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STABILITY PROPERTIES OF AREDUCED GROUP C*-ALGEBRAS FOR TWISTED C*-CROSS PRODUCT

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Abstract

In this paper, we study the ideal structure for reduced group C*-algebras and its associated reduced twisted C*- cross product .We investigate the stability properties of this groups and show that the non-abelian free group on two generators is C*-simple. Our result now generalizes the results of Kalantar et al., and many others in the literatures.

Keywords:C*- simple group, twisted C*-Cross product, twisted C*-dynamics, stability properties,
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1. INTRODUCTION

Let G be a discrete group. Let the group algebra $\ell^1(G)$ equipped with the following product and involution

$$(xy)(s) = \sum x(g)y(g^{-1}s), \qquad x^*(s) = \overline{x(s^{-1})}, \quad x, y \in \ell^1(G), \ s \in G$$

This product is known as the convolution of two functions $x, y: G \to \mathbb{C}$ with respect to these operations and the usual 1-norm, $\ell^1(G)$ is a Banach *-algebra with identityδ_1. The characteristic function $\delta_g \in \ell^1(G)$satisfy$\delta_g\delta_g^* = \delta_g^*\delta_g = \delta_1$, the self adjoint subalgebra $c_c(G)$ of finitely supported function on (G) constitute a dense subset of $\ell^1(G)$. It is clear that any *-homomorphism from a Banach *-algebra to c^*-algebra is contractive [1]. We define a norm $\|\cdot\|_u$on $\ell^1(G)$ by setting $\|x\|_u = sup\|\pi(x)\|$for $x \in \ell^1(G)$ where π runs through all non-degenerate representations of $\ell^1(G)$ on a Hilbert space. Completing $\ell^1(G)$ with respect to $\|\cdot\|_u$, we obtain the unital c^*-algebra, as the full group c^*-algebra denoted by $C^*(G)$. It is well known that any non-degenerate representation of $\ell^1(G)$ on a Hilbert space H extends to a non-degeneraterepresentation of $C^*(G)$ on$H$. Thus, this correspondence of representation is one-to-one.A unitary representation of G is a group homomorphism of G into the group $\mathcal{U}(H)$ of unitary operators on some Hilbert space H. There is one-to-one correspondence between unitary representations of G and non-degenerate representations of$\ell^1(G)$, given by mapping nonlinear function to the operator respectively

$$f = \sum_{g\in G} f(g)\delta_g \in \ell^1(G)$$
$$= \sum_{g\in G} f(g)\pi_g \in \mathcal{B}(H)$$

Where$\pi : g \to \pi_g$ is a unitary representation of G on the Hilbert space$H$. Precisely, any unitary representation of G can be used to construct a c^*-algebra. Indeed, if $\pi : c^*(G) \to \mathcal{B}(H)$ is the non-degenerate representation induced by a unitary representation $\pi : G \to \mathcal{U}(H)$, then the$c^*$-algebra associated to π is given by $c^*(G) = \pi(c^*(G))$. Next, we considerregular (left) regular representation λ in the unitary group of $\ell^2(G)$ given by left translation:

$$[\lambda_g\xi](s) = \xi(g^{-1}s), \quad g, s \in G, \quad \xi \in \ell^2(G)$$

With respect to the canonical orthogonal basis $\{\delta_1 \mid s \in G\}of\ell^2(G)$,where$\lambda$satisfies$\lambda_g\delta_s = \delta_{gs}, \quad g, s \in G$. The reduced group c^*-algebra$C_r^*(G)$ is the c^*-algebra $C_\lambda^*(G)$ associated to λ , and $C_r^*(G)$is therefore the norm-closure in$\mathcal{B}\left(\ell^2(G)\right)$ of the set of operator of the form$\sum_{g\in G}\eta_g\lambda_g, \ \eta_g \in \mathbb{C}$a non-zero for finitely many$g \in G$. Moreover, $C_r^*(G)$ is equipped with a faithful tracial stateτ, given by $\tau(x) = \langle x\delta_1, \delta_1\rangle$. We refer to τ as the canonical tracial state on $C_r^*(G)$.

Definition 1.1 A discrete group G is said to bec^*-simple if $C_r^*(G)$ is simple c^*-algebra with unique trace property and $C_r^*(G)$ admits tracial state.

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The theory of simple c^*-algebra first introduced by Bédos[1] was later reconstructed by De La Harp [2, pp. 13]. Since then, many mathematical philosophers had made huge progress (cf. [3, 4, 5, 6] etc).

C*-Simplicity and Boundary Actions ([7]). Let$\ell^1(G,A)$ denote the space of functions $x : G \to A$ satisfying$\sum_{g\in G}\|x(g)\| < \infty$. From now on, we write the notation $x = \sum_{g\in G} x_g\delta_g$ for a function$x \in \ell^1(G,A)$, where $x_g = x(g)$ for $g \in G$. We next equip $\ell^1(G,A)$ with a product and involution by defining

$$(xy)(s) = \sum_{g\in G} x(g)\big(gy(g^{-1}s)\big), \quad x^*(s) = sx(s^{-1})$$

So that $\ell^1(G,A)$ becomes a Banach *-algebra in the 1-norm. We identify A with the image of A under the *-homomorphism $a \mapsto a\delta_1$. Obviously, the subset $C_c(G,A)$ of finitely supported functors $G \to A$ is a dense *-subalgebra of $\ell^1(G,A)$ and that an approximate identity (e_i) in A yields an approximate identity $(e_i\delta_1) \in \ell^1(G,A)$. It is well known that a covariant representation of the c*-dynamical system (A,G,α) is a triple (π,u,H), where H is a Hilbert space, $\pi : A \to \mathcal{B}(H)$ is a non-degenerate representation and $u : G \to \mathcal{U}(H)$ is a unitary representation such that $\pi(ga) = u_g a u_g^*$for$g \in G$ and$a \in A$. We often suppress the Hilbert space H from the notation if is clear from the context. The associated integral form of covariant representation (π,u) is the map $\pi \times u : \ell^1(G,A) \to \mathcal{B}(H)$ define by

