Transactions of the Nigerian Association of Mathematical Physics
Volume 12, (July – Sept., 2020 Issue), **pp53 – 62**
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FLOW FIELD AND GROUP ENERGY TRANSPORT FOR THE PROGRESSIVE OCEAN WATER WAVES.

Stephen Ehidiamhen Uwamusi

Department of Mathematics, Faculty of Physical Sciences, University of Benin, Benin City, Edo State, Nigeria.

Abstract

The paper presents mathematical formulation for the governing equations in the linearized ocean surface water waves where gravity may be important. Using Euler's equation, we obtain the group velocity for the propagating wave which moves in a circular motion when the depth of water with respect to wavelength is taken into consideration. Equation for a circle describing this phenomenon is given describing the statistical mechanics via least squares approach. It is also analyzed for the speed of ocean waves when tension is present on the surface water as a further demonstration of what is intended to achieve in the paper. It is remarked that most of the equations obtainable in hydrodynamics are amenable to aerodynamics.

Keywords**:** ocean water waves, gravity waves, Euler's equation, surface tension, group energy transport, statistical mechanics
MSC 2010 Category: 76B15, 76B07, 76B99

1.0 Introduction: What causes Ocean water waves

The study of ocean water waves for progressive wave train dates back to centuries with significant interests to civil, mechanical, electrical, petroleum engineering practices and of course military Navy. It also constitutes major course of studies in applied mathematics and physics. In a nutshell, there is a synergy between most of the equations used in water engineering practices (hydro-engineering) with those found in aerodynamics (Kinematic movement of object in air). These are linked together by Euler equations for fluids and Stokes wave's theory.

Waves are created when there is a disturbance on the surface water. Waves at the ocean surface are orbital waves called interface waves. Waves are always subject to the influence of ambient current as they propagate on it. We define still water level as the level in which gravity waves are absent.

A typical example is the splash wave, due to long wave-length propagated when an object splashes into water [1]. The commonly known causes of waves other than artificially known phenomenon of object splashing into surface water are as follow:

∗ The wind: The wind is known to create waves in the Ocean which moves with energy travelling along the interface between ocean and atmosphere and, with a transfer of energy from a storm far out at sea over distances of several thousands of kilometers [2,3].

∗ The movements of fluids of different densities: Waves are also created by the movement of fluids of different densities which travel along the interface and boundary of the two different fluids, [2,4,5].

∗ The air: The general description of Waves is created by the wind between and within these fluids in the form:

(i) Along the air-water interface, the movement of air across the ocean creates ocean waves;
(ii) Along the air –air interface, the movement of different air masses creates atmospheric waves in the form of ripple-like clouds in the sky forming as cold fronts i.e. high density air invading an area;
(iii) Along water –water interface, the movement of water of different densities creates internal waves. These waves, travel along the boundaries between waters of different density associated with pycnocling.
(iv) Tidal movement, turbidity currents, wind stress, or even passing ships at the surface currents create internal waves.

Besides these three itemized above as causes of waves, there is what we call gravity wave. The regular gravity waves on the surface of water are a restoration force that keep waves going [3, 4,5,6, 7]. This applies to wavelengths larger than few centimeters as opposed to shorter waves, created by the capillary force that acts on the water surface. We also have other

Correspondence Author: Stephen E.U., Email: Stephen_uwamusi@yahoo.com, Tel: +2348020741193

sources of waves such as mass movement into ocean due to landslides and calving icebergs-splash wave. Naturally known cause of catastrophic waves which is the Sea floor movement that creates changes in the shape of ocean floor responsible for the releases of large amount of energy to the entire water column builds up to large waves**.**

Tides waves are created due to gravitational pull of the moon and sun tug on the part of earth oceans. These are periodic rise and fall of still water level occasioned by differential attraction of sun and moon caused by gravity and centrifugal forces in which the moon plays more dominating role than the sun as a result of its proximity to the earth. The period of tide is 12 hours (semi-diurnal tide) or 24 hours (diurnal tide) wherefore, the tidal range being the distance between high water highest elevation in one cycle and the low water lowest elevation in one tidal cycle.

Motivated in this direction, from Newton's law, the tidal mechanics [5] is given by

$$F = \left(\frac{m_1 m_2}{d^2}\right)v \qquad (1.1)$$

Here, F is the force of attraction between two bodies, v is the universal constant, m_1, m_2 are their respective masses of the two bodies and d is the distance between them.

Nuclear testing of detonation of nuclear substances at or near the sea level is also largely responsible for the release of huge amount of energy that creates waves of large magnitude. Determination of mean water depth is obtained from the knowledge of Astronomical tide (mean tide) can be obtained based on the factors like tsunamis, wave set up, wind set up and pressure set up [5].

We thus, give the components that make up a regular wave. These are the amplitude a , the wavelength λ , the propagation direction and phase speed at a given location and time.

