Transactions of the Nigerian Association of Mathematical Physics
***Volume 12, (July – Sept., 2020 Issue)*, pp187 – 194**
© Trans. of NAMP

SUSTAINED OSCILLATION IN SOLUTIONS OF LINEAR PIECEWISE CONTINUOUS ORDINARY DELAY INTEGRO-DIFFERENTIAL EQUATION BY BROWNIAN WHITE NOISE

Augustine Omoghaghare Atonuje and Newton I.Okposo

Department of Mathematics, Delta State University, Abraka, Nigeria.

Abstract

The paper studies the oscillatory behavior of the solution of a linear piece-wise continuous deterministic ordinary delay integro-differential equation with time lags. The system is disturbed by an Ito-type Brownian white noise to form a stochastic delay integro-differential equation (SDIDE). Although, it is a known fact that oscillation in solutions of deterministic ordinary delay differential equations is solely caused by the existence of time delays, it is established in this note that even if the deterministic system, in which the noise is absent allows a non-oscillatory solution, the presence of noise in the stochastic delay integro-differential equation will continue to sustain oscillation.

***Keywords*:** Oscillation, piece-wise continuous, deterministic integro-differential equation, time lags, Brownian white noise, stochastic delay integro-differential equations.

1. INTRODUCTION:

Since Sturm, in 1836 introduced the concept of oscillation, when he tried to solve the problem of heat transmission, the study of oscillatory behavior of deterministic and stochastic delay differential equations has become the centre of many research works, due to the fact that these classes of equations find application in many fields such as Engineering, physics, population dynamics, medical sciences and biological sciences. Whether oscillation in solutions of deterministic ordinary and partial differential equations, existed, is still a matter of a search into the very important articles, textbooks and monographs of authors, which include [1 – 12] and some of the references therein. Over the past few decades, the bulk of the work on oscillation had focused mainly on setting necessary and sufficient conditions for oscillation of solutions of deterministic delay differential equations with little attention on the influence of noise as well as the existence of positive and negative solutions of such equations. It appears that research into the combined influence of time delays and stochastic perturbation on oscillation of stochastic delay differential systems is either scanty or quite missing from the literature of stochastic analysis and applications, hence making the topic an interesting area to research on. The first result on the contribution of noise to oscillation of stochastic differential equations to the best of other authors' knowledge was [13]. Later, [14-16] presented certain results relating to the effects of noise on the oscillation and non- oscillation of solutions of both nonlinear and linear delay differential equations.

In the present paper, we establish that, if a deterministic delay integro-differential equation (DIDE), where noise is non-existent, admits a non-oscillatory solution under some conditions, the presence of noise will sustain oscillation in the solution associated with the corresponding stochastic delay integro-differential equation (SDIDE).We shall apply a technique of conjugation relationship found in [17], which will enable us to analyze the role played by a multiplicative white noise as it complements the efforts of time lags in sustaining oscillation in solution of the newly formed SDIDE.

2. PRELIMINARIES:

Armed with the fact that first order ordinary differential equations (ODEs) do not generally possess oscillatory solutions, one can easily see that oscillation in first order delay differential equations is generated by delays or deviating arguments. Moreover, when such equations are perturbed by some external or internal noise, the resulting equations are called stochastic delay differential equations. We shall consider the deterministic linear piecewise delay integro-differential equation whose right hand side is piecewise continuous of the form

$$\left.\begin{aligned} x^{\prime}(t) &= \sum_{i=0}^{m} f_i(t)x(t-\lambda_i) + \int_{t_0}^{t} g(u,t)x(u)du + h(t)\operatorname{sgn}\left(s[t-r]\right), & t \geq 0 \\ x(t) &= \phi(t), & t \in \left[t_0-\lambda_m - r, t_0\right] \end{aligned}\right\} \qquad (1)$$

Correspondence Author: Atonuje O.A., Email: austine412@yahoo.com, Tel: +2348035085758

Where $r > 0$ is a constant called switching delay, λ_i are the usual time lags

$$0 = \lambda_0 < \lambda_1 < . . . < \lambda_m \qquad (2)$$

$f_i(t), \ i = 0, 1, . . ., m$ and $g(u,t)$ are $n \times m$ matrix functions analytic for $u, t \geq t_0$, $h(t)$ is an $n \times r$ function analytic for $t \geq t_0$, $S(.)$ is an r – dimensional column vector function, analytic in the n- dimensional Euclidean space E^n *and* $x(t)$ is an n – dimensional column vector. Define the vector sgnS (signum S) by

$$\text{sgn } S = (\text{sgn } S_1, \text{ sgn } S_2, . . ., \text{sgn } S_r) \qquad (3)$$

Where $S_1, \ S_2, \ , S_r$ are the components of S and

$$Sgn \ S_k = \begin{cases} +1, & for \ S_k > 0 \\ undefined & , for \ S_k = 0 \\ -1, & for \ S_k < 0 \end{cases} \qquad (4)$$

The analytic hyper surfaces $S_k \in E^n$ denoted by

$$S_k = \{S : S_k(x) = 0\}, \ k = 1, 2, . . ., r \qquad (5)$$

are called switching. Moreover, we decompose the Euclidean space E^n into two domains D_k^+ *and* D_k^- in which $S_k(x) \geq 0$ respectively and S_k is the common boundary of the domains. Again we let $\psi = \bigcup_{k=1}^{r} S_k$ *and* $\xi = E^n \setminus S$

Definition 2.1

A continuous vector $x(t) = x(t,u) \in C([t_0, -\Gamma], \Re)$ *for* t_0, *where* $\Gamma = \max_{1 \leq t \leq m} \left(\max_{1 \leq t \leq m} \lambda_i, r \right)$ is called the solution of Eq.(1) if (i) $x(t)$ satisfies Eq.(1) in ξ together with the initial function (ii) $x(t)$ has only isolated points in ψ .