$$(\pi \times u)(x) = \sum_{g\in G} \pi(x_g)u_g, \quad x \in \ell^1(G,A)$$

The full cross product of (A,G,α), denoted by $A \rtimes_\alpha G = A \rtimes G$ is the completion of $\ell^1(G,A)$ or $C_c(G,A)$ with respect to the norm

$$\|x\|_u = sup\|(\pi \times u)(x)\|, \quad x \in \ell^1(G,A)$$

The supremum taken over all (cyclic) covariant representations $(\pi,u,H)$of(A,G,α). To define the reduced crossed product, we assume that $A \subseteq \mathcal{B}(H)$ is faithfully represented, hence the map

$$\pi : A \to \mathcal{B}\left(H \otimes \ell^2(G)\right)$$
$$\lambda : G \to \mathcal{B}\left(H \otimes \ell^2(G)\right)$$

via

$$\pi(a)(\xi \otimes \delta_s) = (s^{-1}a)\xi \otimes \delta_s,$$
$$\lambda_g(\xi \otimes \delta_s) = \xi \otimes \delta_{gs},$$

For all$a \in A, \xi \in H, gs \in G$.

It is verifiable, $\left(\pi,\lambda,H \otimes \ell^2(G)\right)$ is a covariant representation of (A,G,α) call a regular representation of the c*-dynamical system. Again, λ is actually an amplification of the left regular representation of Gon$\ell^2(G)$. The associated form $\pi \times \lambda : \ell^1(G) \to \mathcal{B}\left(H \otimes \ell^2(G)\right)$ is faithful, and the reduced crossed product $A \rtimes_{\alpha,r} G = A \rtimes_r G$ is the completion of $\ell^1(G,A)$ or $C_c(G,A)$ in the reduced norm

$$\|x\|_r = \|(\pi \times \lambda)(x)\|_{\mathcal{B}\left(H\otimes\ell^2(G)\right)}, \quad x \in \ell^1(G,A)$$

Equivalently, $A \rtimes_r G$ (cf. [8, Chapter 4.1]) can be taken to be the norm closure of the image of $\pi \times \lambda$or $\pi \times \lambda|_{C_c(G,\mathcal{A})}$. Clearly, $A \rtimes_r G$ does not depend on the choice of faithful representation $A \subseteq \mathcal{B}(H)$ (see for example., [9, Proposition 4.1.5]). We now define a G-action on $A \rtimes_r G$ by means of the inner automorphism$g \mapsto Ad(\lambda_g)$, so that the inclusion $A \subseteq A \rtimes_r G$ is G-equivariant. Identifying A via its image underπ, then the reduced crossed product also has the nifty property of admitting a faithful conditional expectation$E_A : A \rtimes_r G \to A$ that is G-equivariant and uniquely satisfies$E_A(x) = x_1$ for all$x = \sum_{g\in G} x_g\lambda_g \in \ell^1(G,A) \subseteq A \rtimes_r G$.

We referred to the above inclusion as the canonical conditional expectation and write E instead of E_A if the dynamical system is clear from the context. The existence of a faithful conditional expectation of $A \rtimes_r G$ onto A also characterizes the reduced crossed product among c*-algebras generated by the image of the integrated form of covariant representation of (A,G,α), (cf. [10, Theorem 4.22]. In fact, it holds in more general, if $H \subseteq G$ is a subgroup, then there exists an injective *-homomorphism $A \rtimes_r H \to A \rtimes_r G$ that extends the inclusion $C_c(H,A) \to C_c(G,A)$. Again, if we identify $A \rtimes_r H$ with its image under this *-homomorphism, then there exists a faithful conditional expectation $E_H : A \rtimes_r G \to A \rtimes_r H$ that uniquely satisfy $E_H(\lambda_g) = 0$, for all$g \notin H$. We shall proof in the more general case using reduced twisted crossed products in Theorem 3.2 inspired by [1, Theorem 2.2]. In particular, we obtain the generalization of [11, Theorem 7.1] with the aid of Corollary 3.3, 3.4, and 3.5, while Lemma 3.6 is the generalization of [11, Theorem 7.2]. Again, Theorem 3.7 is the generalization of [12, Proposition 3.13]. Furthermore, we not that if $A, \mathcal{B}$ are $G - c^*$-algebras and $\varphi: A \to \mathcal{B}$ is a G −equivariant c.c.p. map, then the map $\tilde{\varphi}: \ell^1(G,A) \to \ell^1(G,\mathcal{B})$ given by $\varphi(x)_g = \varphi(x_g), \quad x \in \ell^1(G,\mathcal{A}), \quad g \in G$extends to a c.c.p. map $\tilde{\varphi}: A \rtimes_r G \to \mathcal{B} \rtimes_r G$. Thus, $\tilde{\varphi}$ uniquely satisfies$\tilde{\varphi}(a\lambda_g) = \varphi(a)\lambda_g, \quad a \in A,\ g \in G$. It turns out that a property of φ is inherited by $\tilde{\varphi}$. It is easy to show that this include faithfulness, surjectivity and being a *-homomorphism.

Definition 1.2. For an action of a discrete group G on a topological free space X, define $X^g = \{x \in X \mid gx = x\}, \quad g \in G$, we say that the action of G on X is topologically free if X^g has empty interior for all $g \in G \setminus \{1\}$.

Definition 1.3. Let $A$and $\mathcal{B}$ be c*-algebras and let φ be a c.c.p map $\varphi: A \to \mathcal{B}$. The multiplicative domain mult(φ) of φ is the subset of A given by

$\text{mult}(\varphi) = \{a \in A \mid \varphi(a^*a) = \varphi(a)^*\varphi(a),\ \varphi(aa^*) = \varphi(a)\varphi(a)^*\}$

From the result of Choi [13, Theorem 3.1],

$\text{mult}(\varphi) = \{a \in A \mid \varphi(ax) = \varphi(a)\varphi(x),\ \varphi(xa) = \varphi(x)\varphi(a)\},\ \forall x \in A$

Moreover, if $\mathcal{B} \subseteq A$ is a c*-algebras and $\varphi: A \to \mathcal{B}$ is a c.c.p map that restricts to the identity map on $\mathcal{B}$, thenφ is in fact, a conditional of A onto $\mathcal{B}$.