The remaining part in the paper is categorized as follows: In section 2 the mechanisms in the movement of waves have been presented. The speed of ocean water waves in the presence of surface tension is discussed in section 3. Section 4 gives the statistical mechanics for the least squares equations for the circle in the sense of [8,9]. The group energy and wave flux for the ocean waves have been presented in section 5. Section 6 gives the conclusion of the paper.

2.0 Mechanisms in the Movement of Waves

As is well known, waves are energy in motion. Waves transmit energy by means of cyclic movement through matter. We give equations of these circular movements in section three in this paper. The simple progressive waves are waves that oscillate uniformly and progress or travel without breaking**.** Progressive waves are categorized as being longitudinal, transverse, or a combination of two motions called orbital [1, 3,6]. Longitudinal waves have possession of particles that vibrate push and pull in the same direction whose energy is travelling like a spring and its coils are alternately compressed and expanded [6] as the waves progress.

The transverse waves travel at right angles to the direction of the vibrating particles (side-side waves). The body waves consist of both longitudinal and transverse waves and these two waves carry energy through the body matter. The ocean waves transmit energy along an interface between atmosphere and the ocean.

The following factors determine the amount of energy in waves**:**

(i) wind speed,

(ii) The length of time during which the wind blows in one direction,

(iii) The fetch, that is, the distance over which the wind blows in one direction.

Some destructive behaviors of ocean water waves are described herein. Characteristically, the Rogue waves are massive, whereas solitary waves can reach enormous height and often occur at times when normal ocean waves are not usually large [2]. Rogue waves are usually found in the middle of ocean and are very destructive. When waves are reflected in right angles to a barrier, they produce the kind of waves known as standing waves. It is a sum of two waves with the same wavelength moving in opposite directions producing no net movement. Wave refraction is the bending of waves when waves slow in shallow water. Thus, wave reflection is the bouncing back of wave energy caused when waves strike a hard barrier. Dispersion in fluid mechanics in the case of water waves, refers to frequency dispersion. This applies to the waves of different wavelengths travelling at different phase speeds.

It beholds that, substances which allow waves to pass through them are known as dispersive media, e.g., water, air.

We give the prototype of propagating wave of unchanging form whose water elevation profile is

$$\eta(x,t) = A\sin\left(\theta(x,t)\right) \qquad \mathbf{(2.1)}$$

Here, A is the amplitude (measured in meters), $\theta = \theta(x,t)$ is the phase function (in radians) which depends on the horizontal position (x , in meters) and time (t , in seconds).

Usually, $\theta = 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right) = kx - \omega t$, where ,

$k = \frac{2\pi}{\lambda}$ and, $\omega = \frac{2\pi}{T}$ **,**

λ = wavelength (in meters),

T = period (in seconds),

k = wave number (in radians/s)

ω = angular frequency (in radians/s).

Now in the motivation of our work, we shall assume that
curl $q = 0$, (that is, irrotational fluid) , $q = -\nabla \phi$, where ϕ is a velocity potential.
Then the relevant Euler's equations are :

$$\frac{\partial p}{\partial t} + \nabla(\rho v) = 0 \qquad \textbf{(Mass conservation)} \qquad (2.2)$$

$$\frac{\partial \vec{v}}{\partial t} + \left(\vec{v}\,\nabla\right)\vec{v} = -\frac{1}{\rho}\nabla p + \vec{f} + \frac{\eta^*}{\rho}\nabla^2 \vec{v} \qquad \textbf{(Momentum conservation)} \qquad (2.3)$$

In Equation (2.3), $\vec{v}$ is the fluid velocity, i.e., $v_i = \frac{\partial \phi}{\partial x_i}$, ρ is the pressure, $\vec{f}$ is the body force acting on the fluid ,

η^* is the coefficient of viscosity known as kinematic viscosity**.**

Let the surface of the water be perturbed such that the vertical displacement $\eta(x, y, t)$ of each point of the surface is created.
We then define the vorticity of the flow as the quantity $\nabla \times \vec{v}$. Thus, for $\nabla \times \vec{v} = 0$, the flow is irrotational. But the velocity of the flow is a potential field, and for gravity waves, we have that $\vec{f} = -g\,\vec{e}_y$ where**,** g is the acceleration due to gravity and where $\vec{e}_y = (0,1,0)$.

Thus, for $q = 0$, div grad $\phi = \nabla^2\phi = 0$, this means that, the velocity potential is

$$\nabla^2\phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = 0 \quad , \qquad (2.4)$$

$$0 < x < \infty, \eta(x,t) \geq y \geq -h$$

With boundary condition, $\frac{\partial \phi}{\partial y} = 0$ at $y = -h$, where h being the depth of water.

We express the pressure equation[5] due to Euler in the form:

$$\frac{p}{\rho} = \frac{\partial \phi}{\partial t} - gy \;,\quad \left(q^2 \approx \textit{negligible}\right) \qquad (2.5)$$

The free surface, which is a surface of equal pressure (p ~ constant), is given by the equation

$$\frac{dp}{dt} = 0 = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} \quad , \qquad (2.6)$$

for $\underline{q} = u_i + v_j = -\frac{\partial \phi}{\partial x} i - j\frac{\partial \phi}{\partial y}$.