Moreover, since the matrix functions $f_i, \ i = 0, 1, . . ., m; \ g(u,t)$ *and* $h(t)$ are analytic for $u, t \geq 0, \ x(t) = x(u,t)$ is analytic in the neighborhood of $t \geq t_0$.

Definition 2.2

(i) A function f is said to be piecewise continuous in a domain D, if it is continuous except at a finite number of points in D. If a function f can be locally represented by a power series then it is said to be analytic.

(ii) Suppose that solution $x(t)$ exists on some sub- infinite ray $[T_x, \infty)$ and satisfies $Sup \ \{|x(t)| : t \geq T\} > 0$ *for every* $T \geq T_x$ then $x(t)$ is called a regular or non-trivial solution. That is $|x(t)| \neq 0$ in any infinite interval $[T, \infty)$.

(iii) A non-trivial or regular solution $x(t)$ of Eq.(1) is said to be oscillatory if and only if it has arbitrarily large zeros for $t \geq t_0$. That is, there exists a sequence of zeros $(\{t_n\} : x(t_n) = 0)$ *of* $x(t)$ *such that* $\underset{n \to \infty}{Limit} \ t_n = +\infty$ .If a solution is not oscillatory, then it is called non-oscillatory.

(iv) A non-trivial solution $x(t)$ defined on a half-line $[0, \infty)$ with differentiable components is called eventually positive if there exists $t_1 > t_0$ such that $x(t) > 0$, *for* $t \geq t_1$. Also, a non-trivial solution $x(t)$ defined on a half-line $[0, \infty)$ with differentiable components is said to be eventually negative if there exists $t_1 > t_0$ such that $x(t) < 0$, *for* $t \geq t_1$. If $x(t)$ is either eventually positive or eventually negative, then it is said to be non – oscillatory.

The reason for our work is to consider how, the solution of the deterministic DIDE (1) behave in terms oscillation or non-oscillation, when perturbed stochastically into a stochastic delay SDIDE of the form:

$$\left. \begin{aligned} X(t) &= \left(\sum_{i=0}^{m} f_i(t) X(t - \lambda_i) + \int_{t_0}^{t} g(u,t) X(u) du \ + \ h(t) \text{sgn} \ (s[t - r]) \right), dt + \sigma X(t) dB(t) \ t \geq 0 \\ X(t) &= \phi(t), \qquad\qquad\qquad\qquad\qquad\qquad t \in [t_0 - \lambda_m - r, t_0] \end{aligned} \right\} \qquad (6)$$

Where σ is the noise scaling parameter, $\{B(t)\}_{t \in [0,T]} = (B_1(t), B_2(t), B_3(t) . . ., B_m(t))^T$ is an m-dimensional standard Brownian motion defined on a complete probability space (Ω, F, P), with filtration $\{F_t\}_{t \geq 0}$, which is right continuous and of which each F_0 contains all P-null sets. Brownian motion was put on a solid mathematical foundation by [18] when it was established the existence of a complete mass distribution P(B) with total mass +1 defined on the class of all continuous paths $0 < t \to b(t)$ by

$$P[b(t)\in A / b(r): r < s] = \int_A \frac{\exp\left[-(x-y)^2 / 2(t-s)\right]}{\left[2\pi(t-s)\right]^{1/2}} dy$$

For $t > s,\ x = b(s)\ and\ \ A \subset \Re^1$. It was also established in [19] that the Brownian motion is continuous but is nowhere differentiable. Throughout this work, we shall contrast the oscillatory and non-oscillatory behavior of the SDIDE (6) with those of the comparable deterministic delay system in Eq. (1) which has the same initial function ϕ .

Definition 2.3 An $\Re - valued$ stochastic process $X = X(t): [-\Gamma, T] \times \Omega \to \Re$, is called the solution of the SDIDE (6) if X is a measurable sample – continuous process, such that $X / [0, T]\ is\ \{F_t\}_{t\in[0,T]}$- adapted and satisfies the equation together with the initial function $\phi(t)$. Moreover, $f_i, g\ and\ h$ are piecewise continuous matrix functions.

A non-trivial continuous function $x(t) = x: [t_0, \infty) \to \Re$ is called oscillatory if and only if, there exists a sequence of zeros

$\tau_x = (\{t_n\}: x(t_n) = 0,\ for\ all\ t \geq t_1,\ for\ t_1 \geq t_0)\ such\ that\ Sup\ \tau_x = \underset{n\to\infty}{Limit}\ t_n = +\infty$

Extending this in an autonomous manner to stochastic processes: a stochastic process $\{X(t, w)\}_{t\geq 0}$ defined on a probability space (Ω, F, P) is said to be oscillatory, almost surely if and only if there exists a sub-sample space $\Omega_1 \subseteq \Omega\ with\ \mathrm{P}[\Omega_1] = 1$ such that for all $w \in \Omega_1$ the path $X(., w)$ is oscillatory. If X is not oscillatory, then it is called non-oscillatory.

