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INSTABILITY OF THE PERTURBED GENERALIZED PHOTO-GRAVITATIONAL RESTRICTED THREE BODY PROBLEM WITH POYNTING-ROBERTSON DRAG

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Abstract

This work investigates the effects of small perturbations in the Coriolis and centrifugal forces on the stability around the triangular libration points of the Restricted Three-Body Problem (RTBP) under the influence of oblateness, radiation pressure and Poynting-Robertson (PR) drag force. Using Routh and Hurwitz's criteria for stability, the nature of the eigenvalues of the perturbed system is determined and hence the conclusion that the system remains unstable even with the significant influence of perturbations due to the strong destabilizing effect of the PR-drag force. The results are verified using astrophysical data for the Kruger 60 and RXJ0450, 1-5658 binary systems. This work, as a generalization of the classical case and the work of others, will, therefore, serve as a form of reference to achieving better results in Space Dynamics and also an added value to designers of space crafts and aerospace agencies.

Keywords: Restricted Three Body Problem; Libration points; Stability; Perturbation; Poynting-Robertson Drag

1. Introduction

The classical problem assumes that the primaries are spherical, however, due to the advancement in astro-physical studies, other properties (oblateness, triaxiality, surface area light, and force other than the gravitational force, Coriolis and centrifugal forces, atmospheric drag, solar wind e.t.c.) of the planet and extrasolar bodies became clearer.

In view of these properties, many authors have generalized the classical RTBP. These generalizations made the problem more realistic by incorporating the force of radiation pressure; oblateness/triaxiality and Poynting-Robertson (PR) drag effect investigations.

The motion of RTBP was studied and the rational equations of motion established [1]. Later, the circular RTBP was investigated to show that the collinear libration points existed and that the triangular points make an isosceles triangle with the primaries and continued by studying the motion of three rigid bodies whose elementary particles act upon each other according to arbitrary laws of forces along the straight line joining them [2]. The classical RTBP was found to possess three unstable collinear points and two triangular points which are seen to be stable [3]. Analysis on the stability of the classical RTBP was made by formulating the equations of motion of the system using the Langrange's-Hamiltonian technique [4].

One of the immediate generalizations of the RTBP was the study of the photo-gravitational effects. For small particles like asteroids and binary stars, light can cause a significant change in the altitude and direction of motions over a large period when rotating relative to the sun. This energy radiated from the celestial bodies known as the photo-gravitational effect was put into consideration when formulating and establishing the stability of RTBP. Here, linear stability of the problem was examined and found to have five libration points [5]. The stability in the Lyapunov sense with one of the primaries radiating was investigated and it was discovered that the system remain stable [6]. The Astronomical application of the stability of RTBP was examined by looking into the influence of radiation pressure from the sun in the planet-satellite-particle system. It was observed that the radiation pressure force affects the triangular libration points but does not destabilize the system [7]. The complete solution of the RTBP was obtained and the existence of the equilibrium points with their linear stability was discussed for all values

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of radiation pressure from both radiating bodies and for all values of mass ratio [8]. It was discovered that the motion of the triangular libration points of the RTBP when the attracting primaries are radiating under the non-resonance cases was stable, for all values of the mass value, $0 < \mu \leq \frac{1}{2}$ and mass reduction factor due to radiation pressure, q_1, q_2 [9]. The photo-gravitational effect on the infinitesimal mass from both primaries was investigated and it was observed that the system remained stable [10]. Considering the shape of the primaries, the stability of the collinear and triangular libration points for the averaged equations of motion of the elliptic photo-gravitational RTBP to a first approximation was examined [11]. The stability of the relative equilibrium positions (collinear libration points) of the circular photo-gravitational RTBP, in which a point mass is passively experiencing the Newtonian gravitational force and also experiencing forces of light pressure from each of the main bodies (stars) was investigated. The collinear points were found to be unstable for all values of the mass ratio, μ while the non-collinear points are stable and form isosceles triangles with the primaries [12, 13].

The spectacular results of the effect of radiation pressure on RTBP prompted Poynting and Robertson to generalize previous works further by considering the radiation pressure force produced from the absorption and subsequent re-emission of sun rays, striking small particles orbiting it thereby retarding the motion of the particle thus lowering the angular momentum consequently making the particle to spiral towards the sun. This process is known as the Poynting-Robertson drag effect. This was the first to consider by Poynting and later modified by Robertson [14, 15]. He established the expression for the net drag force which opposes the direction of motion using a precise relativistic treatment of the first order in the ratio of the velocity of the particle to the speed of light [15]. The effect of radiation pressure and PR-drag on the RTBP was examined [16]. The existence of six libration points of which one lie out of the orbital plane was established on studying the RTBP and it was observed that due to the PR-drag effect, the triangular libration points are unstable [17, 18]. The general dynamical effect of the drag on the planar circular RTBP was also studied [19]. The effect of radiation, PR and solar wind drag in the RTBP was also examined [20]. Related to this modification, the equations of motion when the primaries are radiating with PR-drag effect from the expression of the net force acting on the system was established and was studied numerically. It was discovered that the collinear points deviate from the axis while the triangular points are no longer symmetrical [21]. The study on the effect of the PR-drag on the triangular Lagrangian points of the RTBP was investigated but in the spatial, elliptic orbit [22]. The diagonalizable Hamiltonian for the photo-gravitational RTBP with the PR-drag effect was established [23]. The linear stability of the generalised photo-gravitational RTBP was examined when the smaller primary is considered to be oblate spheroid and the bigger one radiating with PR-drag [24, 25]. The out of plane equilibrium points of a passive micro size particle was obtained and their stability examined in the field of radiating binary stars [26]. Previous work, [19] was extended to study the effect of PR-drag on the triangular libration points in the framework of elliptic RTBP [22]. The photo-gravitational RTBP when the bigger primary is oblate and the smaller one a source of radiation with PR-drag was investigated [27]. Using analytical and numerical methods, the triangular libration points of the RTBP was obtained and were found to move towards the line joining the primaries in the presence of perturbations (such as oblateness up to J_4 of the less massive primary, electromagnetic radiation of the more massive primary and potential from the belt), except in the presence of oblateness up to J_4 where the points move away from the line joining the primaries and examined their linear stability [28]. A practical application of their model is the study of the motion of a dust particle near a radiating star and an oblate body surrounded by a belt.

Their work was extended to understand the effects of various perturbing factors on the dynamics of a particle orbiting the primaries and it was concluded that the P-R drag renders unstable those libration points that are conditionally stable in the classical case [29]. The effect of P-R and solar wind drag on space debris was studied [30]. Many researchers have used various binary stars such as Procyon, Kruger, RW-Monocerotis, Achird, Luyten, α Cen AB, Xi-Bootis, Algol etc to verify their results [31].