Therefore,

$$\frac{\partial p}{\partial t} - \frac{\partial \phi}{\partial x}\frac{\partial p}{\partial x} - \frac{\partial \phi}{\partial y}\frac{\partial p}{\partial y} = 0 \qquad (2.7)$$

We eliminate the pressure p , in Equation (2.5) using Equation (2.7) and it is written as

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x \partial t} - \frac{\partial \phi}{\partial y}\left(\frac{\partial^2 \phi}{\partial y \partial t} - g\right) = 0 \qquad (2.8a)$$

By further neglecting the nonlinear terms in Equation (2.8a) we obtained that

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial y} = 0 \,. \qquad (2.8b)$$

From Equation (2.4), it would hold that equation for free ocean surface water waves [7,10] is given in the form:

$$\frac{d^2 f}{dy^2} - m^2 f = 0 \qquad (2.9a)$$

In Equation (2.9), we solve for $f(y)$ and is expressed in the form:

$$f(y) = Ae^{my} + Be^{-my}. \tag{2.9b}$$

Because $\frac{\partial \phi}{\partial y} = 0$ at $y = -h$, then it follows that

$mAe^{mh} - mBe^{-mh} = 0$, so that $Ae^{mh} = Be^{-mh} = \frac{c_0}{2}$ giving $A = \frac{1}{2}c_0 e^{mh}; B = \frac{1}{2}c_0 e^{-mh}$.

Coupling together, we obtain that

$$\phi(x, y, t) = \frac{c_0}{2}\left(e^{m(h+y)} + e^{-m(h+y)}\right)\left(\cos(mx - \omega t)\right) \tag{2.10}$$

The waves propagates in the x - direction and uniform in the y - direction. Using that

$$\phi(x,t) = \bar{A}(x, y)\sin(kx - \varpi t) \quad , \tag{2.11}$$

by substituting Equation (2.11) in Equation (2.3) , we obtain in a unique way the surface displacement and the velocity potential ϕ [2] for the propagating wave profile:

$$\eta = A\cos(kx - \omega t) \quad , (x \text{ - direction}) \tag{2.12}$$

$$\phi = \frac{\omega A\cosh(k(y+h))}{k\sinh(kh)}\sin(kx - \omega t) \quad , \tag{2.13}$$

where,

$A = \frac{2ak}{\omega}\exp(-kh)\sin(kh)$, and a , is a constant of integration, A, k and ω are respectively, the amplitude, the wave number, and wave frequency.

We give the dispersion relation for the small amplitude surface water waves as well as their velocity v and group velocity v_g in the form:

$$\omega^2 = gk\tanh(kh) \quad , \tag{2.14}$$

$$v = \sqrt{\frac{g}{k}\tanh(kh)}, \tag{2.15}$$

and,

$$v_g = \frac{v}{2}\left(1 + \frac{2kh}{\sinh(2kh)}\right) \quad . \tag{2.16}$$

The waves going to the right and to the left respectively are written in the form:

$\omega = +\sqrt{gk\tanh(kh)}$ (waves going to the right) ;

$\omega = -\sqrt{gk\tanh(kh)}$ (Waves going to the left).

For small values of the argument, it can be derived that

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\left(1 + x + O(x^2)\right) - \left(1 - x + O(x^2)\right)}{\left(1 + O(x)\right) + \left(1 + O(x)\right)}.$$
$$= x + O(x^2)$$

By taking limit as x approaches infinity, then we have that

$\tanh(x) = \lim_{x\to\infty}\frac{e^x - e^{-x}}{e^x + e^{-x}} \to 1$.But $k = \frac{2\pi}{\lambda}$, and $kh = \frac{2\pi h}{\lambda}$, for $kh \approx$ small, $h << \lambda$ .That is, $\frac{h}{\lambda} << 1$ (shallow water).

Therefore ,

$\frac{\omega^2}{m^2} = c^2 = \frac{g}{k}\tanh(kh) = \frac{\lambda g}{2\pi}\tanh\left(\frac{2\pi h}{\lambda}\right)$. Therefore, for water with infinite depth, $\frac{h}{\lambda} >> 1$,

$$\tanh\left(\frac{2\pi h}{\lambda}\right) \approx \frac{\sinh\left(\frac{2\pi h}{\lambda}\right)}{\cosh\left(\frac{2\pi h}{\lambda}\right)} \approx 1.$$

Hence,

$c^2 = \frac{\lambda g}{2\pi} = \frac{g}{k}$ (the wave speed in deep water).

