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ON 2 −GROUPS G SUCH THAT WHENEVER $H < G$ AND H IS MINIMAL NON-ABELIAN, THEN H IS AN $M_2(2,2)$ −GROUP

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Abstract

In studying the structure of a non-abelian p-group G, the minimal non-abelian subgroups of G play an important role since they generate the group G. In this article, we determine the 2-groups G with the property that if $H < G$ and H is minimal non-abelian, then $H = M_2(2,2)$.
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1. Introduction

To determine a finite group G by using its subgroups structures is an important theme in group theory. In this article, only finite *p-groups*with $p = 2$ are considered and our notations are standard for finite group theory. A group G is said to be a minimal non-abelian group if it is non-abelian but all its proper subgroups are abelian. Every finite non-abelian *p-group* contains a minimal non-abelian subgroup. The smallest order of a minimal non-abelian *p-group* is p^3. A finite non-abelian *p-group* is generated by its minimal non-abelian subgroups. In a sense, a minimal non-abelian subgroup is a "basic element" of a finite *p-group*. The structures of finite *p-groups* depend essentially on their minimal non-abelian subgroups. The group presentation for a minimal non-abelian *p-group*was given in [1]. The minimal non-abelian and maximal subgroups of finite *p-group*were classified in [2]. In [3], the structure theorem for the central automorphism groups of a finite minimal non-abelian *p-group*was given. Some results about the order of the automorphism group, Aut(*G*), of a finite minimal non-abelian *p-group G* were given in [4] when $p > 2$. The finite groups *G*in which every proper subgroup is either abelian or minimal non-abelian were determined in [5]. The non-abelian finite *p-groups G*such that whenever *A* is a maximal subgroup of any minimal non-abelian subgroup *H*in *G*, then *A* is also a maximal abelian subgroup in *G*were classified in [6]. Finite *p-groups*which are the central extension of a cyclic *p-group*, and elementary abelian *p-groups*by minimal non-abelian groups were respectively classified in $[7-10]$. In this article, we determine the structure of *p-groups*(with $p = 2$) in which every minimal non-abelian subgroup has the structure of an $\mathrm{M}_2(2,2)$-group.

2. Preliminaries

In this preliminary section, we give some definitions and some known results about minimal non-abelian *p-groups*which will be used later.

Definition 2.1.[11]. A minimal non-abelian *p-group G* is said to be metacyclic if it has a cyclic normal subgroup *N*such that the quotient $G/_N$ is cyclic.

Lemma 2.1. [5]. If *G*is a nilpotent minimal non-abelian group, then *G* is a *p-group*,$|G'| = p,\ Z(G) = \Phi(G)$is of order p^2 in *G*and one of the following holds:

(i) $p = 2$ and *G*is the ordinary quaternion group;

(ii) $G = \langle\, a, b | a^{p^m} = b^{p^n} = 1, a^b = a^{1+p^{m-1}}, m > 1 \,\rangle$ is metacyclic of orderp^{m+n};

(iii) $G = \langle a, b, c | a^{p^m} = b^{p^n} = c^p = 1,\ [a,b] = c,\ [a,c] = [b,c] = 1\rangle$ is non-metacyclic of order p^{m+n+1}.

Corollary 2.2. [8]. Let G be a *p-group*, then the following statements are equivalent:

(1) G is minimal non-abelian;

(2) $d(G) = 2$ and $|G'| = p$;

(3) $d(G) = 2$ and $Z(G) = \Phi(G)$.

Definition 2,2.[11]. Let*G*be a minimal non-abelian *p-group.*If *G*is non-metacyclic and such that $G/_{G'}$ is abelian of the type (p^m, p^n), then G is called an -group. A metacyclic group *G*is said to be an $M_p(m,n)$-group if $G = AB,\ A \lhd G,\ A \cong C_{p^m}$ and $B \cong C_{p^n}$ (here, C_t is a cyclic group of order t).

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One has that $\left|M_p(m,n,1)\right| = p^{m+n+1}$ and $\left|M_p(m,n)\right| = p^{m+n}$. The groups $M_p(m,n,1)$ and $M_p(m,n)$are determined up to isomorphism by the parameters m,n. Thus, if *G* is a minimal non-abelian *p-group*, then $d(G) = 2, |G{:}\,Z(G)| = p^2, |G'| = p$ and $G \in \left\{M_p(m,n,1),\ M_p(m,n),\ Q_8\right\}$.[11].

3. Finite 2-Groups Whose Minimal Non-Abelian Subgroups Are $M_2(2,2)$-Groups

In Theorem 3.0, we state the main result of this article.

Theorem 3.0. Let *G*be a finite 2-group such that whenever $H < G$ and *H*is minimal non-abelian, then H is an $M_2(2,2)$-group. Then *G*has the following presentation:

$$G = \langle\, a, b | a^{2^m} = b^4 = 1, a^b = a^{\frac{2^m}{2}-1}, \qquad m > 2 \rangle$$

Proof:

Let *H* be proper minimal non-abelian subgroup of the group *G*. Then by the statement of the theorem, $H = M_2(2,2)$. Since *G*is a metacyclic group, there are cyclic subgroups $C_{2^m}$and C_4 in *G*generated by a and b respectively. For each positive power k of 2 that divides$\frac{2^m}{2}$, there is at least a cyclic normal subgroup C_k of order k in G. These subgroups are abelian (every cyclic group is abelian). We stated earlier that $\left|M_p(m,n)\right| = p^{m+n}$. So $|M_2(2,2)| = 2^{2+2} = 16$. From the presentation of *G*in Theorem 3.0,$m > 2$. Also, since$H$ is a proper subgroup of Gthenthere is a k for which $|G{:}\,C_k| = 16$,where k is a power of 2 that divides $\frac{2^m}{2}$. Since the order of *H* is fixed ($|H| = 16$) and the order of *G*varies according to the powers of a, there is always a cyclic normal subgroup $C_k$for which we have the following short exact sequence

$$\{\,1\,\} \to C_k \to G \to H \to \{\,1\,\}$$

In other words,

$$G/_{C_k} \cong H$$

Thus, *G*is a central extension of its minimal non-abelian subgroup $M_2(2,2)$ by its cyclic subgroup C_k of index 16.

Conclusion

In this article, we have shown that the *2-groups*, *G*, all of whose minimal non-abelian subgroups are $M_2(2,2)$-groups are generated by these $M_2(2,2)$-groups. We also gave a structure theorem for the presentation of*G*. These groups are central extensions of the $M_2(2,2)$-groups by an abelian (cyclic) subgroup of index 16 in *G*.

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