \dots, x_n) dx_1 \otimes \dots \otimes dx_n.$$

We now proceed to verify whether every $f \in L^2(G)$ defines a γ -invariant distribution in both cases. We now consider the solution of the cohomological equation.

5.2 Solution of the Cohomological Equation

Smooth periodic functions on \mathbb{R}^n satisfy the cohomological equation since the differential forms associated with these functions are closed and exact on the torus \mathbb{T}^n and also invariant under translation by element \mathbb{Z} or invariant under the diffeomorphism γ when defined by a Diophantine vector. Hence, they satisfy the necessary condition for g to be of the form $f - f \circ \gamma$ which is $g_0 = 0$, which is equivalent to the condition $I(g) = 0$. This is shown as follows.

Being periodic functions on \mathbb{T}^n a Sobolev space, f and g are distributions and can be expanded in Fourier series to obtain $f(x) = \sum_{m \in \mathbb{Z}^n} F_m e^{2i\pi(x,m)}$, $g(x) = \sum_{m \in \mathbb{Z}^n} g_m e^{2i\pi(x,m)}$ and $f(ax) = \sum_{m \in \mathbb{Z}^n} f_m e^{2i\pi(x,m)}$ substituting these in the cohomological equation, it reduces to

$$(1 - e^{2i\pi(x,m)})f_m = g_m \text{ with } m \in \mathbb{Z}^n.$$

Put in this form, Kastznelson [17] sums that the question of solution of the cohomological equation becomes that of identifying (pairs of) spaces on which $(1 - e^{2i\pi(m,a)})$ is a Fourier multiplier. This depends on the Diophantine property of the vector a which controls the rate of growth of $(1 - e^{2i\pi(m,a)})$. The necessary condition for the solution of the equation which is $\int_{\mathbb{T}^n} g(x)dx = (g) = 0$, is satisfied since when $m = 0$, $(g) = 0$. The formal solution is then given as

$$f_m = \begin{cases} 0 & \text{if } m = 0 \\ \frac{g_m}{1 - e^{2i\pi(m,a)}} & \text{if } m \neq 0 \end{cases} \tag{5.4}$$

In the general case where the orbit space is a fibre of each $u \in B$ the f_m and g_m are parametrized by $u \in B$. Hence, we have

$$f_m(u) = \begin{cases} 0 & \text{if } m = 0 \\ \frac{g_m(u)}{1 - e^{2i\pi(m,a)}} & \text{if } m \neq 0 \end{cases} \tag{5.5}$$

With the assumption that the vector a is Diophantine, there exist real numbers $C > 0, \tau > 0$ such that $|1 - e^{2i\pi(m,a)}| \geq C/|m|^\tau$ for any $m \in \mathbb{Z}^n$. From this we have

$$|m|^{2r}|f_m|^2 = |m|^{2r} \left| \frac{g_m}{1 - e^{2i\pi(m,a)}} \right|^2 \leq \frac{1}{C^2} |g_m|^2 |m|^{2(r+\tau)}.$$

As differential forms on \mathbb{T}^n , g is C^∞ , so, the series $\sum_{m \in \mathbb{Z}^n} |g_m|^2 |m|^{2(r+\tau)}$ converges. Which implies that $\sum_{m \in \mathbb{Z}^n} |m|^{2r(r+\tau)} |f_m|^2 < \infty$ defines a distribution on G , which is a C^∞ function f on G .

For the case when γ is defined by a transcendental or Liouville vector $a \in G$. We show that the cohomological equation is solvable only when the measure is ergodic.

5.3 Cohomological Equation for Liouville Vector

Before we consider the solution of the cohomological equation for the case of Liouville vectors, we recall that the measure μ_n , the Lebesgue measure on \mathbb{R}^n which is a Haar probability measure on \mathbb{T}^n , is shown to be ergodic when γ is defined by a Liouville vector. In this case, the rotation $\gamma: x \mapsto x + (a + \mathbb{Z}^n)$ is minimal, which implies that it has dense orbit in \mathbb{T}^n . Assuming μ_n is not ergodic. This implies that μ_n is decomposable into smaller Haar probability measures on \mathbb{T}^n as shown in [8]. What effect will this have on the solution of the cohomological equation?