The trajectory of the fluid particle is determined by taking $x(t)$ and $y(t)$ as the position of the particles with respect to the mean position $\bar{x}$ and $\bar{y}$. Now from well known facts in wave mechanics, differentiate with respect to x and y the potential function

$$\phi(x,y,t)=\frac{ga}{\omega}\frac{\cosh\left(y+h\right)\cos(kx-\omega t)}{\cosh(kh)} \qquad (2.17)$$

We thus have that

$$\bar{X}=-\frac{\partial\phi}{\partial\bar{x}}=\frac{\omega a\cosh k\left(y+h\right)\sin\left(kx-\omega t\right)}{\cosh kh}\ , \qquad (2.18)$$

$$\bar{Y}=-\frac{\partial\phi}{\partial\bar{y}}=-\frac{\omega a\sinh k(y+h)\cos(kx-\omega t)}{\cosh kh}\ , \qquad (2.19)$$

The flux line is given by the equation:

$$\frac{\bar{Y}}{\bar{X}}=\frac{\omega a\sinh k\left(y+h\right)\cos\left(kx-\omega t\right)}{\omega a\cosh k\left(y+h\right)\sin\left(kx-\omega t\right)}$$

For small value of kh ,the energy is almost everywhere moving forward to the wave in shallow water.

Equations (2.18) and (2.19) can be integrated with respect to $\bar{x}$ and $\bar{y}$, from which the equation of Ellipse [1,2,6,5,7,6,] is given to be

$$\frac{X^2}{\cosh^2 k(y+h)}+\frac{Y^2}{\sinh^2 k(y+h)}=\frac{a^2}{\sinh^2 kh}\ , \qquad (2.20)$$

which describes the circle path of the particle about the mean position.

However, empirical evidence showed that for a given particle, the axes of the ellipse would vary with depth (i.e. y and h) , at the bottom where , $y=-h$, the ellipse degenerates to a straight line since, $\sinh^2 k(y+h)=0, \cosh^2 k(y+h)=1$, $\sinh(y+h)\approx\cosh k(y+h)$.

In the case of Stokes waves described for nonlinear deep water, a prototype equation for velocity of the Stokes waves to a second order in steepness is

$$v=\sqrt{\frac{g}{k}(1+\frac{k^2A^2}{2})}\ , \ (A \text{ is the wave amplitude}) \qquad (2.21)$$

For the dispersion relation corresponding to Equation (2.21) the equation is in the form:

$$\omega=\sqrt{gk\left(1+\frac{k^2A^2}{2}\right)} \qquad (2.22)$$

Defining the free surface elevation for the modulated Stokes wave train, we have that:

$$\eta=\text{Re}[A(X,T)\exp\left(i(\omega_0 t-k_0 t)\right)] \qquad (2.23)$$

Where , $X=\varepsilon t\ (\varepsilon<<1)$, $T=\varepsilon t$ .Then, we write that

$$\omega=\sqrt{gk\left(1+k^2|A|^2\right)} \ (A \text{ is amplitude for Stokes wave and the envelope}) \qquad (2.24)$$

By expanding Equation (2.24) in Taylor series about $A=A_0=0$ one obtains an equation [10,11] in the form

$$\omega=\omega_0+\frac{\partial\omega}{\partial k}(k-k_0)+\frac{1}{2}\frac{\partial^2\omega}{\partial k^2}(k-k_0)^2+\frac{\partial\omega}{O|A|^2}\left(|A|^2-|A_0|^2\right) \qquad (2.25)$$

Equation (2.25) is the Kelvin equation for Stokes waves in deepwater.

For aerodynamics, we give the following well known equations which are available in standard texts for comparison:

Compressible equations:

∗ Conservation of mass:

$$\frac{d}{dt}\iiint\limits_{volume}\rho dv+\iint\limits_{surface}\rho\vec{u}.\vec{n}\,ds=0 \qquad (2.26)$$

$$\frac{\partial p}{\partial t}+\nabla(\rho.\vec{u})=0$$

$$\frac{Dp}{Dt}+\rho\nabla.\vec{u}=0$$

$\nabla.\vec{u}=0$ (incompressible)

∗ Conservation momentum:

$$\frac{d}{dt}\iiint\limits_{volume}+\iint\limits_{surface}\rho u\,\vec{u}.\vec{n}\,ds=-\iint\limits_{surface}\rho\,\bar{n}.\bar{i}ds+\iint\limits_{surface}\psi.\vec{i}\,ds \qquad (2.27)$$

Where,

$$\vec{\psi}\, ds = \left(\tau_{xx} + \tau_{yx} dy + \tau_{zx} dz\right)\vec{i} + \left(\tau_{xy} dx + \tau_{yy} dy + \tau_{zy} dz\right)\vec{j} + \left(\tau_{xz} dx + \tau_{yz} dy + \tau_{zz} dz\right)\vec{k}$$

$$\frac{\partial(\rho\mu)}{\partial t} + \nabla.\left(\rho u \vec{u}\right) = -\frac{\partial p}{\partial x} + \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)$$ (non vacuous force)

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

Where,

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \delta_{ij}\lambda\nabla.\vec{u}, \tag{2.28}$$

$$\mu = \mu(t), \lambda = -\frac{2}{3}\mu .$$

Besides the mentioned facts as earlier given, it is true that

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -g\eta_x , \tag{2.29}$$

holds as well in the shallow water wave, [7,11].