Lemma 1.4 ([14, Theorem 1]). Let X be a compact G-space in which the action of G is topologically free. If $I \subseteq C(X) \rtimes_r G$ is closed ideal such that$I \cap C(X) = \{0\}$, then $I = \{0\}$.

Following the original article [14], one can easily verify that Lemma 1.4 holds true for topologically free action on c*-algebras that are possibly non-unital and non-commutative.If X is compact G-space and $x \in X$, then by composing the faithful conditional expectation, *-homomorphism, Gx^0-equivariant , *-homomorphism $\delta_x: C(X) \to \mathbb{C}$, we obtains a u.c.p map

$E_{G_x^0}: C(X) \rtimes_r G \to C(X) \rtimes_r G_x^0,$

$C(X) \rtimes_r G_x^0 \to C_r^*(G_x^0)$

$E_x: C(X) \rtimes_r G \to C_r^x(G_x^0),$

Satisfying

$E_x(f\lambda_g) = f(x)E_{G_x^0}(\lambda_g),\ f \in C(X), \quad g \in G$

The following result is a reformulation of Kawabe [15]

Theorem 1.5 ([15, Lemma 2.4]): Let X be a compact G-space for which $\{x \in X \mid G_x^0 is amenable\}$ is dense in X. If the action of G on X is not topologically free, then there exists a non-zero closed ideal $I \subseteq C(X) \rtimes_r G$ for which$I \cap C(X) = \{0\}$.

Theorem 1.6 ([16, Theorem 3.1]): Let X be a Stonean space. If $f: X \to X$ is homeomorphism, then the fixed point set of f is clopen. In particular, a group action on X is topologically free if and only if it is free.

Boundary action are intimately connected with several commutative C*-algebras that are of interest in the study of c*-simple groups (i.e. groups with simple reduced group c*-algebras) can be found in the literatures [2, 4, 7, 17, 18, 19, 20, and 21]. The concept of boundary action was originally introduced by Furstenberg [17]. The main idea is to describe to what degree a fixed group of homeomorphism of space (i.e., a fixed non-trivial translation of $\mathbb{R}$ to any bounded subset) can map any or at least some points in the boundary of $\mathbb{R} \in \mathbb{R}^*$, namely $(-\infty, +\infty)$ in space. It is clear that any non-trivial translation of $\mathbb{R}$ with positive derivative move any point in $\mathbb{R} \cup \{\infty\}$ closer to $+\infty$, and that$\{\pm\infty\}$ are the only fixed point. The study of boundary actions and ideal structure of reduced crossed products have recently be linked to the study of c*-simple group (cf. [4]). Furthermore, a discrete group can only be c*-simple when the c*-algebra associated to its regular representation is simple. This property for discrete group pioneered by powers is one of the focuses of this paper. In particular, our motivations are the advances in [5, 22 and 23] which were later elaborated in [24].It is our purpose in this paper to extend the results in [7, Theorem 6.2] that characterizes c*-simplicity in terms of boundary actions to the equivalence of topological freeness in [11, Theorem 3.1] and then generalized some of their results. Characterization of c*-simplicity have been consider by Kennedy and Kalantar, a few of which we now review;

I. **Simplicity of reduced crossed products([11, Theorem 7.1])**. Proved that c*-simple discrete groups have the property that a reduced crossed product $\mathcal{A} \rtimes_r G$ of a unital G-c*-algebra by G is simple if and only if $\mathcal{A}$ is G-simple, which means that $\mathcal{A}$ has no non-trivial G-invariant closed ideals. This settled de la Harpe and Skandalis conjecture in [5]. We generalize this result in section 3.

II. **An averaging property([25, see also 26])**. Haagerupand Kennedy proved that a discrete group G is c*-simple if and only if for all $t_1, t_2, \dots, t_m \in G \setminus \{1\}$ and $\epsilon > 0$, there exists $s_1, \dots, s_n \in G$ such that $\left\| \frac{1}{n} \sum_{k=1}^{n} \lambda_{s_k t_j s_k^{-1}} \right\| < \epsilon$

Clearly, this is an important characterization, because many previously study classes of c*-simple groups were always shown to satisfy at most minor variant of the latter property. In fact, it is nonetheless part of the original proof of powers that F_2 is c*-simple. We shall prove in section 3 that the reduced crossed products of c*-simple groups satisfy a similar property. We also record that the above property is a group c*-algebra variant of the Dixmier property. A unital c*-algebra A is said to satisfy the Dixmier property if the closed convex hull of $\{uau^* \mid u \in \mathcal{U}(A)\}$ intersects the centre of A for all $a \in A$. It was proved in [27]that a unital, simple c*-algebra$A$ always satisfies the Dixmier property, and that the intersection of the aforementioned closed convex hull and the centre always reduces to a point, if c*-algebra has a unique tracial state.

III. **Recurrent subgroups ([26])**. Independently, Kennedy obtained an algebraic characterization of c*-simplicity using the notion of recurrence for subgroups, a notion for topological dynamical of uniformly recurrent subgroup. A subgroup H of a group G is recurrent if there exists a final subset $F \subseteq G \setminus \{1\}$ such that $F \cap gHg^{-1} \neq \square$ for all$g \in G$. A discrete group is c* simple if and only if it has no amenable, recurrent subgroups.

Theorem 1.7([11, Corollary 4.3]). Let G be a discrete group with amenable radical$R(G)$. Then $g \in G$ satisfies $\tau(\lambda_g) = 0$ for all tracial states τ on $C_r^*(G)$ if and only if $g \notin R(G)$. In particular, G has the unique trace property if and only if $R(G) = \{1\}$

The proof of the implication of the infamous result (Theorem 1.7) requires generalization. We deffer these until section 3 (Theorem 3.8). However, Theorem 1.7 partially settles de la Harpe conjecture; whether there exist c*-simple groups without the unique trace property. Conversely, by composing the conditional expectation $C_r^*(G) \to C_r^*(R(G))$ with trivial representation $C_r^*(R(G)) \to \mathbb{C}$(i.e., an existence result which follows from the amenability of $R(G)$ [9, Theorem 2.6.8]), yields a state $\tau: C_r^*(G) \to \mathbb{C}$ such that $\tau(\lambda_g) = 1$ for all $g \in R(G)$. Since $R(G)$ is normal, then for any two $g, h \in G$ we have $gh \in R(G)$ if and only if $hg \in R(G)$, implying $\tau(\lambda_g\lambda_h) = \tau(\lambda_h\lambda_g)$. Hence τ is a tracial state on $C_r^*(R(G))$.