The significance of the study of the effects of small perturbations in the Coriolis and centrifugal forces on the instability of the libration points of the generalised photo-gravitational RTBP with Poynting-Robertson drag cannot be over-emphasized because of the peculiar nature. The Coriolis force has a stabilizing effect while the centrifugal force a destabilizing nature. And also in the presence of oblateness, radiating and PR-drag properties of the primaries the level of destibilization is assessed.

It was shown that the stability of the two equilateral points was due to the presence of the Coriolis parameter in the equation of motion [32]. A relation between the critical value of the mass parameter μ_c and the perturbation in the Coriolis (ϵ) force keeping the centrifugal force constant for the triangular points of the RTBP was established and it was discovered that the Coriolis force is a stabilizing force [33]. It was shown that with oblateness, the Coriolis force is not always a stabilizing force [34]. Their work was extended to include the centrifugal (ϵ') force and it was discovered that the collinear points remain unstable while the triangular points are stable and that the increase or decrease in the range of stability depends upon the points (ϵ, ϵ') [35]. The effect of small perturbations in the Coriolis and the centrifugal forces on the location of libration points in the RTBP with variable mass and in line with other results, they established that the range of stability of the triangular points increases or decreases depending on whether the perturbation point (ϵ, ϵ') lies in either of the two parts in which (ϵ, ϵ') plane is divided by the line $36\epsilon - 19\epsilon' = 0$ [36]. Building upon previous works, the combined effects of small perturbations in the Coriolis and centrifugal forces, radiation and oblateness on the stability of the libration points the RTBP have ben investigated and it was discovered that the collinear points remain unstable while the range of stability of the triangular points decreases as seen in the critical mass value μ_c obtained [37]. The non-linear stability of the triangular equilibrium points under the effects of small perturbations in the Coriolis and the centrifugal forces, together with the effects of oblateness and radiation pressures of the primaries was considered [38 39]. The influence of small perturbations in the Coriolis and centrifugal forces was examined both analytically and numerically on the stability of circular RTBP with PR-drag in both primaries. This result was shown for the binary systems Luyten 726-8 and Kruger 60 [40]. The effect of perturbations on the stability of the libration points on RTBP with a triaxial primary and radiating secondary was considered and the range of stability were found to be affected by the perturbing parameters ((ϵ, ϵ'), (q_1, q_2) *and* (σ_1, σ_2)) for the Coriolis and centrifugal forces, mass reduction factor due to radiation pressure force and triaxiality respectively as seen in the relation for the critical mass value [41]. The effect of small perturbations in Coriolis and centrifugal forces on the equilibrium points and examined stability in Robe's circular RTBP when the hydrostatic equilibrium figure of the massive primary is an oblate spheroid; the shape of the less massive primary is a triaxial rigid body was examined and It was discovered that the locations of the axial equilibrium points were only influenced by a small change in the centrifugal force [42]. Many other researchers have introduced and studied the effects of the Coriolis and centrifugal forces, radiation pressure force, oblateness, on the stability of the RTBP, observed that Coriolis force has a stabilizing tendency while the centrifugal force, radiation pressure force and oblateness have destabilizing effect ([43], [44], [45], [46], [47], [48] etc.).

Due to the remarkable overall non-stabilizing effects these perturbing forces (Coriolis and centrifugal forces, shape of the primaries, radiation pressure, solar wind drag, Poynting-Robertson drag etc.) on the motion around the orbit of the RTBP (satellite, both natural and artificial), the equilibrium position of the triangular libration points of the perturbed generalized photo-gravitational restricted three body problem with Poynting-Robertson drag effect was established [49]. In this study, the effect of small perturbations in the Coriolis and centrifugal forces on the stability of these points is investigated under the influence of oblateness and PR-drag force from both radiating primaries.

The paper is organized in five sections, namely:

Section 1 is the introduction. Section 2 presents the perturbed equations of motion for the generalized RTBP under the influence of the PR-drag and their libration points. Section 3 discusses the stability around the triangular points. Here, the characteristics equations corresponding to the equations of motion are established and the nature of their eigenvalues is determined using Routh and Hurwitz criteria for stability both analytically and numerically. Section 4 is the conclusion.

2. Equations of Motion and their Triangular points

With reference to an inertial or fixed coordinates OXYZ, let (x, y, z), ($-a$, 0, 0) and (b, 0, 0) be the coordinates of the infinitesimal body and primaries with masses m, m_1 and m_2 respectively. In addition let r_1, r_2 be the distances between each of the primary and the infinitesimal while r be the distance between the primaries.

Introducing a rotating coordinate system Oxyz with the origin O at the barycenter of the primaries in which the axis rotate relative to the inertial space with an angular velocity $\omega = nk$, the vector equations of motion of the generalised photo-gravitational RTBP are,

$$\bar{a} = -\bar{\omega}\times(\bar{\omega}\times\bar{r}) - 2\bar{\omega}\times\bar{v} + G\left[\frac{m_i q_i}{r_i^3}\bar{r}_i + \frac{3m_i A_i}{2r_i^5}\bar{r}_i + \frac{m_i(1-q_i)}{r_i^2}\left(\frac{(\bar{r}_i + \bar{\omega}\times\bar{r}_i)\bar{r}_i\bar{r}_i}{cr_i^2} + \frac{(\bar{r}_i + \bar{\omega}\times\bar{r}_i)}{c}\right)\right]$$

where,

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k},\ \bar{r}_1 = (x+a)\hat{i} + y\hat{j} + z\hat{i},$$

$$\bar{r}_2 = (x-b)\hat{i} + y\hat{j} + z\hat{k},\quad \dot{\bar{r}} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k},$$

$$\bar{r}_1^{\,2} = (x+a)^2 + y^2 + z^2,\ \bar{r}_2^{\,2} = (x-b)^2 + y^2 + z^2,$$
$$v = \dot{\bar{r}} + \omega \times \bar{r} \quad and \quad i = 1,2,$$

Following Szebehely's terminologies [33], the distance between the primaries along the x- axis is taken to be equal to one. The sum of the masses of the primaries is also taken to be 1 so that if $m_2 = \mu$ then $m_1 = 1-\mu$ and the origin is the barycenter of the masses. Which implies that $m_1(-a) + m_2(b) = 0$ so that $a = \mu$ and $b = 1-\mu$, where, $\mu = \frac{m_2}{m_1+m_2}$ is the mass ratio parameter.

The unit of time is chosen so as to make the gravitational constant G to be equal to unity. The speed of light c is given as $c = c_d$. Assuming that $q_i\,(i = 1,2)$ are constant (neglecting fluctuations in the beam of solar radiation and the effect of the planet shadow).

