Journal of the Nigerian Association of Mathematical Physics
Volume 8 (November 2004)

The Construction of an automorphism with a continuous spectrum and no square root

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Abstract

An automorphism S is called a square root of an automorphism T if $S^2 = T$. The Problem of describing the square root of a given automorphism T is completely solvable only when T has a discrete [1], [2] ***or guasi-discrete spectrum. Katok and Stepin*** [3] ***gave a general construction of an automorphism with a continuous spectrum but no square root. In this work, we construct a particular example of this kind of automorphisms using the following result.***

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1.0 Introduction

With the successful construction of the general form of automorphism with a continuous spectrum and no square root, Katok and Stepin has opened the way for Researchers to explore the results obtained, in constructing typical examples of such automorphisms.

The Automorphism we shall construct in this work with the properties stated above shall be a typical example of what was described by Katok and Stepin. It is important to Note that the following theorems, lemma and definitions will aid our work greatly.

Theorem 1

If an automorphism T *admits a cyclic a.p.t with speed* $\frac{1}{n}$ *and the function* w(x) *is such that the set* $W^{-1}(-1)$ *is oddly approximated with respect to* ε_n *with speed* $\left(\frac{1}{n}\right)$*, the spectrum of the operator* U_T *is simple [3].*

Theorem 2

If an automorphism T *admits a cyclic a.p.t with speed* $\left(\frac{1}{n}\right)$*, the operator* U_T *has a continuous spectrum and the function w(*x*) is such that the sets* $W^{-1}(-1)$ *and* $W^{-1}(1)$ *are oddly approximated with respect to* ε_n *with speed* $\left(\frac{1}{n}\right)$*, then the spectrum of the operator* U_T^1 *is continuous [3].*

Lemma 1

If an automorphism T *with a simple spectrum admits a* Z_2 *fibring J, then* $\sqrt{T}$ *admits J if* $\sqrt{T}$ *exists.*

Definition

1. We say that the ordered pair of numbers (A, B) in [0, 1] satisfies condition C if A is irrational and there exist a sequence of irreducible fractions $\frac{P_n}{q_n}$ such that

C. 1: $$\left|\frac{P_n}{q_n} - A\right| = \circ\left(\frac{1}{q_n^2}\right) \qquad (1.1)$$

C. 2: There exists $r > 0$ such that for all integers t,

$$\left|\frac{t}{q_n} - B\right| > \frac{r}{q_n} \text{ holds} \qquad (1.2)$$

Suppose that for the pair of numbers

$$A = \frac{1-\alpha}{1+\beta-\alpha} \quad , \qquad B = \frac{\beta-\alpha}{1+\beta-\alpha} \tag{1.3}$$

The sequences of rational numbers $\frac{P_n}{q_n}$ and $\frac{y_n}{q_n}$ exists satisfying the conditions

B.1: P_n and q_n are relatively prime and $q_n \to 0$

B.2: $$\left|\frac{P_n}{q_n} - A\right| = \circ\left(\frac{1}{q_n^2}\right) \tag{1.4}$$

B.3: $$\left|\frac{y_n}{q_n} - B\right| = \circ\left(\frac{1}{q_n}\right) \tag{1.5}$$

Then the automorphism $P_{\alpha,\beta}$ admits a cyclic a.p.t with speed $\circ\left(n^{-1}\right)$. Now our construction begins, let (M,μ) be the direct product of [0, 1] with Lebesque measure and the two-point space $Z_2 = \{1,-1\}$ with measure $\left(\frac{1}{2},\frac{1}{2}\right)$. We consider the automorphism S of M that is a fibre bundle with base $P_{\alpha,\beta}$ defined as

$$P_{\alpha,\beta}(x) = \begin{cases} x+1-\alpha & for\ \ 0 \le x < \alpha \\ x+1-\alpha-\beta & for\ \ \alpha \le x < \beta \\ x-\beta & for\ \beta\ \le x \le 1 \end{cases} \tag{1.6}$$

and function

$$w(x) = \begin{cases} -1 & for\ \ 0 \le x < (\beta-\alpha+1)^{-1} \\ 1 & for\ \dfrac{1}{1+\beta-\alpha}\ \le x < 1 \end{cases} \tag{1.7}$$

for $\alpha = \frac{1}{2\sqrt{2}}$ and $\beta = \frac{2}{3}$. Now let S be a fibre bundle with base $P_{\frac{1}{2\sqrt{2}},\frac{2}{3}}(x)$ define as

$$P_{\frac{1}{2\sqrt{2}},\frac{2}{3}}(x) = \begin{cases} x+1-\dfrac{1}{2\sqrt{2}} & \text{for } 0 \le x < \dfrac{1}{2\sqrt{2}} \\ x+1-\dfrac{1}{2\sqrt{2}}-\dfrac{2}{3} & \text{for } \dfrac{1}{2\sqrt{2}} \le x < \dfrac{2}{3} \\ x-\dfrac{2}{3} & \text{for} \le x < 1 \end{cases} \tag{1.8}$$

and function

$$w(x) = \begin{cases} -1 & \text{for}\ \ 0 \le x < (\beta-\alpha+1)^{-1} \\ 1 & \text{for}\ \dfrac{1}{1+\beta-\alpha}\ \le x < 1 \end{cases} \tag{1.9}$$

operating in the space M. Note that since the function $w(x)$ has values ± 1 and since ε_n is the decomposition of [0, 1] into intervals

$$C_{n,i} = \left[\frac{i-1}{q_n},\frac{i}{q_n}\right] \text{ for } i = 1, 2, ---, q_n \tag{1.10}$$