The rest of this paper is organized as follows. In section 2, we give some preliminary results which we shall need later. In section 3, we proof our main results. Precisely, we prove Theorem 3.1, 3.2, 3.7, and 3.8. In section 4, we study stability properties that our results and many others in the literatures satisfy. Specifically, we give some examples of what stability properties that classes of c*-simple groups and groups with trivial amenable radical satisfy. Furthermore, we establish stability criterion which automatically satisfies stability properties for other classes of groups. Finally, we give in section 5, some examples of c*-simple groups, mainly using the characterization of c*-simplicity arising from Theorem 3.1.

2. PRELIMINARIES

We shall need the following Lemmas. We prove Lemma 2.11 for the sake of completeness.

Lemma 2.1 ([28, 1(1957), pp. 509 – 544]). Let N be a closed**,** normal, amenable subgroup of a locally compact group G and let X be a G-bonudary. Then N acts trivially on X.

Lemma 2.2 ([23]). Let G be a non-elementary hyperbolic group. Then the action of G on itself by left translation induces a boundary action of G on∂G.

Lemma 2.3 ([17]). Let G be a Hausdorff topological group. $X'$is a compact minimal G-space and X is a G-boundary, then there is atmost one G-equivariant u.c.p map $C(X) \to C(X)$ and it is an injective *-homomorphism.

Lemma 2.4 ([27, Proposition 7]). Let G to be a locally compact group. Then$R(G) = \cap_{x \in \partial_F G} G_x$. In particular, $R(G) = \{1\}$ if and only if G admits a faithful boundary action. Moreover, $\partial_F G$ is G-equivariantly homeomorphicto$\partial_F(G/R(G))$.

Lemma 2.5 ([30, see also 17, Lemma 4.1]). Let G be a Hausdorff topological group and let X be a minimal, proximal compact G-space. If X has an isolated point, then X is a one-point space.

Lemma 2.6 ([30]). For any discrete group G and any $x \in \partial_F G$, the stabilizer G_x is an amenable subgroup of G.

Lemma 2.7 ([24]). Let X be a minimal compact G-space. If the action of G on X is proximal, then the only G-equivariant continuous map is the identity map.

Lemma 2.8 ([24]). Let G be a topological group, let $(X_i)_{i \in I}$ be a family of compact G-spaces and let $X = \Pi_{i \in I} X_i$ be the product space equipped with the diagonal G-action. Then the action of G on X is proximal (resp. strongly proximal) for all $i \in I$.

Lemma 2.9 ([7, see also 11, Proposition 2.5]). Let G be a Hausdorff topological map and let A be a unital G-invariant C*-subalgebra of a unital$G$-C*-algebra $\mathcal{B}$. Then any G-equivariant u.c.p map $A \to C(\partial_F G)$ extends to a G-equivariant u.c.p map $\mathcal{B} \to C(\partial_F G)$.

Lemma 2.10 ([31, proposition 3.1, see also 32, Proposition 4.26]): Let T be a countable, leafless tree and let G be a discrete group acting minimally on T by automorphisms without inversion. If the action G on ∂T is non-elementary, then $\overline{\partial T}$ is a G-boundary in the shadow topology.

Lemma 2.11 (Special case of [14, Theorem 1]). Let (A, G, α, β) be a unital twisted c*-dynamical system and let X be the maximal ideal space of the centre $Z(A)$of A. Assume that the action of G on X is free. If J is a closed ideal in $A \rtimes_{\alpha,r}^{\beta} G$, then for $J_A = J \cap A$ we have $J_A \rtimes_{\alpha,r}^{\beta} G \subseteq J_A \overline{\rtimes}_{\alpha,r}^{\beta} G$.

Proof. Let $I_A = I \cap A$ and let $\pi: A \to A/I_A$ be the quotient map. We assume that $\rho: A/I_A \to B(H)$ is an irreducible representation of A/I_A. Now, consider the representation $A + I \to (A + I)/I \cong A/I_A \overset{\rho}{\to} B(H)$. By Arveson's extension theorem, this map extends to a u.c.p map $\varphi: A \rtimes_{\alpha,r}^{\beta} G \to B(H)$ such that $\varphi(I) = 0$and $A \subseteq mult(\varphi)$, since $\varphi\,|_A = \rho \circ \pi$. By irreducibility, the restriction of φ to $Z(A) \cong C(X)$ is a point mass on X, i.e., $\varphi\,|_{Z(A)} = \delta_x$ for some $x \in X$. Let $g \in G \setminus \{1\}$, then there exists $f \in C(X)$ such that $f(g^{-1}x) \neq f(x)$. This implies $\varphi\big(\lambda_\beta(g)(f(x))1_H\big) = \varphi\big(\lambda_\beta(g)f\big) = \varphi\left(gf\lambda_\beta(g)\right) = f(g^{-1}x)\varphi$. Therefore, $\varphi\left(\lambda_\beta(g)\right) = 0$. Let $E_A: A \rtimes_{\alpha,r}^{\beta} G \to A$ be the canonical conditional expectation, it follows that $\varphi = \varphi \circ E_A$. Hence,$\rho\left(\pi\big(E_A(I)\big)\right) = \varphi\big(E_A(I)\big) = \varphi(I) = \{0\}$. Since ρ was arbitrary, $\pi\big(E_A(I)\big) = \{0\}$, so that $E_A(I) \subseteq I$. For any positive element $x \in I$, let ℓ be the image of x under $\tilde{\pi}: A \rtimes_{\alpha,r}^{\beta} G \to (A/I_A) \rtimes_{\alpha,r}^{\beta} G$, let $E_{A/I}: (A/I_A) \rtimes_{\dot{\alpha},r}^{\dot{\beta}} G \to A/I_A$be the canonical faithful conditional expectation since $E_{A/I} \circ \tilde{\pi} = \pi \circ E_A$, it follows that $E_{A/I}(\ell) = 0$ since $E_A(x) \in I \cap A$, $E_{A/I}$ is faithful, ℓ=0 and $x \in I_A \overline{\rtimes}_{\alpha,r}^{\beta} G$. □