3 USEFULOSCILLATORY CRITERIA AND COMPOSITE RELATIONSHIP:

In this section, we introduce a linear scalar random delay differential equation

$$Z^{1}(t) = -\sum_{I=1}^{n} P_i(t) Z(t-\delta_i) + \sum_{I=1}^{n} Q_i(t) Z(t - r_i) \qquad (7)$$

where $P_i(t),\ Q_j(t) \in C([t_0, \infty), \Re^+)$ are random positive continuous functions and $\delta_i,\ r_j \in (0, \infty)\, i = 1, 2, \ldots, n,\ \ j = 1,\ 2,\ \ldots, m$ are positive constant time lags. By a solution $Z(t)$ of Eq. (7), we mean a continuously differentiable function $Z \in C([\bar{t} - \Gamma, \infty), \Re)\ for\ some\ \bar{t},\ where\ \Gamma = \max_{1\leq i\leq m}\left\{\max_{1\leq i\leq n} \delta_i, \max_{1\leq j\leq m} r_j\right\}$, which satisfies Eq. (7).

Moreover, $P_i(t) > 0,\ Q_j(t) > 0$ are random non-negative continuous functions defined on $\Omega_1 \subset \Omega$ by

$$P_i(t, w) = \begin{cases} f_i(t) e^{-\lambda\sigma_i} e^{(-\lambda)(B(t)(w) - B(t-\sigma_i)(w))}, & for\ \ t > \underline{t} \\ -\ f_i(t) e^{-\lambda t - \mu B(t)(w)}, & for\ \ t \leq \underline{t} \end{cases} \qquad (8)$$

$$Q_J(t, w) = \begin{cases} -\ h(t) e^{-\lambda r_j} e^{(-\lambda)(B(t)(w) - B(t - r_j)(w))}, & for\ \ t > \underline{t} \\ -\ h(t) e^{-\lambda t - \mu B(t)(w)}, & for\ \ t \leq \underline{t} \end{cases} \qquad (9)$$

Where $\lambda = \left(g(t) - \frac{\mu^2}{2}\right),\ \underline{t} = \inf\ \{t > 0 : t - \sigma_i = 0,\ t - r_j = 0\}$ such that for all $t > \underline{t}$,

$t - \sigma_i \geq 0, t - r_j \geq 0\ \ and\ \ w \in \Omega$. We see that $P_i(t, w)\ and\ \ Q_j(t, w)$ rely on the increments of a standard m-dimensional Brownian motion $\{B(t)\}_{t\geq 0}$. The sufficiently large differences in such increments make it possible for the $P_i\ \ and\ \ Q_j$ to be large enough to propel oscillation in the random linear scalar DDE (7)

Moreover, due to the fact that our oscillatory results shall always be compared with the oscillatory behavior of the deterministic Equation (1), the equation can be written in pure delay terms. We now express the solution of the stochastic delay integro-differential Eq. (6) in terms of the solution of the deterministic random DDE. (7). This will be achieved through a technique, which relates the solution of the stochastic delay integro-differential equation (6), the solution of the random deterministic system (7) and a conjugation random flow similar to the type found in [17]. We next recall for application, on a path-wise sense, some of the widespread oscillatory results in the classical theory of oscillation of delay differential equations.

The result below concerning oscillation of delay differential equations appeared in [8], (Theorem 3.2):

Proposition 3.1 Assume that the following are satisfied:

$(w_1)\ \ P_i(t),\ Q_j(t) \in C([t_0,\ \infty), \Re^+),\ \delta_i,\ r_j \in (0, \infty),\ i = 1, 2, \ldots, n\ , = 1,\ 2,\ \ldots, m$

(w_2) There exists a positive number $p \leq n$ and a partition of the set $(1,\ 2,\ \ldots, m)$ into p disjoint subsets $J_1, J_2, \ldots, J_p\ \ such\ \ that\ \ each\ \ j \in J_i$ implies that $r_j \leq \sigma_i$.

$(w_3)\ P_i(t) \geq \sum_{k \in J_i} Q_k(t + r_k - \sigma_i),\ for\ \ t \geq t_0 + \sigma_i - r_k\ \ and\ \ i = 1,\ 2,\ .\ .\ .,\ \rho,\ \ \rho = \max(\sigma_k)$

$(w_4)\ \sum_{i=1}^{p} \sum_{k \in J_i} \int_{t-\sigma_i}^{t-r_k} Q_k(s)ds \leq 1,\ \ for\ \ t \geq t_0 + \sigma_i\ .\ Let\ \ x(t),\ \sum_{i=1}^{\rho} \int_t^{t+\sigma_i} \left(P_i(s) - \sum_{k \in J_i} Q_k(s + r_k - \sigma_i) \right) ds > 0$

$for\ \ t \geq t_{0,}, t_0 > 0.,\ \ Also\ \ assume\ \ that$

$(w_5)\ \underset{t \to \infty}{Limit}\ \ Sup \int_t^{t+\sigma_\Gamma} \left(P_i(s) - \sum_{k \in J_i} Q_k(s + r_k - \sigma_i) \right) ds\ > 0\ \ and$

$(w_6)\ \int_{t_0}^{\infty} \left(\sum_{i=1}^{\Gamma} \left(P_i(s) - \sum_{k \in J_i} Q_k(s + r_k - \sigma_i) \right) ds \right) In \left(e \sum_{i=1}^{\Gamma} \int_t^{t+\sigma_i} \left(P_i(s) - \sum_{k \in J_i} Q_k(s + r_k - \sigma_i) \right) ds \right) dst\ = \infty$

. Then every solution of

$$x^{|}(t) = -\sum_{I=1}^{n} P_i(t)x(t - \delta_i) + \sum_{I=1}^{n} Q_i(t)x(t - r_i) \qquad (10)$$

oscillates.