In the dimensionless synodic (x-y plane) coordinate system, the equations of motion of the generalized photo-gravitational RTBP, with small perturbations in the Coriolis and centrifugal forces, using the parameter ϕ and ψ respectively such that

$$\begin{aligned} \phi &= 1+\varepsilon \quad |\varepsilon| = 1 \\ \psi &= 1+\varepsilon' \quad |\varepsilon'| = 1 \end{aligned} \tag{1}$$

is given by

$$\begin{aligned} \ddot{x} - 2n\phi\dot{y} &= \Omega_x = \Omega^*_x + \Omega_{PR_x} \\ \ddot{y} + 2n\phi\dot{x} &= \Omega_y = \Omega^*_y + \Omega_{PR_y} \end{aligned} \tag{2}$$

where,

$$\Omega^* = \frac{n^2\psi}{2}(x^2+y^2) + \frac{(1-\mu)q_1}{r_1} + \frac{\mu q_2}{r_2} + \frac{(1-\mu)A_1}{2r_1^3} + \frac{\mu A_2}{2r_2^3} \tag{3}$$

$$\Omega_{PR} = W_1\left[\frac{(x+\mu)\dot{x} + y\dot{y}}{2r_1^2} - n\arctan\left(\frac{y}{x+\mu}\right)\right] + W_2\left[\frac{(x+\mu-1)\dot{x} + y\dot{y}}{2r_2^2} - n\arctan\left(\frac{y}{x+\mu-1}\right)\right] \tag{4}$$

$$W_1 = \frac{(1-\mu)(1-q_1)}{c_d},\ W_2 = \frac{\mu(1-q_2)}{c_d}, \tag{5}$$

$$r_1^2 = (x+\mu)^2 + y^2 \quad and \quad r_2^2 = (x+\mu-1)^2 + y^2 \tag{6}$$

and the mean motion, n is

$$n^2 = 1 + \frac{3A_1}{2} + \frac{3A_2}{2} \tag{7}$$

Ω^* and Ω_{PR} are the negative of the gravitational potential due to attraction on the infinitesimal body under the influence of radiation, oblateness and P-R drag from the primaries. These are functions of position, velocity and dependent on the small perturbation in the centrifugal forces.

Solving the Equations of motion (2) at equilibrium, that is when the velocity and acceleration are equal zero and assuming that, $q_1 = (1-w_1),\ q_2 = (1-w_2)$, where w_1 and w_2 are very much less than 1, $(i.e.\ |w_1|,|w_2| \ll 1)$ we get the coordinates of the triangular libration points, $L_4(x,+y)$ and $L_5(x,-y)$ as;

$$x = \frac{1}{2} - \mu - \frac{w_1}{3\psi^{\frac{2}{3}}} + \frac{w_2}{3\psi^{\frac{2}{3}}} + \frac{A_1}{2} - \frac{A_2}{2} - \frac{W_1(2-\mu\psi^{\frac{2}{3}})}{3\mu(1-\mu)\psi^{\frac{4}{3}}\sqrt{4-\psi^{\frac{2}{3}}}} + \frac{W_2(2-\psi^{\frac{2}{3}}+\mu\psi^{\frac{2}{3}})}{3\mu(1-\mu)\psi^{\frac{4}{3}}\sqrt{4-\psi^{\frac{2}{3}}}} \tag{8}$$

$$y = \pm\frac{\sqrt{4-\psi^{\frac{2}{3}}}}{2\psi^{\frac{1}{3}}}\left\{1 - \frac{2}{4-\psi^{\frac{2}{3}}}\left[\frac{w_1}{3} + \frac{w_2}{3} + \frac{(2-\psi^{\frac{2}{3}})A_1}{2} + \frac{(2-\psi^{\frac{2}{3}})A_2}{2} - \frac{W_1(2-\mu(4-\psi^{\frac{2}{3}}))}{3\mu(1-\mu)\psi^{\frac{2}{3}}\sqrt{4-\psi^{\frac{2}{3}}}} + \frac{W_2(2-\psi^{\frac{2}{3}}-\mu(4-\psi^{\frac{2}{3}}))}{3\mu(1-\mu)\psi^{\frac{2}{3}}\sqrt{4-\psi^{\frac{2}{3}}}}\right]\right\}. \tag{9}$$

Putting Equation (1) in (8) and (9) neglecting product of ε' with other small quantities $(|w_1|,|w_2|,|A_1|,|A_2|,|W_1|,|W_2| \ll 1)$, since they are very negligible the coordinates become

$$x = \frac{1}{2} - \mu - \frac{w_1}{3} + \frac{w_2}{3} + \frac{A_1}{2} - \frac{A_2}{2} - \frac{W_1(2-\mu)}{3\mu(1-\mu)\sqrt{3}} + \frac{W_2(1+\mu)}{3\mu(1-\mu)\sqrt{3}} \tag{10}$$

$$y = \pm\frac{\sqrt{3}}{2}\left[1-\frac{4\varepsilon'}{9}-\frac{2w_1}{9}-\frac{2w_2}{9}-\frac{A_1}{3}-\frac{A_2}{3}+\frac{W_1(2-3\mu)}{9\mu(1-\mu)\sqrt{3}}+\frac{W_2(1-3\mu)}{9\mu(1-\mu)\sqrt{3}}\right] \qquad (11)$$

Equations (10) and (11) are the coordinates of the triangular libration points, $L_4(x,+y)$ and $L_5(x,-y)$.

In order to appreciate the impact of the centrifugal force on the location of the libration points, the product of ε' with the small quantity parameters is further considered, taking only the first order terms in ε'. The coordinates are obtained as

$$x^* = \frac{1}{2}-\mu-\frac{w_1(3-2\varepsilon')}{9}+\frac{w_2(3-2\varepsilon')}{9}+\frac{A_1}{2}-\frac{A_2}{2}-\frac{W_1[18-22\varepsilon'-\mu(9-5\varepsilon')]}{27\mu(1-\mu)\sqrt{3}} - \frac{W_2[9-17\varepsilon'+\mu(9-5\varepsilon')]}{27\mu(1-\mu)\sqrt{3}} \qquad (12)$$

$$y^* = \pm\frac{\sqrt{3}}{2}\left\{1-\frac{4\varepsilon'}{9}-\frac{2(9-2\varepsilon')w_1}{81}-\frac{2(9-2\varepsilon')w_2}{81}-\frac{(9-8\varepsilon')A_1}{27}-\frac{(9-8\varepsilon')A_2}{27} + \frac{W_1[18-14\varepsilon'-\mu(27-54\varepsilon')]}{81\mu(1-\mu)\sqrt{3}}+\frac{W_2[9-11\varepsilon'-\mu(27-23\varepsilon')]}{81\mu(1-\mu)\sqrt{3}}\right\} \qquad (13)$$

In line with the work of [27], [31], [47] and [50], a range of values for the parameters ε' are used in studying the effect of small perturbation in the centrifugal force on the location around the triangular libration points. Specifically for the binary system Kruger - 60 $(\mu = 0.3937,\ c_d = 48002.33,\ q_1 = 0.99992,\ q_2 = 0.99996)$ and RXJ 0450, 1-5856 $(\mu = 0.0967,\ cd = 299792458,\ q_1 = 0.9963,\ q_1 = 0.9965)$ with the aid of micro-soft Excel and Maple 18 Mathematical Software. The values obtained are given in Table 1 and 2 .