For waves with large amplitude, it holds that

$$\frac{\partial}{\partial x}(u + (h + \eta)) = -\eta_t \tag{2.30}$$

If we further ignore nonlinear terms in equation (2.29), we have that

$$\frac{\partial u}{\partial t} = -g\eta_x \tag{2.31}$$

The approximation is true since $\frac{\eta_0\lambda^2}{h^3} << 1$ for h = water depth, λ =wavelength, and η_0 is the wave amplitude.

Now ,

$$\frac{\partial}{\partial x}(uh) = -\eta_t \tag{2.32}$$

$$\frac{\partial}{\partial t}u = -g\eta_x = u_t \tag{2.33}$$

We further differentiate equation (2.32) with respect to t to have

$$\frac{\partial}{\partial t}(u_t h) = -\eta_{tt} \tag{2.34}$$

We also multiply equation (2.33) by h and differentiate with respect to x , to have

$$\frac{\partial}{\partial x}(u_t h) = -gh\eta_{xx} = -g\frac{\partial}{\partial x}(h\eta_x) .$$

Due to the similarity in the two preceding equations, we therefore, write that $g\frac{\partial}{\partial x}(h\frac{\partial \eta}{\partial x}) = \eta_{tt}$. By differentiating equation (2.33) again with respect to t gives that

$$\frac{1}{g}u_{tt} = -\eta_{tt} \tag{2.35}$$

It follows that

$g\frac{\partial^2}{\partial x^2}(uh) = -u_{tt}$. Thus, $\frac{\partial^2\eta}{\partial x^2} = \frac{1}{gh}\eta_{tt}$ has a speed of $\sqrt{gh}$.

From equations (2.29-2.31), we now write that

$$c_x = \frac{(g\eta_x + gh_x)}{2c} , \tag{2.36}$$

$$c_t = \frac{g\eta_t}{2c} \tag{2.37}$$

Then, following [11] we have

$$u_t + uu_x + 2cc_x - H_x = 0 , \qquad (H = gh) \tag{2.38}$$

$$2c_t + 2uc_x + cu_x = 0 \tag{2.39}$$

We are interested in the state where, $\eta = 0, u = 0$. Verification that c is the propagation speed follows from inductive analysis [11] in the form:

$c = c_0 + \varepsilon(x,t), c_0 = \sqrt{gh}, \varepsilon \approx$ a small quantity of first order. u and its derivative is small, and h is constant.

Therefore,

$$u_t + 2c_0\varepsilon_x = 0 , 2\varepsilon_t + c_0 u_x = 0 . \tag{2.40}$$

By retaining the first order term and eliminating ε , we have that

$$u_{tt} - c_0^2 u_{xx} = 0 \cdot \quad (2.41)$$

Equation (2.41) is the classical linear equation all solutions of which are expressed in the form:

$$\bar{u} = u(x \pm c_0 t) \quad (2.42)$$

Solution of equation (2.41) expressed in the form of equation (2.42) means that the motions are suposition of waves with constant propagation speed $c_0 = \sqrt{gh}$. Thus

$$u - uc_0 t = u(x) \Rightarrow u(1 - c_0 t) = u(x) \cdot$$

3.0 Speed of Ocean Water Waves In The Presence of Surface Tension.

When surface tension is present in the water, the compressibility of water is thereby included in the calculation. The velocity potential $u = \nabla \phi$ is now given in the form:

$$\nabla^2 \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} \quad , \quad (3.1)$$

Where, the expression for $c = \sqrt{\frac{dp}{d\rho}}$ is the speed of sound.

We give the ratio for velocity potential in the presence of surface tension to the velocity potential without tension [4] in the form:

$$\frac{\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}}{\nabla^2 \phi} \sim \frac{\omega^2 / k^2}{c^2} \ , \quad (3.2)$$

for $\frac{\omega}{k} = \sqrt{gh}$ = phase speed of the fastest waves, and g is the acceleration due to gravity, h is the sea depth. Here, we supposed that the vertical velocity at the sea water surface must be equal to the vertical velocity of the surface itself. That is,

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial y}, \qquad y = 0 \quad (3.3)$$

Equation (3.3) gives the kinematic condition and, for small amplitudes test wave; the linearized momentum balanced:

$$\rho \frac{\partial u}{\partial t} = -\nabla p - \rho g e_y \qquad (e_y = (0,1,0)) \quad (3.4)$$

We split p in the form: $p = p_0 + p$, where, p_0 is the hydrostatic pressure and then write the equation

$p_0 = -\rho g y$, and $0 = -\nabla p_0 - \rho g e_y \cdot$

It follows $\rho \frac{\partial u}{\partial t} = \rho \frac{\partial \nabla \phi}{\partial t} = -\nabla p$, and hence, the value of the water pressure is calculated from

$$p = -\rho \frac{\partial \phi}{\partial t} \cdot \quad (3.5)$$

Equation (3.5) relates the dynamic pressure with velocity potential. The remarkable statement here, is that for free surface, the hydrostatic pressure is constant, by ignoring surface tension, this gives

$$p = p_0 + p = 0, y = -h \cdot \quad (3.6)$$

The dynamic boundary condition is

$$\rho g h + \rho \frac{\partial \phi}{\partial t} = 0, \quad h = 0 \quad (3.7)$$