Also Qian and Ladas [1] stated a necessary and sufficient condition for oscillation of Eq.(10). As follows:

$If\ \ t \to \infty Lim\ \inf \int_{t-\Gamma}^{t} [P(s) - Q(s + r - \sigma)]ds > \frac{1}{e}\ \ when\ \ n = m = 1$, then every solution of Eq.(10) oscillates.

We also have extensive non – oscillatory results which can be recalled on a path-wise basis for use in the analysis of the effects of multiplicative noise perturbation on oscillatory behavior of delay differential systems. The following is a special case of the result found in [20], (Theorem 4):

Proposition 3.2 Assume that

(i) $P_i, Q_j \in C([t_0, \infty), \Re^+)\ and\ \ \sigma_i, r_j \in [0, \infty),\ \ for\ \ i = 1,\ .\ .\ .,\ n\ \ and\ \ j = 1,\ .\ .\ .,\ m$

(ii) $r_j > \sigma_i$

(iii) $P^* = P_i(t) - Q_j(t + \sigma_i - r_j) > 0,\ \ for\ \ t \geq t_1 = t_0 + r_j - \sigma_i\ or\ \ Q_j > 0$

If there exists a positive constant $\lambda^* > 0$ such that

$$\int_{t-r_j}^{t-\sigma_i} Q_j(s + \sigma_i)e^{\lambda^*(t-s)}ds + \int_t^{\infty} P^*(s)e^{\lambda^*(t-s+r_j)} \leq 1,\ t \geq t_1$$

then Eq.(10) has an eventually positive solution $x(t)\ on\ [t_1, \infty)$, which tends to zero exponentially and hence Eq. (8) has a non-oscillatory solution.

The main focus here is to establish that whenever $t \to t - \Gamma,\ where\ \ \Gamma = \max_{1 \leq t \leq m}\left(\max_{1 \leq t \leq m} \lambda_i, r\right)$ satisfies the conditions of Proposition 1 and $f_i(t) < 0,\ h(t) < 0$, then the solution of the SDIDE (6) is oscillatory with a very positive probability and for the choice of any initial function ϕ . Also by applying Proposition 2, the classical delay integro-differential equation (1) can admit a non-oscillatory for sufficiently small coefficient functions.

The proof of the main results depends on the transformation of the solution of the stochastic delay integro-differential equation (6) into a composite relationship using a stationary bijective random process and the continuously differentiable solution Z(t) of the scalar random delay differential Eq. (7).

To this end, we consider for each $u \in \Re^m,\ t \in \Re,\ \ and\ \ each\ \ w \in \Omega$ a random stationary coordinate change $\{(\wedge(t, .))\}_{t \in \Re}$.

As for the process $\{(\wedge(t, .))\}_{t \in \Re}$, the following properties in the Lemma are true:

Lemma 3.3

$C_1 : \{(\wedge(t, .))\}_{t \in \Re}$ is a continuous $C^{k+1,\varepsilon}$ semi-martingale with $0 \leq \varepsilon \leq \delta\ \ such\ \ that\ \ \ for\ \ all\ \ w \in \Omega,\ \Re^m \ni u \to \wedge(t, u) \in \Re^m$ is a C^{k+1} diffeomorphism of $\Re^m$.

$C_2 :\ For\ \ all\ \ t \geq s, v \in \Re^m,\ and\ \ w \in \Omega,\ \wedge(t, v) + \int_s^t \mu(du, \wedge(u, v)) + \int_s^t \Gamma(u, v)du$

$where\ \ \{\Gamma(t, .)\}_{t \in \Re}\ \ is\ \ a\ \ continuous\ \ \ C^{k.\varepsilon}\ \ semi - martingale\ \ .$

C_3 : The processes $\{\wedge(t, .)\}_{t \in \Re}$ and $\{\Gamma(t, .)\}_{t \in \Re}$ are perfectly stationary. That is,

$\wedge(t, v, w) = \wedge(0, v, \theta(t, w))\ \ and\ \ \Gamma(t, v, w) = \Gamma(0, v, \theta(t, w)),\ \ for\ \ all\ \ t \in \Re^m,\ \ w \in \Omega$

Proof: (See [17]).

The next step is to use the stationary coordinate change $\{\wedge(t,.)\}_{\in\Re}$ to establish a conjugation or composite relationship, which holds for all $t \geq 0,\ w \in \Omega$:

$$X(t,.,w) = \wedge(0,.,\ \theta(t,w)) \circ Z(t,.,w) \circ \wedge^{-1}(0,.,w) \qquad (11)$$

where, $X(t,,,w)$ is the solution of the stochastic delay integro-differential equation (6) and $Z(t,.,\ w)$ is the continuously differentiable solution of the deterministic random delay differential equation.

A method of obtaining a composite relationship between the solution of a stochastic delay differential equation and that of a random deterministic delay differential equation was employed in [13], by using a generic random process.

4. ANALYSIS OFNOISE CONTRIBUTION TO OSCILLATION OF STOCHASTIC DELAY INTEGRO-DIFFERENTIAL EQUATION:

We need the following lemmas to enable us establish the main results:

Lemma 4.1

Assume that $P_i(.)\ and\ r_i \in C([t_0,\infty),\ \Re^+),\ for\ i = 1, 2, \ldots, n$, then the differential inequality $x^{|}(t) + \sum_{i=1}^{n} P_i(t)\, x(t - r_i) \leq 0,\ t \geq 0$ has eventually positive solution if and only if the differential equation $x^{|}(t) + \sum_{i=1}^{n} P_i(t)\, x(t - r_i) = 0,\ t \geq t_0$ has an eventually positive solution.