Table 1: Effects of ϵ' on $L_{4,5}$ for Kruger - 60 $(\mu = 0.3937, c_d = 48002.33, q_1 = 0.99992, q_1 = 0.99996)$ binary system, $x_c = 0.1063$ $y_c = \pm 0.866025$

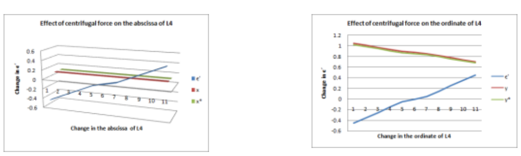
ϵ'	x	x^*	$\pm y$	$\pm y^*$
-0.45	0.101286665	0.101282665	1.049792146	1.027080725
-0.35	0.101286665	0.101283554	1.011302128	0.989361021
-0.25	0.101286665	0.101284443	0.97281211	0.951641317
-0.15	0.101286665	0.101285332	0.934322092	0.913921612
-0.05	0.101286665	0.101286221	0.895832074	0.876201908
0	0.101286665	0.101286666	0.876587065	0.857342056
0.05	0.101286665	0.10128711	0.857342056	0.838482204
0.15	0.101286665	0.101287999	0.818852038	0.800762499
0.25	0.101286665	0.101288888	0.780362020	0.763042795
0.35	0.101286665	0.101289777	0.741872002	0.72532309
0.45	0.101286665	0.101290666	0.703381984	0.687603386

Table 2: Effects of ϵ' on $L_{4,5}$ for RXJ 0450, 1-5856 $(\mu = 0.0967, c_d = 299792458, q_1 = 0.9963, q_1 = 0.9965)$ binary system, $x_c = 0.4033$ $y_c = \pm 0.866025404$

ϵ'	x	x^*	$\pm y$	$\pm y^*$
-0.45	0.397183333	0.396848333	1.029790808	1.034331668
-0.35	0.397183333	0.396922778	0.99130079	0.99611557
-0.25	0.397183333	0.396997222	0.952810772	0.957899473
-0.15	0.397183333	0.397071667	0.914320754	0.919683376
-0.05	0.397183333	0.397146111	0.875830736	0.875437175
0	0.397183333	0.397183333	0.856585727	0.856585727
0.05	0.397183333	0.397220556	0.837340718	0.837734278
0.15	0.397183333	0.397295000	0.798850700	0.805035084
0.25	0.397183333	0.397369444	0.760360682	0.766818986
0.35	0.397183333	0.397443889	0.721870664	0.728602889
0.45	0.397183333	0.397518333	0.683380646	0.690386792

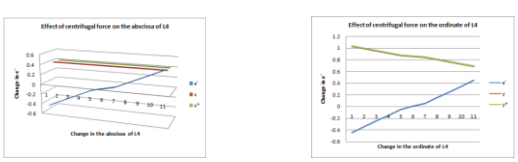
*the subscript c indicates the coordinate evaluation for the classical case.

It is observed from the Tables 1 and 2 that there is a significant change in the value of x_c and y_c due to the presence of all the perturbing factors. It is also seen that as ε' is increasing the values of the x coordinate is not affected by it, but when x is extended to accommodate more of ε' up to the first order product of ε' and other small quantities $(w_1, w_2, A_1, A_2, W_1, W_2)$ as seen in Equations (12) and (13), there is a significant increase in the value of x as ε' is increasing. On the other-hand the values of y is seen to be decreasing as ε' is increasing at the same rate. This can be seen in the Fig. 1 and 2 below:



(a) Graph of ϵ' vs x-axis (b) Graph of ϵ' vs y-axis.

Figure 1: Effect of the centrifugal force on coordinates of the Triangular Libration points for Kruger-60



(a) Graph of ϵ' vs x-axis (b) Graph of ϵ' vs y-axis.

Figure 2: Effect of the centrifugal force on coordinates of the Triangular Libration points for RXJ 0450, 1-5856 binary system.

3. Stability around the Triangular Points

Let (x_0, y_0) be the coordinate of the triangular libration points and $\xi, \eta << 1$ be the small displacements such that $(x_0+\xi, y_0+\eta)$ is a point in the vicinity of (x_0, y_0), with velocity component $(\dot{\xi}, \dot{\eta})$. Using the Taylor series expansion gives variational equation of motion corresponding to equations of motion as,

$$\begin{aligned} \ddot{\xi} - 2n\phi\dot{\eta} &= \xi\Omega^0_{xx} + \eta\Omega^0_{xy} + \dot{\xi}\Omega^0_{x\dot{x}} + \dot{\eta}\Omega^0_{x\dot{y}} + 0(2) \\ \ddot{\eta} + 2n\phi\dot{\xi} &= \xi\Omega^0_{yx} + \eta\Omega^0_{yy} + \dot{\xi}\Omega^0_{y\dot{x}} + \dot{\eta}\Omega^0_{y\dot{y}} + 0(2) \end{aligned} \tag{14}$$

where the superscript $(^0)$ indicate that the partial derivatives are evaluated at the libration points, which implies that $\Omega^0_x = \Omega^0_y = 0$. Note that the second and higher order terms, 0(2) are truncated.

Let $\xi = Ae^{\lambda t}$, $\eta = Be^{\lambda t}$ be the trial solutions of Equation (14), then the variational equations can be written as,

$$(\lambda^2 - \lambda\Omega^0_{x\dot{x}} - \Omega^0_{xx})A + [-(2n\phi + \Omega^0_{x\dot{y}})\lambda - \Omega^0_{xy}]B = 0$$
$$[(2n\phi - \Omega^0_{y\dot{x}})\lambda - \Omega^0_{yx}]A + (\lambda^2 - \lambda\Omega^0_{y\dot{y}} - \Omega^0_{yy})B = 0$$

and solving,

$$\begin{vmatrix} \lambda^2 - \lambda\Omega^0_{x\dot{x}} - \Omega^0_{xx} & -(2n\phi + \Omega^0_{x\dot{y}})\lambda - \Omega^0_{xy} \\ (2n\phi - \Omega^0_{y\dot{x}})\lambda - \Omega^0_{yx} & \lambda^2 - \lambda\Omega^0_{y\dot{y}} - \Omega^0_{yy} \end{vmatrix} = 0$$

yields the characteristic equation corresponding to the variational equation of motion (14) as,

$$\lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d = 0 \tag{15}$$

where

$$\left.\begin{aligned} a &= -(\Omega^0_{x\dot{x}} + \Omega^0_{y\dot{y}}) \\ b &= 4n^2\phi^2 + 2n\phi\Omega^0_{x\dot{y}} - 2n\phi\Omega^0_{y\dot{x}} - \Omega^0_{x\dot{y}}\Omega^0_{y\dot{x}} + \Omega^0_{x\dot{x}}\Omega^0_{y\dot{y}} - \Omega^0_{xx} - \Omega^0_{yy} \\ c &= \Omega^0_{x\dot{x}}\Omega^0_{yy} + \Omega^0_{xx}\Omega^0_{y\dot{y}} - 2n\phi\Omega^0_{yx} - \Omega^0_{yx}\Omega^0_{x\dot{y}} + 2n\phi\Omega^0_{xy} - \Omega^0_{xy}\Omega^0_{y\dot{x}} \\ d &= \Omega^0_{xx}\Omega^0_{yy} - \Omega^0_{yx}\Omega^0_{xy} \end{aligned}\right\} \tag{16}$$