Summing up these, we have that

$$\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial y} = 0, \qquad y = 0 \quad (3.8)$$

From [2,,4,6,10] , by taking T as the surface tension per unit length, and for a thin film covering surface water with the horizontal rectangle $dxdy$ on the free surface, the net vertical force from the four sides is given in the form

$$\left(T \frac{\partial \eta}{\partial x}\bigg|_{x+dx} - T \frac{\partial \eta}{\partial x}\bigg|_{x} \right) dy + \left(T \frac{\partial \eta}{\partial y}\bigg|_{y+dy} - T \frac{\partial \eta}{\partial x}\bigg|_{y} \right) dx \quad (3.9)$$

$$= T \left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} \right) dxdy$$

From Equation (3.6), the continuity of vertical pressure [4,6,7] on a unit area of the surface is given by

$$p_0 + p + T\left(\frac{\partial^2\eta}{\partial x^2} + \frac{\partial\eta}{\partial y^2}\right) = 0 \cdot \qquad (3.10)$$

Therefore, using Equation (3.10) we have

$$-\rho g\,\eta - \rho\frac{\partial\phi}{\partial t} + T\left(\frac{\partial^2\eta}{\partial x^2} + \frac{\partial^2\eta}{\partial y^2}\right) = 0,\ y = 0 \qquad (3.11)$$

The combined kinematic condition becomes

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial y} - \frac{T}{\rho}\nabla^2\frac{\partial\phi}{\partial y} = 0, \quad y = 0 \cdot \qquad (3.12)$$

If the viscosity is neglected, the normal fluid velocity vanishes on the rigid seabed and hence,

$$n.\nabla\phi = 0 \qquad (3.13)$$

Where the unit normal is defined to be

$$n = \frac{(h_x, h_y, 1)}{\sqrt{1 + h_x^2 + h_y^2}} \qquad (3.14)$$

4.0 The Least Squares Equations of Circle As Statistical Mechanics.

In what follows, we give [8,9] the statistical least squares equations for fixing a circle to the progressive wave train. Firstly, we consider points in two –space dimension $(x_{1,}y_1),(x_2, y_2),\ldots,(x_n, y_n)$.

It is aimed to construct a circle like pattern for the waves train using the equation

$$(x-a)^2 + (y-b)^2 = r^2 \qquad (4.1)$$

In Equation (4.1), a, b are respectively, the center for x and y.

Using least squares approach [1,2 and 8] for example, we write that

$$SS(a,b,r) = \sum_{i=1}^{n}\left(r - \sqrt{(x_i - a)^2 + (y_i - b)^2}\right)^2 \qquad (4.2)$$

From elementary calculus, for stationary values it is that

$$\frac{\partial SS}{\partial r} = -2\sum_{i=1}^{n}\sqrt{(x_i - a)^2 + (y_i - b)} + 2nr \qquad (4.3)$$

$$\frac{\partial SS}{\partial a} = 2r\sum_{i=1}^{n}\frac{x_i - a}{\sqrt{(x_i - a)^2 + (y_i - b)^2} - 2n\bar{x} + 2na} \qquad (4.4)$$

$$\frac{\partial SS}{\partial b} = 2r\sum_{i=1}^{n}\frac{y_i - b}{\sqrt{(x_i - a)^2 + (y_i - b)^2} - 2n\bar{y} + 2nb} \qquad (4.5)$$

Because, it might be difficult equating these derivatives to a real number 0 with a mission to obtaining these parameters, instead, we introduce in the sense of [8,9] adopt the following equations for a useful purpose.

Setting as

$$a_M = \frac{DC - BE}{AC - B^2} \quad , \qquad (4.6)$$

$$b_M = \frac{AE - BD}{AC - B^2} \ , \qquad (4.7)$$

Where,

$$A = n\sum_{i=1}^{n}x_i^2 - \left(\sum_{i=1}^{n}x_1\right)^2 = n(n-1)S_x^2 \qquad (4.8)$$

$$B = n\sum_{i=1}^{n}x_i y_i - \left(\sum_{i=1}^{n}x_i\right)\left(\sum_{i=1}^{n}y_i\right) = n(n-1)S_{xy} \qquad (4.9)$$

$$C = n\sum_{i=1}^{n}y_i^2 - \left(\sum_{i=1}^{n}y_i\right)^2 = n(n-1)S_i^2 \qquad (4.10)$$

$$D = 0.5\left\{n\sum_{i=1}^{n}x_i y_i^2 - \left(\sum_{i=1}^{n}x_i\right)\left(\sum_{i=1}^{n}y_i^2\right) + n\sum_{i=1}^{n}x_i^3 - \left(\sum_{i=1}^{n}x_i\right)\left(\sum_{i=1}^{n}x_i^2\right)\right\} \qquad (4.11)$$