Moreover, fore any (A, G, α, β) twisted c*-dynamical system and any G-invariant, closed ideal $I \in A \overline{\rtimes}_{\alpha,r}^{\beta} G$, the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I \rtimes_{\alpha,r}^{\beta} G & \longrightarrow & A \rtimes_{\alpha,r}^{\beta} G & \longrightarrow & A/I \rtimes_{\dot{\alpha},r}^{\dot{\beta}} G\tilde{\pi} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I & \longrightarrow & A\tilde{\pi}\, A/I & \longrightarrow & 0 & \longrightarrow & \end{array} \qquad (2.1)$$

arise when π induces a surjective *-homomorphism $\tilde{\pi}: A \rtimes_{\alpha,r}^{\beta} G \to A/I \rtimes_{\dot{\alpha},r}^{\dot{\beta}} G$ at the level of crossed products. This yields the identity

$$\big(I \overline{\rtimes}_{\alpha,r}^{\beta} G\big) \cap A = I \qquad (2.2)$$

Thus, E_I, E_A and $E_{A/I}$ denote the canonical conditional expectation respectively. Furthermore, for any G-boundary X. If A is unital, then for the natural extension $(A \otimes C(X), G, \mu, \gamma)$, we found that if $K \subseteq \big(A \otimes C(X)\big) \rtimes_{\mu,r}^{\gamma} G$ is a closed ideal and $K_A = K \cap \big(A \otimes C(X)\big)$, then there is a commutative diagram of *-homomorphism

$$\begin{array}{ccc} A \rtimes_{\alpha,r}^{\beta} G & \longrightarrow & \big(A \otimes C(X)\big) \rtimes_{\mu,r}^{\gamma} G \\ \downarrow & & \downarrow \\ A/(K \cap A) \rtimes_{\dot{\alpha},r}^{\dot{\beta}} G & \longrightarrow & \big(A \otimes C(X)\big)/K_A \rtimes_{\mu,r}^{\gamma} G \end{array}$$

where the horizontal arrows are injective. It follows that

$$\big(K_A \overline{\rtimes}_{\mu,r}^{\gamma} G\big) \cap \big(A \rtimes_{\alpha,r}^{\beta} G\big) = (K \cap A) \overline{\rtimes}_{\alpha,r}^{\beta} G \qquad (2.3)$$

3. MAIN RESULTS

We now prove the following

Theorem 3.1 (Main Theorem). Let G be a discrete group. Then the followingare equivalent, simple (**I** – **IV**), and topologically free (**V and VI**).

I. G
II. $C(X) \rtimes_r G$, for some G-boundary X
III. $C(X) \rtimes_r G$, for all G-boundary X
IV. $C(\partial_F G) \rtimes_r G$
V. The action of G on some X
VI. The action of G on $\partial_F G$

Proof: Clearly, III implies II, III imply IV and IV implies V are trivial. Now, by Theorem 1.6, the action of G on $\partial_F G$ is topologically free if and only if it is free, since $\partial_F G$ is Stonean (cf. [33, Theorem 3.1]). Again V imply II, VI imply IV follow from Lemma 1.4. If $C(\partial_F G) \rtimes_r G$ is simple, then all stabilizer subgroups for the G-action on $\partial_F G$ are amenable by Lemma 2.10. The action of G on $\partial_F G$ is topologically free by Theorem 1.5, thus proving IV which implies VI.

Next, we need to prove that I imply III. Let X be a G-boundary, using Lemma 2.6, we may assume that there is a G-equivariant unital c*-algebra inclusion $C(X) \subseteq C(\partial_F G)$. Let $\pi: C(\partial_F G) \rtimes_r G \to \mathcal{B}$ be a unital *-homeomorphism. The action of G on $\mathcal{B}$ may be defined by means of inner automorphisms$Ad\left(\pi(\lambda_g)\right)$ of $\mathcal{B}$, so that π becomes G-equivariant. Using the inclusion $\mathbb{C} \subseteq C(X)$, we realize $C_r^*(G)$ as a unital$G$-invariant c*-subalgebra of $C^*(X) \rtimes_r G$. If $C_r^*(G)$ is simple, then $\pi \mid_{C_r^*(G)}$ is injective, so that canonical tracial state of $\tau: C_r^*(G) \to \mathbb{C} \subseteq C(\partial_F G)$ extend to G-equivariant u.c.p map $\tau: \mathcal{B} \to C(\partial_F G)$ such that $\tilde{\tau} \circ \pi \mid_{C_r^*(G)} = \tau$ by Lemma 2.7. Using Lemma 2.6 once again, we find that the map $\tilde{\tau} \circ \pi \mid_{C(X)}: C(X) \to C(\partial_F G)$ is the inclusion $C(X) \to C(\partial_F G)$. Precisely, $C(X) \subseteq mult(\tilde{\tau} \circ \pi)$. If $E: C(X) \rtimes_r G \to C(X)$ is the canonical faithful conditional expectation, the $\tilde{\tau}\left(\pi(f\lambda_g)\right) = f\tau(\lambda_g) = E(f\lambda_g)$ in $C(\partial_F G)$ for all $f \in C(X)$and $g \in G$. Hence, $\tilde{\tau} \circ \pi = E$, meaning that π is faithful and therefore injective. Henceforth, $C(X) \rtimes_r G$ is simple.