Lemma 4.2

If $x^{|}(t) + \sum_{i=1}^{n}\left(P_i(t) - \sum_{k\in J_i} Q_k(t)(t + r_k - \sigma_i)\right) x(t - r) = 0,\ t > t_0$ has an eventually positive solution, then

$\int_t^{t+r_k}\left(P_i(t) - \sum_{k\in J_i} Q_k(t)(t + r_k - \sigma_i)\right) ds < 1\, i = 1,\ 2, \ldots, n$, eventually.

Lemma 4.3

Assume that $\underset{t\to\infty}{Lim\ Sup}\int_t^{t+r_i}\left(P_i(s) - \sum_{k\in J_i} Q_k(s)(s + r_k - \sigma_i)\right) ds > 0$ for some i and $x(t)$ is an eventually positive solution of the delay differential equation $x^{|}(t) + \sum_{i=1}^{n}\left(P_i(t) - \sum_{k\in J_i} Q_k(t)(t + r_k - \sigma_i)\right) x(t - r_i) = 0,\ t \geq t_0$, then for some i , $\underset{t\to\infty}{Limit}\ \inf \dfrac{x(t - r_i)}{x(t)} < \infty$.

Proposition 4.4

Assume that $f_i(t) > 0\ and\ h(t) > 0\ for\ t \in [t_0 - \sigma_i - r,\ t_0]$ and an initial function $\phi(t) > 0$. Then Eq.(6) can have an eventually positive solution almost surely (a. s.) on the half-line $(0,\infty)$ and as such is non-oscillatory.

Proof:

Define the probability triple (Ω, F, P) and let $\Omega_1 \subset \Omega$ be an almost sure event on which the solution Z(t) satisfies the relation in Eq.(11).

Define $(t_1, w) = \inf\ \{t \geq 0 : Z(t, w)\} = 0$. Since $Z(t) = \phi(t) > 0\ \ for\ t \in [t_0 - \sigma_i - r,\ t_0]$ and $w \in \Omega_1$. Clearly $\phi > 0,\ t_1(w) > 0$. And by definition, it must be that $Z^{|}(t_1(w), w) \leq 0$. Since $\Gamma = \max\left\{\max_{1\leq i\leq i} \sigma_i, r\right\} > 0,\ Z^{|}(t_1(w)) - Z(t_1(w), w) > 0$ as such $Z(t,w) > 0,\ for\ t < t_1(w)$. Hence, as $\phi(t) > 0,\ f_i(t) > 0\ and\ h(t) > 0$, we see that $Z^{|}(t_1(w), w) > 0$ which contradicts our earlier definition of $Z^{|}(t_1(w), w)$. Therefore, $Z(t,w) > 0$ for all $t \geq 0$. Applying the relation in Eq.(11), we see that $X(t,w) > 0,\ for\ \ all\ \ t \geq 0$, showing that X(t, w) is on a path-wise sense a. s. eventually positive and hence non – oscillatory as required.

In the following result, we establish that if the feedback coefficient functions $f_i(t),\ and\ \ h(t)$ are negative, for some initial function, then the SDIDE (6) can possess an eventually positive solution and as such non-oscillatory.

Proposition 4.5 Assume that $f_i(t) < 0,\ for\ i = 1, 2, \ldots n\ \ and\ \ h(t) < 0$ and $t \to t - \Gamma$ is non –decreasing, where $0 \leq \Gamma < \infty$. Then for the choice of any continuous initial function $\phi(t)$, the solution X(t,w) of the SDIDE (6) is a. s. oscillatory on the half-line $[0,\ \infty)$

Proof: Since the relation $X(t,.,w) = \wedge(0,.,\ \theta(t,w)) \circ Z(t,.,w) \circ \wedge^{-1}(0,.,w)$ holds and following the definition of stochastic processes, we have that

$Y = \{t \geq 0 : X(t) = 0\}$ holds true for $Sup\ Y = \infty$ almost surely if and only if $Y_1 = \{t \geq 0 : Z(t) = 0\}$ satisfies $Sup\ Y_1 = 0$, a.s. Introduce $w \in \Omega\ and\ t \geq 0$ such

$$P(t,w) = -\ f_i(t) \wedge (t - \sigma_i, w) \wedge^{-1} (t,w)\ and\ \ Q(t,w) = -\ g(t) \wedge (t - r_j, w) \wedge^{-1} (t,w)$$

Then on $[0, \infty)$, we see that $P(.)\ and\ Q(.)$ are non-negative continuous functions. Moreover, the solution Z satisfies

$$Z^{|}(t,w) = -\sum_{i=1}^{n} P_i(t,w) Z(t - \sigma_i) + \sum_{j=1}^{m} Q_j(t,w) Z(t - r_j),\ \ for\ \ t > 0 \qquad (12)$$