Differentiating the partial derivatives Ω_x, Ω_y from Equations (3) and (4) with respect to $x, y, \dot{x}, \dot{y}$ respectively and evaluating the second order partial derivatives at libration points using Equations (6) - (9), so that

$$r_1 = \frac{1}{\psi^{\frac{1}{3}}}\left\{1 - \frac{w_1}{3} - \frac{(1-\psi^{\frac{2}{3}})A_1}{2} - \frac{A_2}{2} - \frac{(2-\psi^{\frac{2}{3}})W_1 + 2W_2}{3(1-\mu)\psi^{\frac{2}{3}}\sqrt{4-\psi^{\frac{2}{3}}}}\right\}$$

$$r_2 = \frac{1}{\psi^{\frac{1}{5}}}\left\{1 - \frac{w_2}{3} - \frac{A_1}{3} - \frac{(1-\psi^{\frac{2}{3}})A_2}{2} + \frac{2W_1 + (2-\psi^{\frac{2}{3}})W_2}{3\mu\psi^{\frac{2}{3}}\sqrt{4-\psi^{\frac{2}{3}}}}\right\}$$

and
$$\begin{aligned} \Omega^0_{xx} &= \frac{3}{4}\psi^{\frac{5}{3}} \\ &- \frac{[(2\psi - \psi^{\frac{5}{3}}) - \mu(4\psi - \psi^{\frac{5}{3}})]w_1}{2} + \frac{[2\psi - \mu(4\psi - \psi^{\frac{5}{3}})]w_2}{2} + \frac{3(9\psi^{\frac{5}{3}} - 8\mu\psi^{\frac{5}{3}})A_1}{8} \\ &+ \frac{3(\psi^{\frac{5}{3}} + 8\mu\psi^{\frac{5}{3}})A_2}{8} - \frac{[8\psi^{\frac{5}{3}} - \mu(16\psi^{\frac{1}{3}} - 3\psi^{\frac{5}{3}}) + \mu^2(4\psi - 3\psi^{\frac{5}{3}})]W_1}{4\mu(1-\mu)\sqrt{4-\psi^{\frac{2}{3}}}} \\ &+ \frac{[(8\psi^{\frac{1}{3}} - 4\psi) - \mu(16\psi^{\frac{1}{3}} - 8\psi + 3\psi^{\frac{5}{3}}) - \mu^2(4\psi - 3\psi^{\frac{5}{3}})]W_2}{4\mu(1-\mu)\sqrt{4-\psi^{\frac{2}{3}}}}, \end{aligned} \tag{17}$$

$$\begin{aligned} \Omega^0_{xy} &= \frac{3\psi^{\frac{4}{3}}\sqrt{4-\psi^{\frac{2}{3}}}}{4}\left\{1 - 2\mu - \frac{[(8 - 8\psi^{\frac{2}{3}} + 2\psi^{\frac{4}{3}}) - \mu(4\psi^{\frac{2}{3}} - 2\psi^{\frac{4}{3}})]}{3\psi^{\frac{2}{3}}(4-\psi^{\frac{2}{3}})}w_1\right. \\ &\left.+ \frac{[(8 - 4\psi^{\frac{2}{3}}) - \mu(4\psi^{\frac{2}{3}} - 2\psi^{\frac{4}{3}}]}{3\psi^{\frac{2}{3}}(4-\psi^{\frac{2}{3}})}w_2 + \frac{[(24 - 5\psi^{\frac{2}{3}}) - \mu(32 - 6\psi^{\frac{2}{3}})]}{2(4-\psi^{\frac{2}{3}})}A_1 + \frac{[(8 - \psi^{\frac{2}{3}}) - \mu(32 - 6\psi^{\frac{2}{3}})]}{3\psi^{\frac{2}{3}}(\psi - \psi^{\frac{2}{3}})}A_2\right\} \\ &- \frac{[(16 - 8\psi^{\frac{2}{3}}) - \mu(56\psi^{\frac{2}{3}} - 28\psi^{\frac{4}{3}} + 3\psi^2) + \mu^2(48\psi^{\frac{2}{3}} - 24\psi^{\frac{4}{3}} + 3\psi^2)]W_1}{4\mu(1-\mu)(4-\psi^{\frac{2}{3}})} \\ &- \frac{[(16 - 16\psi^{\frac{2}{3}} + 4\psi^{\frac{4}{3}}) - \mu(40\psi^{\frac{2}{3}} - 20\psi^{\frac{4}{3}} + 3\psi^2) + \mu^2(48\psi^{\frac{2}{3}} - 24\psi^{\frac{4}{3}} + \psi^2)]W_2}{4\mu(1-\mu)(4-\psi^{\frac{2}{3}})} = \Omega^0_{yx}, \end{aligned} \tag{18}$$

$$\Omega^0_{yy} = \frac{(12\psi - 3\psi^{\frac{5}{3}})}{4} + \frac{[(4\psi - 2\psi^{\frac{5}{3}}) - \mu(8\psi - 2\psi^{\frac{5}{3}}]w_1}{4} - \frac{[4\psi - \mu(8\psi - 2\psi^{\frac{5}{3}})]w_2}{4}$$
$$+ \frac{3}{8}(12\psi - \psi^{\frac{5}{3}})A_1 + \frac{3}{8}(12\psi - \psi^{\frac{5}{3}})A_2 + \frac{[8\psi^{\frac{1}{3}} - \mu(16\psi^{\frac{1}{3}} + 4\psi - 3\psi^{\frac{5}{3}}) + \mu^2(8\psi - 3\psi^{\frac{5}{3}})]W_1}{4\mu(1-\mu)\sqrt{4-\psi^{\frac{2}{3}}}}$$
$$+ \frac{[(8\psi^{\frac{1}{3}} - 4\psi) - \mu(16\psi^{\frac{1}{3}} - 12\psi + 3\psi^{\frac{5}{3}}) - \mu^2(8\psi - 3\psi^{\frac{1}{3}})]W_2}{4\mu(1-\mu)\sqrt{4-\psi^{\frac{2}{3}}}}, \qquad (19)$$

$$\left.\begin{aligned}
\Omega^0_{x\dot{x}} &= -\frac{\psi^{\frac{2}{3}}}{4}(4+\psi^{\frac{2}{3}})(W_1+W_2)\\
\Omega^0_{x\dot{y}} &= \frac{-\psi\sqrt{4-\psi^{\frac{2}{3}}}}{4}(W_1 - W_2) = \Omega^0_{y\dot{x}}\\
\Omega^0_{y\dot{y}} &= -\frac{\psi^{\frac{2}{3}}}{4}(8-\psi^{\frac{2}{3}})(W_1+W_2)
\end{aligned}\right\} \qquad (20)$$