$$= 0.5n(n-1)(S_{xy^2} + S_{xx^2})$$

$$E = 0.5\left\{ n\sum_{i=1}^{n} y_i x_i^2 - \left(\sum_{i=1}^{n} y_i\right)\left(\sum_{i=1}^{n} x_i^2\right) + n\sum_{i=1}^{n} y_i^3 - \left(\sum_{i=1}^{n} y_i\right)\left(\sum_{i=1}^{n} y_i^2\right)\right\} \tag{4.12}$$

$$= 0.5n(n-1)\left(S_{yx^2} + S_{yy^2}\right)$$

We compute the radius by the equation

$$r_M = \sum_{i=1}^{n}\left[\sqrt{(x_i - a_M)^2 + (y_i - b_M)^2}\right]/n \tag{4.13}$$

Similarly, extension to 3- dimensional space, we have the expression

$$SSS^*(a,b,r) = \sum_{i=1}^{n}\left(r - \sqrt{(x_i - a)^2 + (y_i - b)^2 + (z_i - c)^2}\right) \tag{4.14}$$

and we compute thus the expressions

$$a_M = \frac{\left\{\begin{matrix}\left(S_{xx^2} + S_{xy^2} + S_{xz^2}\right)\left(S_y^2 S_z^2 - S_{yz}^2\right) + (S_{yx^2} + S_{yy^2} + S_{yz^2})\left(S_{xz}S_{yz} - S_{xy}S_z^2\right) \\ \left(S_{zx^2} + S_{zy^2} + S_{zz^2}\right)\left(S_{xy}S_{yz} - S_{xz}S_y^2\right)\end{matrix}\right\}}{2\left\{S_x^2 S_y^2 S_z^2 + 2S_{xy}S_{yz}S_{xz} - S_x^2 S_{yz^2} - S_y^2 S_{xx}^2 - S_z^2 S_{xy}^2\right\}} \tag{4.15}$$

$$b_M = \frac{\left\{\begin{matrix}\left(S_{xx^2} + S_{xy^2} + S_{xz^2}\right)\left(S_{xx}S_{yz} - S_{xy}S_z^2\right) + \left(S_{yx^2} + S_{yy^2} + S_{yz^2}\right)\left(S_x^2 S_{xz}^2\right) \\ + \left(S_{zx}^2 + S_{zy^2} + S_{zz}^2\right)\left(S_{xy}S_{xz} - S_{yz}S_x^2\right)\end{matrix}\right\}}{2\left\{S_x^2 S_y^2 S_z^2 + 2S_{xy}S_{yz}S_{xz} - S_x^2 S_{yz}^2 - S_y^2 S_{xx}^2 - S_z^2 S_{xy}^2\right\}} \tag{4.16}$$

$$c_M = \frac{\left\{\begin{matrix}\left(S_{xx^2} + S_{xy^2} + S_{xz^2}\right)\left(S_{xy}S_{yz} - S_{xy}S_y^2\right) + \left(S_{yx^2} + S_{yy^2} + S_{yz^2}\right)\left(S_{xz}S_{xy} - S_{yz}S_x^2\right) \\ + \left(S_{zx^2} + S_{zy^2} + S_{zz^2}\right)\left(S_x^2 S_y^2 - S_{yy}^2\right)\end{matrix}\right\}}{2\left\{S_x^2 S_y^2 S_z^2 + 2S_{xy}S_{yz}S_{xz} - S_x^2 S_{yz}^2 - S_y^2 S_{xz}^2 - S_z^2 S_{xy}^2\right\}} \tag{4.17}$$

$$r_k = \left(\sqrt{\sum_{i=1}^{n}\left((x_i - a_M)^2 + (y_i - b_M)^2 + (z_i - c_M)^2\right)}/n\right) \tag{4.18}$$

5.0 The Group Energy and Wave flux for the Ocean Waves.

The theory of transport energy in water waves was greatly advanced by Stokes and Rayleigh in their various independent research works. They showed that the Kinematical result obtained from superposition of two dispersive waves of small differing wavelength is the group velocity and it is the derivative of the wave frequency, ω , with respect to wave number, k , that is, $\frac{d\omega}{dk}$.

The group velocity had been used to treat wave spectra, modulating waves, wave patterns (Kelvin ship waves), and internal waves. The purpose of this presentation is to derive the components of wave energy, that is , the kinematic, gravity potential, and surface tension which are uniformly distributed within the wave. Water waves are often formulated in terms of trigonometric sine and cosine functions. We express the rate at which water waves transmit energy [7] if we take a vertical section of the water at right angles to the direction of propagation. For a constant depth, the velocity equation is expressed by the equation

$$\phi(x,y,t) = \frac{gh}{n}\frac{\cosh\ m(y+h)}{\cosh\ mh}\cos(mx - nt) \tag{5.1}$$