As illustrated in [11], if the vector a is Liouville then there exists $C > 0, m_\tau \in \mathbb{Z}^n$ for any $\tau \in \mathbb{N} \setminus \{0\}$ satisfying:

$$|1 - e^{2i\pi(m,a)}| \leq \frac{C}{|m_\tau|^\tau}.$$

If we then have an increasing sequence $(\tau_k)_k$ in $\mathbb{N} \setminus \{0\}$, with the corresponding integers m_{τ_k} simply given as m_k . A function g is defined by the Fourier coefficients:

$$g_m = \begin{cases} |m_k|^{-\frac{\tau_k}{2}} & \text{if } m = m_k \\ 0 & \text{if } m \neq m_k \end{cases}$$

Then $g \in C^\infty(\mathbb{R}^n)$ and satisfies $\int_{\mathbb{T}^n} g(x)dx = g_0 = 0$. To check the regularity of g , we let $f \in C^\infty(\mathbb{R}^n)$ be such that $f - f \circ \gamma = g$. Then we have

$$|f_{m_k}|^2 = \left| \frac{g_{m_k}}{1 - e^{2i\pi(m,a)}} \right|^2 = \frac{|m_k|^{-\tau_k}}{|1 - e^{2i\pi(m,a)}|^2} \geq \frac{1}{C^2} |m_k|^{\tau_k}.$$

This shows that the coefficient f_m are not of polynomial growth and cannot define a distribution f to be a solution of the cohomological equation. More importantly for us, f has no Fourier expansion in the space. Using this means, we can construct infinitely many linearly independent C^∞ -functions $(g^l)_{l \in \mathbb{N}^*}$ for which the cohomological logical equation has no solution.

Since the operator $\delta: C^\infty(\mathbb{T}^n) \rightarrow C^\infty(\mathbb{T}^n)$ is such that $f \mapsto g$, it follows then that the cokernel of δ given as $H^1(\gamma, C^\infty(\mathbb{T}^n)) = C^\infty(\mathbb{T}^n)/B(\gamma, C^\infty(\mathbb{T}^n))$ is not Hausdorff and is infinite dimensional. This corresponds to the infinitely many non-ergodic probability measures into which μ_n can be decomposed. But since a Liouville vector can be approximated by infinitely

many rational numbers $(\frac{p_n}{q_n})_{n \geq 1}$ the sequence of which converges to the vector a . Likewise, the sequence of measures defined by their rotations converges to the ergodic measure μ'_n which is unique to the Liouville vector a . Hence, when μ_n is ergodic, which implies indecomposability or irreducibility of the Haar probability measure, we have g to be a trigonometric polynomial whose constant term $g_0 = 0$. The corresponding f has Fourier series expansion.

Thus, the cohomological equation has a solution for the problem of convergence does not arise since the ergodic measure μ_n is the extremal/limit point of the set of all γ -invariant probability measure on \mathbb{T}^n denoted by $P_\gamma(\mathbb{T}^n)$. Therefore, the image of the operator δ which is $B(\gamma, C^\infty(\mathbb{T}^n))$, is closed since it contains its limit. Thus, the 1-dimensional Hausdorff space $\bar{H}^1(\gamma, C^\infty(\mathbb{T}^n)) \cong \mathcal{D}'_\gamma(\mathbb{T}^n)$ is closed, where $\mathcal{D}'_\gamma(\mathbb{T}^n)$ is generated by the n -form $dx = dx_1 \otimes \dots \otimes dx_n$.

So, when γ is defined by a Liouville vector, the relation of $\mathcal{D}'_\gamma(\mathbb{R}^n)$ to the reduced cohomological group $\bar{H}^1(\gamma, C^\infty(\mathbb{T}^n))$ is given in the following proposition.