Now given that there exists $\Omega_1 \subset \Omega$ such that

$$\Omega_1 = \left\{ w \in \Omega : \int_{t_0}^{\infty} \sum_{i=1}^{\Gamma} \left(P_i(t) - \sum_{k \in J_i} Q_k(t)(t + r_k - \sigma_i) \right) In \left[e \sum_{i=1}^{\Gamma} \int_{t}^{tt+\sigma_i} P_i(s) - \sum_{k \in J_i} Q_k(s)(s + r_k - \sigma_i) ds \right] dt \right\} = \infty$$

$$with\ \ P[\Omega_1] = 1 \qquad (13)$$

It follows that $P(.),\ Q(.)\ and\ \ t = t - \Gamma$, satisfy the axioms of Proposition 1, for $w \in \Omega_1$, indicating that the sample path Z(., w) is oscillatory. Applying the relation in Eq.(11), we see that the solution path X(., w) of the SDIDE (6) is also oscillatory and as such Eq.(13) is an almost sure event. If not, we assume by way of contradiction that Z(t,w) has an eventually positive solution, say $y(t,.)$, which is almost surely decreasing. It follows that $y(t) = x(t) - \sum_{i=1}^{\Gamma} \sum_{k \in J_i} \int_{t-\sigma_i}^{t-r_k} Q_k(s + r_k) x(s) ds$ is an eventually positive. Moreover, since $0 \leq y(t) \leq x(t)$, $y(t)$ satisfies

$$y^{|}(t) + \sum_{i=1}^{\Gamma} \left[P_i(t) - \sum_{k \in J_i} Q_k(t + r_k - \sigma_i) \right] y(t - \sigma_i) \leq 0 \qquad (14)$$

And applying Lemma 2, we see that the delay differential equation

$$y^{|}(t) + \sum_{i=1}^{\Gamma} \left[P_i(t) - \sum_{k \in J_i} Q_k(t + r_k - \sigma_i) \right] y(t - \sigma_i) \leq 0$$

admits an eventually solution. Now put $\beta(t) = -\,y^{|}(t)/\,y(t),\ then\ \ \beta(t) > 0$ and as such there exists $t_1 \geq t_0,\ with\ \ y(t_1) > 0$ such that $y(t) = y(t_1) \exp\left(-\int_{t_1}^{t} \beta(s) ds \right)$. Hence $\beta(t)$ satisfies $\beta(t) = \sum_{i=1}^{\Gamma} P_i(t) - \sum_{k \in J_i} Q_k(t)(t + r_k - \sigma_i) \exp\left(\int_{t-\sigma_i}^{t} \beta(s) ds \right)$. Define

$$M(t) = \sum_{i=1}^{\Gamma} \int_{t}^{t+\sigma_i} P_i(s) - \sum_{k \in J_i} Q_k(s)(s + r_k - \sigma_i) ds .$$

The following generalized characteristics of the exponential function are true:

$$(i)\ e^{px} \leq pe^{x} + -\ p,\ \ p < 1$$

$$(ii)\ e^{px} \geq x + \frac{In(ep)}{p},\ \ if\ \ \ p > 0$$

$$(iii)\ e^{px}\ \geq\ qx + \frac{p}{q}\left[1 - In\ \frac{p}{q} \right],\ \ if\ \ \ p. > 0\ \ and\ \ q > 0$$

Using (ii) above, one gets

$$\beta(t) = \sum_{i=1}^{\Gamma} P_i(t) - \sum_{k \in J_i} Q_k(t)(t + r_k - \sigma_i) \exp\left(M(t).\frac{1}{M(t)} \int_{t-\sigma_i}^{t} \beta(s) ds \right)$$

$$\geq\ \sum_{i=1}^{\Gamma} P_i(t) - \sum_{k \in J_i} Q_k(t)(t + r_k - \sigma_i) \left[\frac{1}{M(t)} \int_{t-\sigma_i}^{t} \beta(s) ds + \frac{In(e\,M(t))}{M(t)} \right]$$

$$Equivalent\ \ \ ly,$$

$$\sum_{i=1}^{\Gamma} \int_{\Gamma}^{t+\sigma_i} \left(P_i(s) - \sum_{k \in J_i} Q_k(s)(s + r_k - \sigma_i) \right) ds\ \beta(t) - \sum_{i=1}^{\Gamma} P_i(t) - \sum_{k \in J_i} Q_k(t)(t + r_k - \sigma_i) \int_{t-\sigma_i}^{t} \beta(s) ds$$

$$\geq \sum_{i=1}^{\Gamma} \left(P_i(t) - \sum_{k \in J_i} Q_k(t)(t + r_k - \sigma_i) \right) \int_{t-\sigma_i}^{t} \beta(s) dsdt$$

Then for N > T, we have that

$$\int_{T}^{N} \beta(t) \sum_{i=1}^{\Gamma} \int_{t}^{t+\sigma_i} \left(P_i(s) - \sum_{k \in J_i} Q_k(s)(s + r_k - \sigma_i) ds \right) dt - \int_{T}^{N} \sum_{i=1}^{\Gamma} \left(P_i(t) - \sum_{k \in J_i} Q_k(t)(t + r_k - \sigma_i) \right) \int_{t-\sigma_i}^{t} \beta(s) dsdt$$

$$\geq \int_{T}^{N} \left(\sum_{i=1}^{\Gamma} P_i(t) - \sum_{k \in J_i} Q_k(t)(t + r_k - \sigma_i) \right)\ In \left(e \int_{t}^{t+\sigma_i} P_i(t) - \sum_{k \in J_i} Q_k(s)(s + r_k - \sigma_i) ds \right) dt \qquad (15)$$

Applying change of order in integration, we have

$$\int_T^N \sum_{i=1}^{\Gamma}\left(P_i(t)-\sum_{k\in J_i}Q_k(t)(t+r_k-\sigma_i)\right)\int_{t-\sigma_i}^{t}\beta(s)\,ds\;dt$$

$$\geq \int_T^{N-\sigma_i}\left(\int_s^{s+\sigma_i}\sum_{i=1}^{\Gamma}\left(P_i(t)-\sum_{k\in J_i}Q_k(t)(t+r_k-\sigma_i)\right)dt\right)ds$$