Using Equation (1), Equations (15) – (18) become

$$\Omega^0_{xx} = \frac{3}{4} + \frac{5}{4}\varepsilon' - \frac{1}{2}(1-3\mu)w_1 + \frac{1}{2}(2-3\mu)w_2 + \frac{3}{8}(9-8\mu)A_1 + \frac{3}{8}(1+8\mu)A_2 - \frac{(8-13\mu+\mu^2)}{4\mu(1-\mu)\sqrt{3}}W_1$$
$$- \frac{(4-11\mu-\mu^2)}{4\mu(1-\mu)\sqrt{3}}W_2, \qquad (21)$$

$$\Omega^0_{xy} = 1 - 2\mu + \frac{11}{9}(1-2\mu)\varepsilon' - \frac{2}{9}(1+\mu)w_1 + \frac{2}{9}(2-\mu)w_2 + \frac{1}{6}(19-26\mu)A_1 + \frac{1}{6}(7-26\mu)A_2$$
$$- \frac{(8-31\mu+27\mu^2)}{9\mu(1\mu)\sqrt{3}}W_1 - \frac{(4-23\mu+27\mu^2)}{9\mu(1-\mu)}W_2$$
$$= \Omega^0_{yx}, \qquad (22)$$

$$\Omega^0_{yy} = 1 - 2\mu + \frac{11}{9}(1-2\mu)\varepsilon' - \frac{2}{9}(1+\mu)w_1 + \frac{2}{9}(2-\mu)w_2 + \frac{1}{6}(19-26\mu)A_1 + \frac{1}{6}(7-26\mu)A_2 \qquad (23)$$
$$- \frac{(8-31\mu+27\mu^2)}{9\mu(1\mu)\sqrt{3}}W_1 - \frac{(4-23\mu+27\mu^2)}{9\mu(1-\mu)}W_2,$$

$$\left.\begin{aligned}
\Omega^0_{x\dot{x}} &= -\frac{5}{4}(W_1+W_2)\\
\Omega^0_{x\dot{y}} &= -\frac{3}{4}(W_1-W_2) = \Omega^0_{y\dot{x}}\\
\Omega^0_{y\dot{y}} &= -\frac{7}{4}(W_1+W_2)
\end{aligned}\right\}. \qquad (24)$$

Substituting these values from Equations (21) to (24) in (16), neglecting second and higher order terms of small quantities and product of ε' with $(w_1, w_2, A_1, A_2, W_1, W_2)$, gives

$$\left.\begin{aligned}
a &= 3(W_1+W_2) > 0\\
b &= 1 + 8\varepsilon - 3\varepsilon' - (3/2 - 3\mu)A_1 + (3/2 - 3\mu)A_2 + \frac{W_1}{\sqrt{3}} - \frac{W_2}{\sqrt{3}}\\
c &= \frac{-3}{4}(4+3\mu)W_1 - \frac{3}{4}(7-3\mu)W_2 < 0\\
d &= \frac{27}{4}\mu(1-\mu) + \frac{33}{2}\mu(1-\mu)\varepsilon' + \frac{3}{2}\mu(1-\mu)w_1 + \frac{3}{2}\mu(1-\mu)w_2 + \frac{117}{4}\mu(1-\mu)A_1 + \frac{117}{4}\mu(1-\mu)A_2\\
&\quad - \frac{W_1(54-81\mu)}{4\sqrt{3}} - \frac{W_2(27-81\mu)}{4\sqrt{3}}
\end{aligned}\right\} \qquad (25)$$

These are all constant coefficients in which a and c are found to depend only on PR-drag force, b depends on the parameters of small perturbations in the Coriolis and centrifugal forces, (ε, ε'), oblateness (A_1, A_2), radiation pressure force (w_1, w_2) with PR- drag force (W_1, W_2) while d depend on all except the parameter of small perturbation in the Coriolis force only. And for $|w_1|, |w_2|, |A_1|, |A_2|, |W_1|, |W_2| \ll 1$, the coefficients $a > 0$ and $c < 0$ but the nature of b and d will be difficult to predict analytically.

In the absence of perturbations in the Coriolis and centrifugal forces, radiation pressure, oblateness and PR-drag effect from both primaries, the characteristics equation of motion obtained in Equation (15) reduces to that of the classical problem [49]:

$$\lambda^4 + \lambda^2 + \frac{27}{4}\mu(1-\mu) = 0$$

with it's four roots as,

$$\lambda = \lambda_n = \pm zi \qquad (n = 1,2,3,4) \tag{26}$$

where,

$$z^2 = \frac{1}{2}\{1 \mp [1-27\mu(1-\mu)]^{\frac{1}{2}}\} \quad [3] \tag{27}$$

Assuming σ_1 and σ_2, (σ_1, σ_2 are small quantities) to represent the net perturbations due to Coriolis, centrifugal, oblateness, radiation and PR -drag effects on the RTBP, then the solutions of Equation (15) are

$$\lambda = \lambda_n(1+\sigma_1+i\sigma_2) = \pm[-\sigma_2 + (1+\sigma_1)i]z \tag{28}$$

Substituting Equation (26) and its multiples in equation (15) , neglecting product of small quantities and comparing coefficients of the real and imaginary part we get

$$\sigma_1 = \frac{-z^4+bz^2-d}{2z^2(2z^2-1)} \quad and \quad \sigma_2 = \frac{\pm az^3 \mp cz}{2z^2(2z^2-1)} \tag{29}$$

where the values of a, b, c, d and z are given in Equations (25) and (27) .

Therefore, the roots of the characteristic equation of motion of the perturbed system (15) gives,

$$\begin{aligned} \lambda_{1,2} &= \pm\gamma\{-az^3+cz+i[3z^4-(2-b)z^2-d]\} \\ \lambda_{3,4} &= \pm\gamma\{az^3-cz+i[3z^4-(2-b)z^2-d]]\} \end{aligned} \tag{30}$$

where $\gamma = \dfrac{1}{2z^2(2z^2-1)}$

The roots, $\lambda_i, i = 1,2,3,4$ are functions of the constants coefficients (a,b,c,d) obtained in Equation (25) . These are seen to be dependent on the parameters of the small perturbations in the Coriolis and centrifugal forces, oblateness, mass reduction factor due radiation pressure, PR-drag force and supposed to be controlled by the discriminant. This shows that the root is influenced by the aforementioned factors.