Next, we need to prove that II implies I. If $C(X) \rtimes_r G$ is simple for some G-boundary X, let $I \subseteq C_r^*(G)$ by a proper closed ideal. If $\varphi: C_r^*(G) \to \mathbb{C}$ is a state then $\varphi(I) = \{0\}$ extend φ to a state on $C(X) \rtimes_r G$. Let (g_i) be a net in G such that $g_i\mu \to \delta_x$ for some $x \in X$where $\mu = \varphi \mid_{C(X)}$. By weak*- compactness we may assume that $\left(\varphi \circ Ad(\lambda_g)\right)$ converges to some state ψ on $C(X) \rtimes_r G$, so that $\psi \mid_{C(X)} = \delta_x$ and $\psi \mid_I = 0$. Thus, $C(X) \subseteq mult(\psi)$. Furthermore, for any $b \in I,\ f_1 f_2 \in C(\partial_F G)$and $f_1 f_2 \in G$, we have that

$\psi(f_1 \lambda g_1) b (f_2 \lambda g_2) = f_1(x) \psi(\lambda g_1 b \lambda g_2) f_2(g_2 x) = 0, \qquad \lambda g_1 b \lambda g_2 \in I$

It is not difficult to see that the ideal generated by I is proper. Therefore we have $I = \{0\}$ because $C(\partial_F G) \rtimes_r G$ was assumed to be simple. □

Remark 3.2. It follows from Theorem 3.1 that any c*-simple discrete group G has trivial amenable radical. Indeed, if the action of G on $\partial_F G$ is free, then$R(G) = \cap_{x \in \partial_F G} G_x = \{1\}$ by Lemma 2.9. Since the result of Kalantar and Kennedy [7], other characterizations of c*-simplicity have been obtained (cf. [11, 12, 25, 26, 27]). Moreover, some of these results in the later literatures required generalization. Precisely, [11, Theorem 7.1, 7.2, Corollary 4.3], [12, Proposition 3.13] and [26, Definition 5.2].

Assume that (A, G, α) is separable. If G is amenable, then every primitive ideal of $A \rtimes_r G$ is an induced primitive ideal. Moreover, if G acts freely on prim(A), then the induce process establishes a bijection between prim$(A \rtimes_r G)$ and the quasi-orbits in prim(A). In particular, if G acts freely and every orbit is dense, then $A \rtimes_r G$ is simple. It is instructive to note that the twisted action and the equivalence of a group being c*-simple admits a free boundary action allows us to generalize many of these results. This we do in the following theorems.

Theorem 3.2. Let (A, G, α, β) be a unital twisted c*-dynamical system where G is c*-simple. For a maximal ideal I of $A \rtimes_{\alpha,r}^{\beta} G,\ \ I \cap A$ is a maximal G-invariant ideal of A. Conversely, for a maximal G-invariant ideal $\gamma$of A, the ideal $\gamma \,\overline{\rtimes}_{\alpha,r}^{\beta}\, G$ of $A \rtimes_{\alpha,r}^{\beta} G$ is maximal. Moreover, the correspondence is bijective.

Proof: Let γ be a maximal G-invariant ideal in A. We claim that the ideal $\gamma \,\overline{\rtimes}_{\alpha,r}^{\beta}\, G$ in $A \,\overline{\rtimes}_{\alpha,r}^{\beta}\, G$ is maximal; assume that J is a proper ideal in $A \,\overline{\rtimes}_{\alpha,r}^{\beta}\, G$ such that $\gamma \,\overline{\rtimes}_{\alpha,r}^{\beta}\, G \subseteq J$. Now, let $(A \otimes C(\partial_F G), G, \nu, \iota)$ of (A, G, α, β) be the natural extension. Let K denote the ideal in $\left(\left(A \otimes C(\partial_F G)\right)\right) \rtimes_{\nu,r}^{\iota} G$ generated by J. By Lemma 2.11, $K \subseteq K_A \,\overline{\rtimes}_{\nu,r}^{\iota}\, G$, where $K_A = K \cap \left(\left(A \otimes C(\partial_F G)\right)\right)$. By (2.3) $J \subseteq K \cap \left(A \rtimes_{\alpha,r}^{\beta} G\right) \subseteq \left(K_A \,\overline{\rtimes}_{\nu,r}^{\iota}\, G\right) \cap \left(A \rtimes_{\alpha,r}^{\beta} G\right) = (J \cap A) \,\overline{\rtimes}_{\alpha,r}^{\beta}\, G$.

On applying (2.2) to γ and $K \cap A$gives $\gamma \subseteq J \cap A \subseteq K \cap A$. Clearly, J is proper. Theorem 3.6 implies that K is proper, so the maximality of γ implies that $\gamma = K \cap A$ since $K \cap A$ is a G-invariant. It follows that $J \subseteq \gamma \,\overline{\rtimes}_{\alpha,r}^{\beta}\, G$, and $\gamma \,\overline{\rtimes}_{\alpha,r}^{\beta}\, G$ is maximal.It suffice to show that the ideal $I \cap A$ is maximal among proper G-invariant ideals in A. Now let I be a maximal ideal in$A \,\overline{\rtimes}_{\alpha,r}^{\beta}\, G$. Let J denote the ideals in $\left(\left(A \otimes C(\partial_F G)\right)\right) \rtimes_{\nu,r}^{\iota} G$ generated by I. By Lemma 2.11 $J \subset J_A \,\overline{\rtimes}_{\nu,r}^{\iota}\, G$where $J_A = J \cap \left(A \otimes C(\partial_F G)\right)$. Hence by (2.3), we have that

$I \subset J \cap \left(A \rtimes_{\alpha,r}^{\beta} G\right) \subset \left(J_A \,\overline{\rtimes}_{\nu,r}^{\iota}\, G\right) \cap \left(A \rtimes_{\alpha,r}^{\beta} G\right) = (J \cap A) \,\overline{\rtimes}_{\alpha,r}^{\beta}\, G$

Since I is proper. Theorem 3.6 implies that $J \cap A$ is proper in A. Thus, maximality of I implies that $I = (J \cap A) \,\overline{\rtimes}_{\alpha,r}^{\beta}\, G$. Now $I \cap A = J \cap A$ follows from (2.2). It follows from our analysis $I = (I \cap A) \,\overline{\rtimes}_{\alpha,r}^{\beta}\, G$ by (3.1). Now, let $\mathbb{F}$ be a proper G-invariant ideal in A such that $I \cap A \subset \mathbb{F}$. Then $\mathbb{F} \,\overline{\rtimes}_{\alpha,r}^{\beta}\, G$ is a proper ideal in $A \,\overline{\rtimes}_{\alpha,r}^{\beta}\, G$ and $I = (I \cap A) \,\overline{\rtimes}_{\alpha,r}^{\beta}\, G \subset \mathbb{F} \,\overline{\rtimes}_{\alpha,r}^{\beta}\, G$.