(where $q^n = 10^n + 1$), the set $W^{-1}(1)$ and $W^{-1}(-1)$ are oddly approximated with speed $\circ\left(n^{-1}\right)$ with respect to the sequence of decomposition ε_n, From theorem 1 together with definitions B1 $\longrightarrow$ B3 the automorphism $P_{\frac{1}{2\sqrt{2}},\frac{2}{3}}(x)$ has a simple spectrum. This is true because if $A = \dfrac{1-\alpha}{1+\beta-\alpha}$, where $\alpha = \dfrac{1}{2\sqrt{2}}$ and $\beta = \dfrac{2}{3}$ then $A = \dfrac{222-24\sqrt{2}}{382}$ and $A^* = 0.492300706$. Also $B = \dfrac{\beta-\alpha}{1+\beta-\alpha} = \dfrac{142-36\sqrt{2}}{382} \Rightarrow B^* = 0.238451047$. Now using

$$\left|\frac{P_n}{q_n} - A\right| \le \frac{1}{q_n^2} \qquad (1.11)$$

With a fixed $q_n = 10^n$ and since between any two real numbers there is a rational number, we can choose positive rationals $P_1, P_2, \ldots, P_n$ such that

$$\begin{aligned} &\left|\frac{P_1}{10} - A\right| \le \frac{1}{10^2} \\ &\left|\frac{P_2}{10^2} - A\right| \le \frac{1}{10^4} \\ &\qquad\vdots \qquad\quad \vdots \\ &\left|\frac{P_n}{10^n} - A\right| \le \frac{1}{10^{2n}} \\ &\qquad\vdots \qquad\quad \vdots \end{aligned}$$

For $n = 1, 2, \ldots$ condition C1 holds: Also since $B^* = 0.238451047$, condition C2 holds with $r = 1$. That is if $r = 1 > 0$ then for all integers t

$$\left|\frac{t}{q_n} - B\right| > \frac{r}{q_n} \qquad (1.12)$$

holds. For conditions B1 $\longrightarrow$ B3 observe that B1 holds since we can choose the P_n's such that $q_n = 10^n + 1$ and P_n are relatively prime with $q_n \longrightarrow \infty \quad as \quad n \longrightarrow \infty$. For B2, we see from the above that

$$\left|\frac{P_n}{q_n} - A\right| = \circ\left(\frac{1}{q_n^2}\right) \qquad (1.13)$$

$$q_n^2\left|\frac{P_n}{q_n} - A\right| \longrightarrow \infty \quad as \quad n \longrightarrow \infty \qquad (1.14)$$

Similarly, B3 holds, since

$$\left|\frac{t_n}{q_n} - B\right| = \circ\left(\frac{1}{q_n^2}\right) \qquad (1.15)$$

$$q_n^2\left|\frac{t_n}{q_n} - B\right| \longrightarrow \infty \; as \; n \longrightarrow \infty \qquad (1.16)$$

Now, since conditions C1 and C2 hold, the automorphism $P_{\frac{1}{2\sqrt{2}},\frac{2}{3}}(x)$ has a simple spectrum and by Theorem 1, together with B1 $\longrightarrow$ B3 the automorphism $P_{\frac{1}{2\sqrt{2}},\frac{2}{3}}(x)$ admits a cyclic a.p.t. with speed $\circ\left(n^{-1}\right)$ Now consider the fibre bundle U with base S and function $w(y) = w(x, j) = j$ with $y = (x, j)t\ m, \quad j = \pm 1$. Again the sets $W^{-1}(1)$ and $W^{-1}(-1)$ are oddly approximate with respect to the sequence of decomposition ε_n of the space M with space $\circ\left(n^{-1}\right)$. By theorems (1) and (2) the automorphism has a continuous spectrum. Also Lemma (1), the square root of U is a fibre bundle with base $\sqrt{S}$ (if it exists) and function F(y) where F(y) must satisfy

$$F(\sqrt{S}\, y)\, F(y) = w(y) \qquad (1.17)$$

Equation (1.17) was obtained from the definition of a fibre bundle T^1 with base T and function $T^1(x, j) = (T_x, w(x)j) x \varepsilon m, j \varepsilon Z_\alpha$ Now $U(y,i) = (Sy, w(y)i), y = (x, j), i \in Z_\alpha$ (1.18)

$$\sqrt{U}(V,L) = \left(\sqrt{S}(V)\right) F(V)L \qquad (1.19)$$

where $S(V,L) = P_{\frac{1}{2}\sqrt{2},\frac{2}{3}}(V), w(V)n, n \in Z_\alpha$ Put V = $\sqrt{S}$ y in (1.19) then we have

$$\sqrt{U}\left(\sqrt{S}y, L\right) = \left(\sqrt{S}\ \ \sqrt{S}(y)\right), F\left(\sqrt{S}(y)\right) L(y) \ = \left(S(y), F\sqrt{S}(y)\ L(y)\right) \qquad (1.20)$$