3.1 Critical Mass

The discriminant Δ is,

$$\begin{aligned} \Delta &= (a^2b^2c^2 - 4a^3c^3 - 4b^3c^2 + 18abc^3 - 27c^4 + 256d^3) \\ &\quad + d(-4a^2b^3 + 18a^3bc + 16b^4 - 80ab^2c - 6a^2c^2 + 144bc^2) \\ &\quad + d^2(-27a^4 + 144a^2b - 128b^2 - 192ac) \end{aligned}$$

Considering only first order term of small quantities and since a and c, given in Equation (25) , are functions of $|W_1|,|W_2|$ □ 1, then the equation above reduces to

$$\Delta = 256d^3 - 128b^2d^2 + 16b^4d \tag{31}$$

and substituting for b and d from Equation (25) give,

$$\begin{aligned} \Delta &= 256\left[\frac{27}{4}\mu(1-\mu)\right]^3\left[1+\frac{22}{3}\varepsilon'+\frac{2}{3}w_1+\frac{2}{3}w_2+13A_1+13A_2-\frac{3W_1(2-3\mu)}{\mu(1-\mu)\sqrt{3}}-\frac{3W_2(1-3\mu)}{\mu(1-\mu\sqrt{3})}\right] \\ &\quad -128\left[\frac{27}{4}\mu(1-\mu)\right]^2\left[1+16\varepsilon-\frac{10}{9}\varepsilon'+\frac{4}{9}w_1+\frac{4}{9}w_2+\left(\frac{17}{3}+6\mu\right)A_1+\left(\frac{35}{3}-6\mu\right)A_2\right. \\ &\quad \left.-\frac{(4-8\mu+2\mu^2)}{\mu(1-\mu\sqrt{3})}W_1-\frac{(2-4\mu-2\mu^2)}{\mu(1-\mu)\sqrt{3}}W_2\right] \\ &\quad +16\left[\frac{27}{4}\mu(1-\mu)\right]\left[1+32\varepsilon-\frac{86}{9}\varepsilon'+\frac{2}{9}w_1+\frac{2}{9}w_2-\left(\frac{5}{3}-12\mu\right)A_1+\left(\frac{31}{3}-12\mu\right)A_2\right. \\ &\quad \left.-\frac{(2-7\mu+4\mu^2)}{\mu(1-\mu)\sqrt{3}}W_1+\frac{(1+\mu-4\mu^2)}{\mu(1-\mu)\sqrt{3}}W_2\right] \end{aligned} \tag{32}$$

Here, the discriminant Δ is a function of the mass parameter μ and other perturbing factors. Δ is studied in the interval $0 \le \mu \le \frac{1}{2}$.

If $\mu = 0$, then $\Delta = 0$

This implies that the discriminant vanishes at this point and since the critical mass value, μ_c is expected to exist when $\Delta = 0$ therefore

$$\mu_c = \mu = 0 \qquad (33)$$

and if $\mu = \frac{1}{2}$, then Equation (32) becomes

$$\Delta = 27\left[\frac{529}{16} - 184\,\varepsilon + \frac{24449}{72}\varepsilon' + \frac{1771}{72}w_1 + \frac{1771}{72}w_2 + \frac{23023}{48}A_1 + \frac{23023}{48}A_2 - \frac{1955}{8\sqrt{3}}W_1 + \frac{1955}{8\sqrt{3}}W_2\right] > 0 \qquad (34)$$

This shows that when $\mu = \frac{1}{2}$, $\Delta > 0$ and it implies that the solution of the characteristics equation (15) would consist of both real and complex conjugate roots (secular terms) and the critical mass value $\mu_c = \mu = 0$ does not exist in the interval $0 < \mu \le \frac{1}{2}$ and hence the triangular libration point will not be unstable.

3.2 Routh and Hurwitz test

The characteristics equation (15) is a constant coefficient equation with coefficients a, b, c, d. The Hurwitz's determinants are:

$$D_1 = a > 0$$

$$D_2 = \begin{vmatrix} a & c \\ 1 & b \end{vmatrix} = ab - c > 0$$

$$D_3 = \begin{vmatrix} a & c & 0 \\ 1 & b & d \\ 0 & a & c \end{vmatrix} = a(bc - ad) - c^2$$

$$D_4 = \begin{vmatrix} a & c & 0 & 0 \\ 1 & b & d & 0 \\ 0 & a & c & 0 \\ 0 & 1 & b & d \end{vmatrix} = a\begin{vmatrix} b & d & 0 \\ a & c & 0 \\ 1 & b & d \end{vmatrix} - c\begin{vmatrix} 1 & d & 0 \\ 0 & c & 0 \\ 0 & b & d \end{vmatrix}$$

$$= abcd - (ad)^2 - c^2 d$$

If all the D_i's $(i = 1, 2, 3, 4)$ possesses positive values then the roots of the characteristics equation (15) will have negative real parts and thus the system will be stable otherwise it is unstable.

The values of the D_i's for the binary systems, Kruger-60 and RXJ0450,1-5856 are computed and presented in Tables 3 and 4 below:

Table 3: Effects of ϵ and ϵ' on the D_i's for kruger-60 ($\mu = 0.3937, c_d = 48002.33, q_1 = 0.99992, q_2 = 0.99996$) binary system

ϵ'	ϵ	D_1	D_2	D_3	D_4
-0.45	0.9	4.01556E-09	4.37195E-08	-2.35037E-16	-1.1371E-17
-0.35	0.7	4.01556E-09	3.609E-08	-2.00507E-16	-8.86715E-17
-0.25	0.05	4.01556E-09	2.84604E-08	-1.65978E-16	-1.38772E-16
-0.15	0.3	4.01556E-09	2.08309E-08	-1.31448E-16	-1.61674E-16
-0.05	0.1	4.01556E-09	1.32013E-08	-9.69183E-17	-1.57376E-16
0	0	4.01556E-09	9.38654E-09	-7.96534E-17	-1.45027E-16
0.05	0.1	4.01556E-09	5.57176E-09	-6.23886E-17	-1.25879E-16
0.15	-0.3	4.01556E-09	-2.05779E-09	-2.78589E-17	-6.7182E-17
0.25	-0.5	4.01556E-09	-9.68735E-09	6.67087E-18	1.87142E-17
0.35	-0.7	4.01556E-09	-1.73169E-08	4.12006E-17	1.3181E-16
0.45	-0.9	4.01556E-09	-2.49465E-08	7.57303E-17	2.72105E-16

Table 4: Effects of ϵ and ϵ' on the D_i's for RXJ 0450, 1-5856 ($\mu = 0.0967, c_d = 299792458, q_1 = 0.9963, q_1 = 0.9965$) binary system