Therefore the maximality of I implies that $I = \mathbb{F} \overline{\rtimes}_{\alpha,r}^{\beta} G$. Hence$I \cap A = \left(\mathbb{F} \overline{\rtimes}_{\alpha,r}^{\beta} G\right) \cap A = \mathbb{F}$. Thus, $I \cap A$ is maximal. Finally, it now clear, the correspondence is bijective follows from the identities (2.2) and (3.1) □

Corollary 3.3. Let (A, G, α, β) be a unital twisted C*-dynamical system where G is C*-simple. Then $A \rtimes_{\alpha,r}^{\beta} G$ is simple if and only if $A$is G-simple.

Corollary 3.4. If G is C*-simple, then the reduced twisted group C*-algebra $C_r^*(G, \beta)$ is simple for every multiplier $\beta: G \times G \to \mathbb{T}$.

Corollary 3.5. Let (A, G, α, β) be as in Corollary 3.3. Let N be a normal subgroup of G. Write (α, β) for the restriction of $(\alpha, \beta)$to N. If G/N is C*-simple, then $A \rtimes_{\alpha,r}^{\beta} G$ is simple whenever $A \rtimes_{\alpha,r}^{\beta} N$ is simple.

Proof: $A \rtimes_{\alpha,r}^{\beta} G \cong \left(A \rtimes_{\alpha,r}^{\beta} N\right) \rtimes_{\nu,r}^{\iota} (G/N)$ follow from the existence of a twisted action (ι, ν) of G/Non$A \rtimes_{\alpha,r}^{\beta} N$. The desire conclusion now follows from Corollary 3.3. □

Remark 3.6: Corollary 3.3 and Corollary 3.4 gives the generalization of [11, Theorem 7.1]. It should be noted that the conclusion of Theorem 3.2 is not true if we allow the underlying c*-algebra to be non-unital. Thurs, $C_0(X)$ is always G-simple, even though $C_0(X) \rtimes_r G$ may contain many ideals. Furthermore, assume that G is a C*-simple group and A, a unital G-C*-algebra. If $Z\big(I(A)\big)$ is G-simple, and A is prime, then the action of G on A has the intersection property (i.e., $Z\big(I(A)\big) \rtimes_r G$ is simple (cf. [15, Theorem 3.4]), and $A \rtimes_r G$ is prime respectively. Indeed, there is an injective map of the set of prime and G-invariant ideals to the set of prime ideals in $A \rtimes_r G$, given by $I \mapsto I \overline{\rtimes}_r G$. In fact, if $I \subseteq A$ is a prime, and G-ivariant ideal, then A/I is prime C*-algebra and $I(A) \rtimes_r G$ is a prime C*-algebra by our analysis above. Thus, $I \overline{\rtimes}_r G$ is a prime ideal of $A \rtimes_r G$, therefore the map $I \mapsto I \overline{\rtimes}_r G$ is well defined, and it is injective since $(I \overline{\rtimes}_r G) \cap A = I$ for each G-invariant ideal $I \subseteq A$.

Theorem 3.7. Let (A, G, α, β) be a unital twisted C*-dynamical system. Let X be a G-boundary and let $(A \otimes C(X), G, \nu, \iota)$ denote the associated natural extension. Let I be a proper ideal in $A \rtimes_{\alpha,r}^{\beta} G$ and let J denote the ideal in $\big(A \otimes C(X)\big) \rtimes_{\nu,r}^{\iota} G$ generated by I. Then J is proper.

Proof: Let φ be a state on $A \rtimes_{\alpha,r}^{\beta} G$ such that $\varphi(I) = 0$. By [10 and 34], there is a state ψ on $\big(A \otimes C(X)\big) \rtimes_{\nu,r}^{\iota} G$, a net $(g_i) \in G$ and $x \in X$ such that$\psi\big|_{A \rtimes_{\alpha,r}^{\beta} G} = lim_j \psi \circ Ad\left(\lambda_\beta(g_j)\right)$ and $\psi\big|_{C(X)} = \delta_x$. It should be noted that $\psi\big|_{A \rtimes_{\alpha,r}^{\beta} G(I)} = 0$, and $C(X) \in \psi$. Hence for $b \in I$, $a_1, a_2 \in A, d_1, d_2 \in C(X)$ and $\varsigma_1, \varsigma_2 \in G$ we have

$$\psi\big((a_1 \otimes d_1)\lambda_\iota(\varsigma_1) b (a_2 \otimes d_2)\lambda_\iota(\varsigma_2)\big)$$
$$= d_1(x)\psi\left(a_1\lambda_\beta(\varsigma_1) b a_1 \lambda_\beta(\varsigma_2)\right) d_2(d_2 x)$$
$$= 0$$

It follows that $\psi(J) = 0$. Hence J is proper. □

Remark 3.8: Theorem 3.7 above generalizes [11, Lemma 7.2].

4. STABILITY PROPERTIES

Here we establish stability criteria to ensure that many stability properties are automatically satisfied for any class of groups.

Theorem 4.1. Let Γ be a property for discrete groups such that

T 1. The trivial group $\{1\}$ has property Γ

T 2. If G has property Γ, then G is ice

T 3. If N is normal subgroup of G, then G has property Γ if and only if $C_C(N)$ have property Γ

Then the following holds;

L 1.$G_1 \times G_2$has property Γ if and only if G_1 and G_2 has property Γ

L 2.Ghas property Γ if and only if $Aut(G)$ has property Γ

L 3. If N is normal subgroup of G such that N and G/Nhave propertyΓ, then G has property Γ

L 4. If H is a finite index subgroup of G, then G has property Γ if and only if G is ice and H has property Γ.