Proposition 5.5. Given $r \in Diff(G)$, then

$$\overline{B(\gamma, C^\infty(\mathbb{T}^n))} = \bigcap_{T \in \mathcal{D}'_\gamma(\mathbb{R}^n)} \ker T.$$

which implies, in particular, that $dim \bar{H}^1(\gamma, C^\infty(\mathbb{T}^n)) = dim \mathcal{D}'_\gamma(\mathbb{R}^n)$.

Therefore, we conclude that the ergodic Haar measure μ_n is the only γ -invariant Borel probability measure on \mathbb{T}^n unique to the Liouville vector a . The γ -invariant distributions on \mathbb{T}^n are constant multiples of the ergodic measure μ_n . This is given as a proposition in [3].

Proposition 5.6. When γ is defined by a Liouville vector, the γ -invariant distributions on \mathbb{T}^n are constant multiples of the unique ergodic probability measure μ_n on \mathbb{T}^n . That is $\mathcal{D}'_\gamma(\mathbb{T}^n) = \mathbb{R}\mu_n$.

Proof: Let $g \in C^\infty(\mathbb{T}^n)$ be such that $\int_{\mathbb{T}^n} g d\mu_n = 0$. Then Fourier expansion of g is given as

$$g(x) = \sum_{m \in \mathbb{Z}^n \setminus \{0\}} g_m e^{2i\pi(m,x)}.$$

Let $f \in C^\infty(\mathbb{R}^n)$ be a solution of the cohomological equation, then its Fourier coefficients must satisfy

$$g_m := \frac{g_m}{1 - e^{2i\pi(m,a)}}, \forall m \in \mathbb{Z}^n \setminus \{0\}$$

If we then have a finite sequence $\{U_k\}_{k \geq 1}$ of subsets of \mathbb{Z}^n such that they cover $\mathbb{Z}^n \setminus \{0\}$, that is, $\cup_{k \geq 1} U_k = \mathbb{Z}^n \setminus \{0\}$ and $U_k \subset U_{k+1}$ for every $k \geq 1$, we define the trigonometric polynomials:

$$g_k(x) = \sum_{m \in U_k} g_m e^{2i\pi(m,x)},$$

$$f_k(x) = \sum_{m \in U_k} f_m e^{2i\pi(m,x)},$$

for $k \geq 1$. Since $f_k - f_k \circ \gamma = g_k$, we have $g_k \in B(\gamma, C^\infty(\mathbb{T}^n))$ and $g_k \rightarrow g$ in the C^∞ -topology. Hence, $g \in \overline{B(\gamma, C^\infty(\mathbb{T}^n))}$. By the preceding proposition, we conclude that $\mathcal{D}'_\gamma(\mathbb{T}^n) = \mathbb{R}\mu_n$, which is 1-dimensional. Therefore, $\mathcal{D}'_\gamma(\mathbb{T}^n)$ is the solution space for the cohomological equation.

6 Interpretation of Result

From the above, it is clear that we have a harmonic analysis of the discrete dynamical system only when γ is defined by a Diophantine vector $a \in G$ and when the Haar measure μ_n on \mathbb{T}^n is ergodic for the case of a Liouville vector $a \in G$. In other words, the cohomological stability of a discrete dynamical system corresponds to its harmonic analysis. To achieve this, we will establish the necessary connection between the two concepts.

In our description of the discrete dynamical system, we worked with a compact connected Lie group G which acts as transformation group of itself. The torus was realized as the orbit space of the action of a vector $a \in G$, by collating elements of the group G fixed by the action, we realized the discrete subgroup $\langle a \rangle$. This discrete subgroup coincides with the lattice \mathbb{Z}^n in the case of \mathbb{R}^n , but is isomorphic to a torus in the general case. Its closure formed a subgroup K of elements, which in addition to the identity element, defines the identity map. Since all the functions of invariants of a transformation are also invariants, the effective transformation group for the space $C^\infty(G)$ was then given as the quotient $B=G/K$.