If L(y) = I F(y) then $|L(y)| = |iF(y)| = |F(y)||i| = 1$ using the fact that $(a_1, b_1) = (a_2, b_2)$ iff $a_1 = a_2$ and $b_1 = b_2$ and comparing (1.18) and (1.20) we have that $F\left(\sqrt{S}(y)\right)F(y) = w(y)$ (1.21)

If we put $J(x, j) = (x, -j)$. Then $w(j\ y) = -w(y)$ which from (1.21) gives

$$F\left(\sqrt{S}(J\ y)\right) F(J\ y) = -w(y) \qquad (1.22)$$

Multiply (1.21) and (1.22) $F(y)\,F(j\ y)\,F\left(\sqrt{S}(y)\right) F\left(\sqrt{S}(j\ y)\right) = -1$ (1.23)

We put $F(y)\,F(j_y) = \theta(y)$ since j is a Z_2 fibering of S and S has a simple spectrum by Lemma (1) the automorphism, j and $\sqrt{S}$ commute (i.e. $j \circ \sqrt{S} = \sqrt{S} \circ j$) and therefore $F\left(\sqrt{S}(y)\right) F\left(\sqrt{S}(j_y)\right) = \theta\left(\sqrt{S}(y)\right)$ from (1.23). We obtain $\theta(y)\,\theta\ \left(\sqrt{S}(y)\right) = -1$ (1.24)

Since $\theta\ (y)$ takes values ± 1, it follows from (1.24) that $\theta\ (y)$ is not a constant. Now $|\ \theta(y)\ | = 1$ for all y, then from (1.24) $\theta\ \left(\sqrt{S}(\,y\,)\right)\ \theta\ \left(\sqrt{S}(\,y\,)\right) = -1$ (1.25)

i.e $\theta\left(\sqrt{S}(\,y\,)\right)\ \theta\left(S(\,y\,)\right) = -1$ (1.26)

Since $\theta\left(\sqrt{S}(y)\right)^2 = 1$, because θ takes only the values 1 and – 1, we have by (1.26) that

$$\theta\left(\sqrt{S}(\,y\,)\right)\ \theta\left(S(\,y\,)\right)\ = -\left(\ \theta\sqrt{S}(\,y\,)\right)^2 \qquad (1.27)$$

Hence $\theta\left(S(\,y\,)\right) = -\theta\left(\sqrt{S}(\,y\,)\right)$ or (1.28)

$$\theta\left(S(y) = \theta\left(\sqrt{S}(y)\right)\right) \qquad (1.29)$$

the latter i.e (1.29) cannot hold because of (1.26), so $\theta\left(S(\,y\,)\right) = -\theta\left(\sqrt{S}(\,y\,)\right)$ (1.30)

Also $-\theta(y)\,\theta\left(\sqrt{S}(y)\right) = 1$ by equation (1.24). So, $-\theta(\,y\,)\theta\left(\sqrt{S}(\,y\,)\right) = \left(\,\theta(\,y\,)\right)^2$ (1.31)

because θ take values ± 1, hence $-\theta\left(\sqrt{S}(y)\right) = \theta(y)$ or $\theta\left(\sqrt{S}(y)\right) = \theta(y)$. Again the latter cannot hold because by (1.24) $\theta(y)$ and $\theta\left(\sqrt{S}(y)\right)$ always have different signs. So $-\ \theta\left(\sqrt{S}(\,y\,)\right) = \theta(\,y\,)$ (1.32)

Thus , $\theta\left(S(\,y\,)\right) = -\ \theta\left(\sqrt{S}(\,y\,)\right)\ = \theta(\,y\,)$ (1.33)

Which contradicts the fact that S is ergodic, because an automorphism on a probability space is weak mixing if and only if it has a continuous spectrum and because a weak mixing automorphism is ergodic.

2.0 Conclusion

Since $\sqrt{U}$ does not exists, $\sqrt{S}$ which is the base of $\sqrt{U}$ does not exists and $P_{\frac{1}{2\sqrt{2}},\frac{2}{3}}$ which is the base of $\sqrt{S}$ does not exists. Therefore $P_{\frac{1}{2\sqrt{2}},\frac{2}{3}}$ with a continuous spectrum does not have a square root.

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