We express the change in pressure δp to a first order approximation as $\delta p = -\rho v = \rho\frac{\partial\phi}{\partial x}$, the variable part of the pressure. But the rate at which pressure is doing work is

$$v\delta p = \rho\left(\frac{\partial\phi}{\partial x}\right)^2 \tag{5.2}$$

To calculate the work done in unit time or the energy carried across unit width of this section by the ideas in {4,6,7,10,11], we are able to write that

$$W = v\delta p = \int_{-h}^{0}\delta p\frac{\partial\phi}{\partial x}dy = \rho\int_{-h}^{0}\left(\frac{\partial\phi}{\partial x}\right)^2 dy \tag{5.3}$$

This leads us to

$$W = \frac{g^2\rho A^2}{n}m\frac{\sin^2(kx - \omega t)}{\cosh^2 kh}\int_{-h}^{0}\cosh^2 k(y+h)dy \tag{5.4}$$

$$= \frac{g^2\rho A^2 k}{\omega}\frac{\sin^2(kx - nt)}{\cosh^2 kh}\left(\frac{\sinh\ 2kh}{4k} - \frac{h}{2}\right)$$

Using the fact that $\omega^2 = gk\tanh\ kh$, such that $k = \frac{\omega^2}{g\tanh\ kh}$, then due to [7], e.g., the work done at a cross section is

$$W = \frac{1}{2} g\rho a^2 \frac{\omega}{k}\left(1 + 2kh\cos ech\, 2kh\right).\sin^2(kx - \omega t)\,. \qquad (5.5)$$

The average of $\sin^2\theta$ over any period is $\frac{1}{2}$. Therefore, the work done by the water wave across a section is now in the form

$$W = \frac{1}{4}\frac{g\rho A^2 n}{k}(1 + 2kh\cos ech\, 2kh) = \frac{1}{2} gA^2\rho U \qquad (5.6)$$

Where , $U = (1 + 2kh\cos ech\, 2kh))$. The energy is transmitted at a speed equal that of group velocity. Thus the transport of the kinetic energy density $\left(\frac{\rho q^2}{2} = \rho\, \vec{u}.\vec{u}\right)$ within an irrotational $\vec{u} = \nabla\phi$, incompressible $\left(\nabla^2\phi = 0\right)$ flow can be obtained in the form:

$$KE = \frac{\rho q^2}{2} = \frac{\rho}{2}\nabla.(\phi\nabla\phi)\,. \qquad (5.7)$$

We give the conservation law for the incompressibility of water as

$$\frac{\partial KE}{\partial t} = \frac{\rho}{2}\nabla.\{\phi_t\nabla\phi + \phi\nabla\phi_t\} = \rho\nabla.\{\phi_t\nabla\phi\}\,. \qquad (5.8)$$

Therefore,

$$\frac{\partial KE}{\partial t} - \nabla.\left\{\rho\phi_t\,\vec{u}\right\} = 0 \qquad (5.9)$$

The energy flux is

$\vec{F}\left(KE, \vec{u}\right) = -\rho\,\vec{u}$ and $Q(KE) = 0$. The Eulerian flux is in the direction of the velocity of the fluid particle. To obtain the Lagrangian flux of the fluid particle the setting $\vec{V}_R = \vec{u}$ was initiated such that

$\vec{F}\left(KE, \vec{u}\right) = -\rho\,\vec{u}\left(\phi_t + \frac{q^2}{2}\right)$ relevant to above sections 2 and 3 in the paper.

We give information on the energy density or wave variance density $E(\omega,\theta)$ as the distribution of wave energy over radian frequency ω moving with current velocity and propagation directions θ , the direction being normal to the wave crest of each spectral component. By designating the evolution of action density as

$N\left(\vec{x}, t, \omega, \theta\right)$ in space $\vec{x}$ and time t . Following [5] we set $N = \frac{E}{\omega}$ wherefrom, the action balance equation is in the form

$$\frac{\partial N}{\partial t} + \nabla_{\vec{x}}.\left[\left(\vec{c}_g + \vec{V}\right)N\right] + \frac{\partial c_\omega N}{\partial\omega} + \frac{\partial c_\theta N}{\partial\theta} = \frac{S_{tot}}{\omega} \qquad (5.10)$$

In equation (5.10), the left-hand side gives the kinematic part while the right hand side is the source or sink term gives all information of the physical processes that generate, dissipate or redistribute wave energy. The term $\vec{c}_g = \frac{\partial\omega}{\partial\vec{k}}$ is the group energy accompanying the dispersion relation

$\omega^2 = g\left|\vec{k}\right|\tanh(\left|\vec{k}\right|h)$ with $\vec{k}$ the wave number.

6.0 Conclusion

The paper considered the mathematical model for discussing progressive ocean water waves for shallow water using Euler equation for the hydrostatic pressure. The statistical least squares equations describing the circles for the movement of the wave train (wave mechanics) when the ellipse degenerates to a circle have been presented. The effect of a higher order nonlinear term on the shallow water with finite amplitude wave was highlighted. The group velocity for the wave train was discussed.

It is herby suggested that most relevant equations obtained in the studies of Ocean waves mechanics are amenable to aerodynamics. We hope to dwell more of this in subsequent studies.

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