ϵ'	ϵ	D_1	D_2	D_3	D_4
-0.45	0.9	3.37839E-11	3.59484E-10	-1.312E-20	-2.39032E-22
-0.35	0.7	3.37839E-11	2.95295E-10	-1.09455E-20	-1.77695E-21
-0.25	0.5	3.37839E-11	2.31105E-10	-8.77103E-21	-2.68807E-21
-0.15	0.3	3.37839E-11	1.66916E-10	-6.59653E-21	-2.97238E-21
-0.05	0.1	3.37839E-11	1.02726E-10	-4.42203E-21	-2.62989E-21
0	0	3.37839E-11	7.06317E-11	-3.33478E-21	-2.22359E-21
0.05	-0.1	3.37839E-11	3.85369E-11	-2.24753E-21	-1.66059E-21
0.15	-0.3	3.37839E-11	-2.56525E-11	-7.30348E-23	-6.44879E-23
0.25	-0.5	3.37839E-11	-5.77472E-11	1.01422E-21	9.68614E-22
0.35	-0.7	3.37839E-11	-5.77472E-11	1.01422E-21	9.68614E-22
0.45	-0.9	3.37839E-11	-1.21937E-10	3.18872E-21	3.50492E-21

* D_i's, $i = 1, 2, 3, 4$ are the Hurwitz determinants

It is observed that D_1 is always positive for the two binary system. It is expected that the nature of the roots of the Equation (15) would be influenced by a change in the value of the perturbation factors, due to the presence of the parameters of the small perturbations in the Coriolis (ε) and centrifugal (ε') forces in the coefficients b and d of the characteristic Equation of motion (14) .

However, as the value of ε' is increasing, ε is deceasing and consequently the values of D_2 changes from positive to negative, D_3 and D_4 from negative to positive. Since there is no point of ε, at which all the D_i's are all positive in the chosen range. This implies that the real part of the roots of the characteristics equation cannot be all negative. Therefore the perturbed RTBP system remains unstable according to Routh and Hurwitz's criteria for stability.

Tables 5 and 6 below shows the effects of small perturbations in the Coriolis, ε and centrifugal, ε' forces on the RTBP under the influence of oblateness and radiation pressure force with PR-drag. The roots of the characteristics equation in (30) are evaluated for Kruger-60 and RXJ 0450,1 – 5856 binary systems. Although the small perturbations in the Coriolis and centrifugal forces affects the value of the characteristic roots, the systems are unstable since for $|\varepsilon|=$ and $|\varepsilon'|=1$, the real part of the eigenvalues λ_i are not all negative.

Table 5: Effects of ϵ and ϵ' on λ_i, $i = 1,2,3,4$ for kruger-60 ($\mu = 0.3937, c_d = 48002.33, q_1 = 0.99992, q_2 = 0.99996$)

ϵ'	ϵ	$\lambda_{1,3}$	$\lambda_{2,4}$
-0.5	1	± 0.1188458174	$-2.259476994E-09 \pm i3.243040908$
-0.1	0.2	$2.387068724E-09 \pm i0.7917180227$	$-4.394848724E-09 \pm i1.508765248$
-0.05	0.1	$0.3858037786 \pm i1.060867358$	$-0.3858037806 \pm i1.060867363$
0	0	$0.6510565577 \pm i0.9620133262$	$-0.6510565597 \pm i0.9620133291$
0.05	-0.1	$0.8348199175 \pm i0.8505993738$	$-0.8348199195 \pm i0.8505993757$
0.1	-0.2	$0.9840108877 \pm i0.7210217945$	$-0.9840108897 \pm i0.7210217957$
0.5	-1	$1.088952547 \pm 0.8723400427I$	$-1.088952549 \pm i0.8723400436$

Table 6: Effects of ϵ and ϵ' on λ_i, $i = 1,2,3,4$ for RXJ 0450, 1-5856 ($\mu = 0.0967$, $c_d = 299792458, q_1 = 0.9963, q_1 = 0.9965$)

ϵ'	ϵ	$\lambda_{1,3}$	$\lambda_{2,4}$
-0.5	1	± 0.07154875190	$-1.861523864E-11 \pm i3.243026245$
-0.1	0.2	$8.491802263E-12 \pm i0.4383589260$	$-2.538375226E-11 \pm i1.649224500$
-0.05	0.1	$2.024011448E-11 \pm i0.6120743754$	$-3.713206448E-11 \pm i1.259946411$
0	0	$0.3940317902 \pm i0.8132103367$	$-0.3940317902 \pm i0.8132103367$
0.05	-0.1	$0.6436279371 \pm i0.6673132109$	$-0.6436279371 \pm i0.6673132109$
0.1	-0.2	$0.8198949364 \pm i0.4777841634$	$-0.819894936 \pm i0.4777841634$
0.5	-1	± 0.4083305543	± 2.884643160

4. Conclusion

The effect of small perturbations in the Coriolis and centrifugal forces on the stability of the libration points (triangular points) of the RTBP were considered when the primaries are taken to be both oblate spheroids, radiating with PR-drag effect.

The values of the coordinate of the triangular libration points, $L_{4,5}(x,\pm y)$ found to be influenced only by the small perturbation in the centrifugal force, ε' given to the presence of its parameter in Equations (10) – (13) were computed for Kruger-60 and RXJ 0450,1 – 5856 binary system using Microsoft Excel Mathematical software. It was observed that the x coordinate is not affected by the change in the value ε' of the centrifugal force while the values of the y coordinate decreases with an increase in the value ε' thereby affecting the isosceles triangle obtained from other generalization.

The characteristics equation (15) corresponding to the variational equations of motion is seen to depend on all the perturbing parameters, a and c are dependent on the PR-drag alone, d are dependent on the centrifugal force, oblateness, radiation with PR-drag force while the coefficient b is dependent on all the aforementioned and the Coriolis force. The roots of the characteristic equation obtained in Equation (30) are affected by these forces.

It was discovered from Equations (33) and (34) that the critical mass value, $\mu_c = \mu = 0$ suggest that the characteristic equation consist of both real and complex conjugate roots and that μ_c does not exist in the interval $0 < \mu < 1/2$ for this particular system which is an indication of instability

The values of the coefficients of the characteristics equation were computed and used to determine the Hurwitz's determinants, $D_i's$.

According to Routh and Hurwitz criteria for stability, it is observed that the perturbed generalized RTBP remains unstable under the influence of small perturbations in the Coriolis and centrifugal forces.

Therefore in line with existing research, results of various generalizations involving small perturbations in the centrifugal force, radiation pressure forces, oblateness of primaries, Poynting-Robertson drag, it has been shown that the system remained unstable even with the significant influence of the small perturbation of the Coriolis force which has a stabilizing tendency owing to the strong destabilizing effect of the PR-drag force.

This research work has improved upon the existing knowledge about the impact of small perturbations in the Coriolis and centrifugal forces on the stability of a small particle to be launched in the vicinity of oblate and radiating bodies under the influence of the PR-drag force.

This work would serve as a form of reference to achieving vital results in the framework of generalizing RTBP.

Data Availability

The data used to support this study are from previously reported studies and are available at: DOI:10.1155/2013/936859; DOI:10.1007/s10509 – 013 – 1707 – 8 and DOI :10.1051/ aas :1997202 . These prior studies (and datasets) are cited at relevant places within the text as references.

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