Proof. L1 is clear from T3. For L2 , one can identify copy of G with the normal subgroup of $Aut(G)$ of linear automorphisms, since G is nice by T2 and T3. As $C_{Aut(G)}(G) = \{1\}$, L2 follows immediately. For L3, we need to show that $C_G(N)$ has property P. Since $NC_G(N)$ and $NC_G(N)/N$ are normal in G and G/N respectively, hence $C_G(N)$ has property P. Because N is centerless and nice due to T2, $NC_G(N)/N$ is isomorphic to $C_G(N)/N \cap C_G(N) = C_G(N)$. For L4, one can assume G is nice by T2, and let $N = \bigcap_{g \in G} gHg^{-1}$. Then N is the kernel of the canonical action of G on the final coset space G/H sometimes called the normal core of $H \in G$, so it is a normal finite-index subgroup of G. It follows from our analysis that any element $x \in C_G(N)$ has finite conjugacy class in G, and $C_G(N) = \{1\}$. Since N, H, G has property P, then $C_H(N) \subseteq C_G(N) = \{1\}$. □

Remark 4.2. It is instructive to note that G could be isomorphic to the direct product of H and a finite cyclic group. Thus, $G \subset H$ is not necessary nice. As immediate consequence of theorem 4.1, we have the following proposition;

Proposition 4.3. Let G be a discrete group with a normal subgroup N. Then G has trivial amenable radical if and only if N and $C_G(N)$ has trivial amenable radical.

Proof. It is clear, the amenable radical is characteristic, i.e., $\alpha(H) = H$ for any automorphisms $\alpha \in Aut(H)$. If $H \subset G$ is a normal subgroup, the conjugation by $g \in G$ is an automorphisms of H, implying $gR(H)g^{-1} = R(H)$. Therefore, $R(H)$ is normal in G and amenable, so that $R(H) \subseteq R(G) \cap H$. Thus, $R(G) \cap H$ is amenable and normal in$H$, $R(H) = R(G) \cap H$. We now prove the claim. Since N and $C_G(N)$ are normal, $R(G) = \{1\}$ implies $R(N) = R\big(C_G(N)\big) = \{1\}$ by our analysis. Conversely, assume that $R(N) = R\big(C_G(N)\big) = \{1\}$. Then$R(G) \cap N = R(N) = \{1\}$, so normality of $R(G)$ and N implies $g(ng^{-1}n^{-1}) = (gng^{-1})n^{-1} \in R(G) \cap N = \{1\}$ for all $g \in R(G)$ and $n \in N$. Therefore, $R(G)$ and N commute, meaning that $R(G) = R(G) \cap C_G(N) = R\big(C_G(N)\big) = \{1\}$. □

5. SOME EXAMPLES OF C*-DISCRETE GROUPS

We shall need the following Lemma.

Lemma 5.1: Let G be a Hausdorff topological group and let X be a minimal proximal compact G-space. If X has an isolated point, then X is a one point space.

In all of the following examples, we have assume$X$ to be boundaries that are not one-point spaces, such that X has no isolated points by Lemma 5.1. Precisely, finite subsets of X have empty interior.

Example I (*Non-abelian free groups of finite rank).*For $n \geq 2$, the action of non-abelian free group $\mathbb{F}_n$ on its boundary $\partial\mathbb{F}_n$ of one-sided reduced infinite words is topological free. This implies that $\mathbb{F}_n$ is C*-simple. Indeed, if A is a free generating set for $\mathbb{F}_n$, let $\Pi = g_1 \ldots g_n \in \mathbb{F}_n/\{1\}$ be a word in a reduced form, where $g_1 \ldots g_n \in A \cup A^{-1}$. We claim that X^{Π} is finite, so that it has empty interior. Taking the conjugation if necessary, assuming that $g_1 g_n \neq 1$since $\Pi \neq 1$, then $gX^{\Pi} = X^{g\Pi g^{-1}}$, $g \in \mathbb{F}_n$ so that X^{Π} has an empty interior if and only if $X^{g\Pi g^{-1}}$ has empty interior. If $\Pi x = x$ for some $x \in \partial\mathbb{F}_n$, assume that the generator is reduced. Then the first n letter of x are the n letters of Π. Since $g_1 g_n \neq 1$, $\Pi^2 x$ is also reduced. Therefore, then next n letters of x are those of Π. On iterating this process, we see that $x = \Pi\Pi\Pi \ldots$, if Πx is not reduced, then let $1 \leq k \leq n$ be the largest such that the first k letters of x, are $g_n^{-1} \ldots g_{n-k+1}^{-1}$. Since the first letter of x is g_n^{-1} and the first letter of $\Pi x$is g_1, assuming that $k < n$, we have $k = n$. Which is the first n letters of Π^{-1}. From our analysis so far, we conclude that $x = \Pi^{-1}\Pi^{-1}\Pi^{-1} \ldots$, finally, X^{Π} consists of two points. Hence $\mathbb{F}_n$ is C*-simple.

Example II (*Projective Special linear Groups).*For $n \geq 2$, the action of $PSL(n, \mathbb{R})$ on real projective $n-1$-space $\Omega = \mathbb{P}^{n-1}(\mathbb{R})$ is topologically free.We need to realize this. Let $\Pi: (\mathbb{R}^n)_o \to \Omega$ be the quotient map and $g \in SL(n, \mathbb{R})$ fixes a non-empty open subset $U \subseteq \mathbb{P}^{n-1}(\mathbb{R})$ pointwise. Let $V \subseteq \Pi^{-1}(U)$ be non-empty open ball in $\mathbb{R}^n$. For $\nu, \varkappa \in V$ such that $\Pi(\nu) \neq \Pi(\varkappa)$ and $\gamma, \gamma' \in \mathbb{R}$ such that $g\nu = \gamma\nu$and $g\varkappa = \gamma'\varkappa$, then by convexity there exists $\gamma'' \in \mathbb{R}$ such that

$\gamma''(\nu + \varkappa) = g(\nu + \varkappa) = \gamma\nu + \gamma'\varkappa$

Since $\nu \notin \mathbb{R}\varkappa$, if follows, $\gamma = \gamma' = \gamma''$. Therefore $g = \gamma 1$on $V \cup \{0\}$. Assuming $a \in V$, then for all $b \in \mathbb{R}^n$ there exists $c \neq 0$ such that $cb + a \in V$. By linearity, $gb = \gamma b$. Since g factors to the identity element in G, it holds for any discrete subgroup $\Phi \subseteq G$ for which Ω is a Φ-boundary C*-simple. Precisely, $PSL(n, \mathbb{Z})$ is C*-simple..

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