The quotient group therefore gives the possible configurations or parameters of the space of transformations which coincided with the n -dimensional torus \mathbb{T}^n in both cases, when n is the dimension of G . In the second case which is a fibre space or bundle, each torus is indexed by $u \in B$ upon which it is fibred; hence they form an indexed family of transformation spaces or orbits. Thus the transformation γ defines a quotient group, whereby every $x \in G$ is mapped to a quotient class $a[x]$ which denotes the image of $x \in G$ under the map $\pi: G \rightarrow \mathbb{T}^n$. The transformation endowed G with more structures/symmetries which are preserved by G , so that the transformations $x \mapsto a[x] \in \mathbb{T}^n$ express the symmetries of discrete dynamical system of G . These symmetries are captured in linear representations of the orbit space \mathbb{T}^n in $C^\infty(G)$.

Since $C^\infty(G)$ is the vector space of complex-valued functions on G which is G -invariant, that is, the translation of f by a is in $C^\infty(G)$ whenever f is in $C^\infty(G)$. The action of G induces a linear translation on $C^\infty(G)$ by a pullback operation. Just as we have already mentioned above, any smooth diffeomorphism $\gamma: G \rightarrow G$ acts linearly on $C^\infty(G)$ by a pullback, whereby for any

$f \in C^\infty(G), \gamma^*(f): C^\infty(\mathbb{T}^n) \rightarrow C^\infty(G)$ define by $f \mapsto f \circ \gamma$. This gives the regular representation of \mathbb{T}^n in $C^\infty(G)$. The adjoint of this action, which is the push-forward operation $\gamma_*: \mathcal{D}'(G) \rightarrow \mathcal{D}'(\mathbb{T}^n)$ defined by

$$(\gamma_* T, \psi) = (T, \psi \circ \gamma) = (T, \gamma^*(\psi))$$

is a representation of \mathbb{T}^n on the dual space of $C^\infty(\mathbb{T}^n)$ which is $\mathcal{D}'(\mathbb{T}^n)$.

This representation on the dual space or the Schwartz space of γ -invariant distributions in \mathbb{T}^n coincides with the harmonic analysis of the orbit space of the discrete dynamical system. This conclusion is evident in the result we have already stated above, the fact that all the distributions in $\mathcal{D}'_\gamma(\mathbb{T}^n)$ have Fourier series expansions. So, every $T \in \mathcal{D}'_\gamma(\mathbb{T}^n)$ is of the form

$$T = \sum_{m \in \mathbb{Z}^n \setminus \{0\}} T_m \theta_m, \text{ where } \theta_m(x) = e^{2i\pi\langle m, x \rangle}.$$

Further, the harmonic analysis is related to the cohomological stability of the discrete dynamical system because as we have established, the smooth functions f_m and g_m in the formal solution of the cohomological equation define distributions only when γ is defined by a Diophantine vector. When this is the case, we have that \mathbb{T}^n is a periodic orbit of the iteration, a connected component of G which forms conjugacy classes of G , generating the subspaces of $C^\infty(G)$ such that every complex-valued smooth function $f \in C^\infty(G)$ admits a Fourier series expansion given as

$$f(x) = \sum_{m \in \mathbb{Z}^n \setminus \{0\}} f_m \theta_m(x),$$

and for each $m \in \mathbb{Z}^n$.

$$f_m = \int_{\mathbb{T}^n} f(x) e^{-2i\pi\langle m, x \rangle} dx \text{ where } dx = dx_1 \otimes \dots \otimes dx_n.$$

We have also seen that the subspace of $C^\infty(\mathbb{T}^n)$ are simple subspaces of smooth functions on the subspaces generated by a_1, \dots, a_n which are components of the vector $a \in G$ and are linearly independent set over \mathbb{Q} . This decomposition of functions in $C^\infty(G)$ into sums and integrals of functions associated with components or subspaces of the vector space $C^\infty(\mathbb{T}^n)$ is a decomposition that simplifies many problems.