Hence

$$\int_T^N\left(\sum_{i=1}^{\Gamma}P_i(t)-\sum_{k\in J_i}Q_k(t)(t+r_k-\sigma_i)\right)\int_{t-\sigma_i}^{t}\beta(s)ds$$

$$\geq\int_T^{N-\sigma_i}\beta(s)\left(\int_s^{s+\sigma_i}\sum_{i=1}^{\Gamma}\left(P_i(t)-\sum_{k\in J_i}Q_k(t)(t+r_k-\sigma_i)\right)dt\right)ds \tag{16}$$

As such,

$$\int_T^N\left(\sum_{i=1}^{\Gamma}P_i(t)-\sum_{k\in J_i}Q_k(t)(t+r_k-\sigma_i)\right)\int_{t-\sigma_i}^{t}\beta(s)dsdt$$

$$\geq\sum_{i=1}^{\Gamma}\int_T^{N-\sigma_i}\beta(t)\left(\int_t^{t+\sigma_i}P_i(t)-\sum_{k\in J_i}Q_k(t)(t+r_k-\sigma_i)ds\right)dt \tag{17}$$

Using (16) and (17), one gets

$$\int_T^N\beta(t)\left(\sum_{i=1}^{\Gamma}\int_t^{t+\sigma_i}\left(P_i(s)-\sum_{k\in J_i}Q_k(s)(s+r_k-\sigma_i)\right)ds\right)dt$$

$$-\int_T^N\beta(t)\int_t^{tt+\sigma_i}\sum_{i=1}^{\Gamma}\left(P_i(t)-\sum_{k\in J_i}Q_k(t)(t+r_k-\sigma_i)\right)dsdt$$

$$\geq\int_T^N\sum_{i=1}^{\Gamma}\left(P_i(t)-\sum_{k\in J_i}Q_k(t)(t+r_k-\sigma_i)\right)\left(Ine\sum_{i=1}^{\Gamma}\int_t^{t+\sigma_i}\left(P_i(t)-\sum_{k\in J_i}Q_k(t)(t+r_k-\sigma_i)\right)ds\right)dt \tag{18}$$

As such

$$\sum_{i=1}^{\Gamma}\int_{N-\sigma_I}^{N}\beta(t)\left(\int_t^{tt+\sigma_i}\left(P_i(s)-\sum_{k\in J_i}Q_k(s)(s+r_k-\sigma_i)\right)ds\right)dt$$

$$\geq\int_T^N\sum_{i=1}^{\Gamma}\left(P_i(t)-\sum_{k\in J_i}Q_k(t)(t+r_k-\sigma_i)\right)\left(In\;\;e\int_t^{t+\sigma_i}\sum_{i=1}^{\Gamma}\left(P_i(t)-\sum_{k\in J_i}Q_k(t)(t+r_k-\sigma_i)\right)ds\right)dt \tag{19}$$

Applying Lemma 3, one gets

$$\int_t^{t+r_k}\left(P_i(t)-\sum_{k\in J_i}Q_k(t)(t+r_k-\sigma_i)\right)ds<1,\, i=1,2,..,,\Gamma, \tag{20}$$

eventually. Using (18) and (19), we have

$$\sum_{i=1}^{\Gamma}\int_{N-\sigma_i}^{N}\beta(t)dt$$

$$\geq\int_T^{tN}\left(\sum_{I=1}^{\Gamma}P_i(t)-\sum_{k\in J_i}Q_k(t)(t+r_k-\sigma_i)\right)In\left(e\int_t^{t+\sigma_i}\sum_{i=1}^{\Gamma}P_i(t)-\sum_{k\in J_i}Q_k(t)(t+r_k-\sigma_i)ds\right)dt$$

Equivalent ly,

$$\sum_{i=1}^{\Gamma}In\,\frac{y(N-\sigma_I)}{y(N)}$$

$$\geq\int_T^N\sum_{i=1}^{\Gamma}\left(P_i(t)-\sum_{k\in J_i}Q_k(t)(t+r_k-\sigma_i)\right)In\left(e\int_t^{t+\sigma_i}\sum_{i=1}^{\Gamma}\left(P_i(t)-\sum_{k\in J_i}Q_k(t)(t+r_k-\sigma_i)\right)ds\right)dt \tag{21}$$

Applying (w_6) in Proposition 1, we have

$$\underset{t\to\infty}{Limit}\prod_{i=1}^{\Gamma}\frac{y(t-\sigma_i)}{y(t)}=\infty \tag{22}$$

which is equivalent to

$$\underset{t\to\infty}{Limit}\frac{y(t-\sigma_i)}{y(t)}=\infty \tag{23}$$

Applying Lemma 4, it follows that

$\underset{t\to\infty}{Limit}\inf\frac{y(t-\sigma_\Gamma)}{y(t)}<\infty$ which contradicts Eq.(23). As such of Eq.(7) is oscillatory.

Now from the dependence of P(.) and Q(.) on the increments of Brownian noise as in Eq.(8) and (9), one gets that

$$\int_t^{t+\sigma_i} P_i(s)ds = \int_t^{t+\sigma_i} - f_i(s)\exp\left(-g(u,s)-\frac{\eta^2}{2}\right)\sigma_i \exp\left(-\eta\left(B(s)-B(s-r_i)\right)\right)ds$$

$$\geq - f_i(s)\max\left(1, \exp\left(-\left(g(u,s)-\frac{\eta^2}{2}\right)\sigma_i\right)\right)\int_{t-\sigma_i}^t \exp\left(-\eta\left(B(s)-B(s-\sigma_i)\right)\right)ds$$

and

$$\int_{t-R}^t Q_J(s)ds = \int_t^{t+\sigma_i} - h(s)\exp\left(-g(u,s)-\frac{\eta^2}{2}\right)r_j \exp\left(-\eta\left(B(s)-B(s-r_i)\right)\right)ds$$

$$\geq - h(s)\max\left(1, \exp\left(-\left(g(s)-\frac{\eta^2}{2}\right)r_j\right)\right)\int_{t-\sigma_i}^t \exp\left(-\eta\left(B(s)-B(s-r_j)\right)\right)ds$$

The sure event Ω_1 as in Eq.(13) exists almost surely, whenever

$$\underset{t\to\infty}{Limit} \int_{t-\sigma_i}^t \exp\left(-\eta\left(B(s)-B(s-\Gamma)\right)\right)ds = \infty. \qquad (24)$$

The proof is complete.

In Proposition 4, the main propelling force which sustains oscillation in the SDID (6) is (24), which makes Eq. (6) satisfy the conditions of Proposition 1.In contrast, in the deterministic delay integro-differential DIDE. (1), where the noise is absent, $\underset{t\to\infty}{Limit} \int_{t-\sigma_i}^t \exp\left(-\eta\left(B(s)-B(s-\Gamma)\right)\right)ds = \Gamma$ and at that point the conditions in Proposition 2 are satisfied, thus making Eq. (1), which is without noise, possess a non-oscillatory solution. The presence of noise is actually responsible for sustaining oscillation in the stochastic system.

REFERENCES:

[1] Qian, X. and Ladas, G. (1990). Oscillation in Differential Equations with Positive and Negative Coefficients. Canadian Math. Bulletin., 33, 442 – 451.

[2] Gyori, I and Ladas, G. (1991). Oscillation Theory of Delay Differential Equations with Applications. Clarendon Press, Oxford.

[3] Ladde, G. S., Lakshmikkantham, V. and Zhang, B. G. (1987). Oscillation Theory of Differential Equations with Deviating Arguments. Marcel Dekker, Inc. New York.

[4] Agwo, H. A. (1999). On the Oscillation of Delay Differential Equations with Real Coefficients. Int. Jour. Math. & Math. Sci. Vol. 22, No. 3, 573 -578.

[5] Ahmad, F. (2003). Linear Delay Differential Equation with a Positive and a Negative Term. Electronic Jour. of Differential Equations, Vol. 2003, No. 92, 1 – 6.

[6] Li, B. (1996). Oscillation of first order Delay Differential Equations. Proceedings of the American Mathematical Society, Vol. 124, No. 12, 3729 – 3737.

[7] Wang, Z; Stavronlaski, I. P. and Qian, X. (2002). A survey on the Oscillation of Solutions of First Order Linear Differential Equations with Deviating Arguments. Applied Mathematics E-Notes, Vol. 2, 171 - 191.

[8] Elabbasy, E. M; Hegazi, A. S. and Saker, S. H. (2000). Electronic Journal of Differential Equations, Vol. 2000, 1 – 13.

[9] Gopalsamy, K and Weng, P. (2004). Oscillation in Linear Neutral Differential Systems with several delays. Bulletin of the Institute of Mathematics, Academia Sinica, Vol. 22, 195 – 205.

[10] Gopalsamy, K. (1992). Stability and Oscillations in Delay Differential Equations of Population Dynamics. Kluwer Academic Publishers, Dordrecht, Boston, London.

[11] Gopalsamy, K. (1978). Oscillatory Properties of Systems of first Order Delay Differential Inequalities. Pacific Jour. of Mathematics, Vol. 128, No. 2, 299 – 305.

[12] Mackean (Jr.), H. P. (1969). Stochastic Integrals. Academic Press, New York.

[13] Appleby, J. A. D. and Buckwar, E.(2005). Noise Induced Oscillation in Solutions of Stochastic Delay Differential Equations. Dynamical Systems and Application Vol. 14, No. 2, 175 – 196.

[14] Appleby, J. A. D. and Kelly, C. (2004). Oscillation and Non-Oscillation in Solutions of Nonlinear Stochastic Delay Differential Equations. Electronic Communications in Probability, Vol. 9, 106 – 118.

[15] Atonuje, A. O. and Ayoola, E. O. (2007). On Noise Contribution to the Oscillatory Behaviour of Stochastic Delay Differential Equations. Jour. Inst. Maths& Comp. Sci. (Computer Science Series), Vol. 18, No. 2, 51 – 59.

[16] Atonuje, A. O. (2015). Issues in the Influence of Ito – type Noise on the Oscillation of Solutions of Delay Differential Pantograph Equations. Journal of Mathematics and System Science, Vol. 5, No. 11, 480 – 487.

[17] Lisei, H. (2001). Conjugation of Flows for Stochastic and Random Functional Differential Equations. Stochastic and Dynamics, Vol. 1, No.2, 283 – 298.

[18] Wiener, N.(1923). Differential space. Journal of Mathematics and Physics, Vol. 2, 132 – 174.

[19] Wiener, N.(1930).Generalized Harmonic Analysis. Acta Math; Vol. 55, 117 – 258.

[20] Yu, J. and Wang, Z. (1994). Oscillation and Existence of Positive solutions in Differential Equations. Soochow Journal of Mathematics, Vol. 20, No.2, 193 – 204.