Thus, we have been able to reinterpret the solution of the cohomological equation in terms of the harmonic analysis of the orbit space of the action of a vector $a \in G$. The harmonic analysis also expresses or implies the cohomological stability of the discrete dynamical system of a connected compact Lie group G and hence the rigidity/stability of the action since only then is dimension of $H^1(\gamma, C^\infty(G))$ is 1.

References

- [1] Rudin, W. *Functional Analysis*, McGraw-Hill Book Company 1973.
- [2] Akhil, M. *The Cohomological Equation for Dynamical Systems*, an internet post available at <http://amathew.wordpress.com/2010/07/17/the-cohomological-equation-for-dynamical-systems/>.
- [3] Avila, A. Kocsard, A. *Cohomological Equations and Invariant Distributions for Minimal Circle Diffeomorphisms*, 2010, available electronically at: arXiv:1002.3392v2[math.DS] 24 Nov. 2010.
- [4] Schmidt, Klaus. *Dynamical Systems of Algebraic Origin*, BirkhauserVerlag, 1995.
- [5] Dwyer, W.G. and Wilkerson, C.W. *The Elementary Geometric Structure of Compact Lie Group*. Available electronically at <http://hopf.math.purdue.edu/Dwyer-Wilkerson/lie/liegroups.pdf>.
- [6] Sepanski, M. *Compact Lie Groups*, Springer 2007.
- [7] Stillwell, J. *Naive Introduction to Lie Theory*, Springer 2000.
- [8] Einsiedler, M. Ward, T. *Ergodic Theory*, Springer, 2011.
- [9] Angell, David. *Irrationality and Transcendence*. A lecture note prepared by David Angell, available at <http://web.maths.unsw.edu.au/angell/5535/>.
- [10] Ash, Robert. *Abstract Analysis: The Basic Graduate Year*, Dover Books, 2000.
- [11] El KacimiAlaoui, A. Hmili, H. *Cohomological Equations and Invariant Distributions on a Compact Lie Group*, Hokkaido Math. Journal 43 (2014), 1-23.
- [12] Kiesenhofer, A. *Haar Measure on Compact Groups*. Available electronically at <http://www.asc.tuwicn.ac.at/funkana/downloads-general/sem-kiesenhofer.pdf>, 2001.
- [13] Richards, I. Youn, H. *Theory of Distributions: A Non-technical Introduction*, Cambridge University Press, 1990.
- [14] Treves, F. *Topological Vector Spaces, Distributions and Kernels*, Academic Press 1967.
- [15] Taylor, M. *Partial Differential Equation I*, 2nd edit., Springer, 2011.
- [16] Al-Gwaiz, M.A. *Theory of Distributions*, Marcel Dekker Inc., 1992.
- [17] Katznelson, Y. *An Introduction to Harmonic Analysis*, Stanford, 2002.
- [18] Alaoui, El Kacimi A. Moses, T. Matsumoto, S. *Currents Invariant by a Kleinian Group*. Hokkaido Mathematical Journal, Vol. 26, (1997), 177-202.
- [19] El KacimiAlaoui, Aziz. *On Holomorphic Cohomological Equations*. Results Math. 63 (2013), No. 1-2, 329-334.
- [20] Leo De Roberto. *Stability of the Cohomological Equation for Regular Vector Fields on the Plan*. Ann. Global Anal. Geom. 39 (2001), No. 3, 231-248.
- [21] De la Llave, R. Marco, J.M. and Moriyon, R. *Canonical Perturbation Theory of Anosov Systems and Regularity Results for the Livsic Cohomological Equation*, Annals of Math. Second Series, Vol. 123, No. 3 (May 1986).
- [22] Lyubich, Y.I. *The Cohomological Equations in Nonsmooth Categories*, available online: arXiv:1211.0229v1 [Maths FA] 1 Nov, 2012.
- [23] Belitskii, G. Lyubich, Y.I. *On the Solvability of Cohomological Equations on Compact Topological Spaces*, Recent Progress in Operator Theory (Regensburg, 1995) 75-87.