Journal of the Nigerian Association of Mathematical Physics
Volume 8 *(*November 2004*)*

On the existence of weak solutions of quantum stochastic differential equations

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Abstract

We establish further results concerning the existence, uniqueness and stability of weak solutions of quantum stochastic differential equations (QSDEs). Our results are achieved by considering a more general Lipschit condition on the coefficients than our previous considerations in [1]. *We exhibit a class of Lipschitzian QSDEs in the formulation of this paper, whose coefficients are only continuous on the locally convex space of the weak solution.*

Key words QSDEs, Fock space Exponential vectors, Lipschitzian, Quantum stochastic processes

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1.0 Introduction

We investigate the existence, uniqueness and stability of weak solutions of QSDEs in integral form given by

$$X(t)=X_o+\int_o^t(E(s,X(s))d\wedge_\pi(s)+F(s,X(s)dA_f(s)+G(s,X(s))dA_g^+(s)$$
$$+H(s,X(s))ds),t\in[0,T]. \qquad (1.1)$$

This paper continues our previous work in [1] in the framework of the Hudsom-Parthasarathy [2] formulation of quantum stochastic calculus. We consider a more general class of Lipschitzian coefficients E, F, G, H. Our approach in this paper is the weak formulation of the method employed in [3]. As in [3], we are able to exhibit a wider class of Lipschitzian QSDE (1.1) whose coefficients are only continuous on the space of our quantum stochastic processes. Our previous works [1,4,5,6] have focused on some qualitative aspects and approximations of the weak solutions of [1.1].

We employ the notations and structures introduced in [1, 5]. We refer the reader to the references for the details of the various spaces employed in this paper. Our results in this paper enables us to accommodate a wider class of equation [1.1] in our approximation theory, satisfying the Lipschitz condition with respect to the weak topology employed in the references [1, 4, 5, 6, 7]. In addition, we refer the reader to [2, 8, 9.] for some interesting accounts of the Hudson–Parthasarathy quantum stochastic calculus. The first and second fundamental theorems of Hudson and Parthasarathy [2] induced the weak and strong topologies employed respectively here and in [3]. For any pair of $\eta,\xi\in ID\underline{\otimes} IE$ such that $\eta, = c\otimes e(\alpha),\xi = d\otimes e(\beta),\alpha,\beta\in L_\gamma^2(\mathrm{IR}_+)$, we shall in what follows, employ the equivalent form of (1.1) given by

$$\frac{d}{dt}<\eta, X(t)\,\xi> = P(t; X(t)\,(\eta,\xi), \qquad X(0) = X_0,\ t\in[0,T]. \qquad (1.2)$$

The explicit form of the sesquilinear form valued map P is of the form $P(t,x)(\eta,\xi) = <\eta,\ p_{\alpha\beta}(t,x)\,\xi>$where the map $P\eta p_{\alpha\beta}:[0,T]\times\tilde{A}\to\tilde{A}$ is given by $P_{\alpha\beta}(t,x)=\mu_{\alpha\beta}(t)E(t,x)+\gamma\beta(t)F(t,x)+\sigma_\alpha(t)G(t,x)+H(t,x)$ for all $(t,x)\in[0,\mathrm{T}]\times\tilde{A}$ (*see* [1,5,7] for details.)

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The rest of the paper is organized as follows: Section 2 is devoted to some fundamental results and assumptions. Our main results concerning the existence, uniqueness and stability of QSDE (1.1) are established in Section 3.

2.0 Preliminary results and assumptions

As explained in [1,4,5,6,7], in this framework, quantum stochastic processes are $\tilde{A}$ - valued maps on [0.7]; the space $\tilde{A}$ is a locally convex space whose topology is generated by a family of semi-norms

$\| x \|_{\eta\xi} = |<\eta, x\xi>|$, $\eta, \xi \in ID \underline{\otimes} IE$. Here, $x \in A = L_w^+(ID \underline{\otimes}$ E, $IR \otimes \Gamma(L_\gamma^2(IR_+)))$.

***Definition* 2.1**

(a) *Let $F\ (ID \underline{\otimes} IE)^2$. denotes the set of all finite subsets of $(ID \underline{\otimes} IE)^2$.* If $x \in \tilde{A}$ *and* $\Theta \in F$ *in* $\left(ID \underline{\otimes} IE\right)^2$, *define* $\|x\|\Theta$ *by* $\|x\|\Theta = \max_{(\eta,\xi)\in\Theta} \|x\|_{\eta,\xi}$. *Then, the set* $\{\|\cdot\|_\Theta : \Theta \in Fin(ID \underline{\otimes} IE)^2\}$ *is a family of semi-norms on* $\tilde{A}$. *We denote by* τ *the topology generated by this family of seminorms on* $\tilde{A}$.

(b) *Let I = [0,T]* $\subseteq IR_+$.. *A map* $\Phi : I \times \tilde{A} \to \tilde{A}$ *will be called Lipschitzian if for each pair* (η, ξ) $\in (ID \underline{\otimes} IE)^2$, *the map satisfies an estimate of the type*

$$\| \Phi(t,x) - \Phi(t,y) \|_{\eta\ \xi} \le K^{\Phi}_{\eta\xi}(t) \| x - y \|_{\Theta\Phi(\eta,\xi)}. \qquad (2.1)$$

for all $x, y \in \tilde{A}$ *and almost all* $t \in I$ *and where* $K^{\Phi}_{\eta,\xi} : I \to (0,\infty)$ *lying in* $L^1_{l0c}(I)$ *and* Θ_Φ *is a map from* $(ID \underline{\otimes} IE)^2$ *into F in* $(ID \underline{\otimes} IE)^2$.

Remark

Let L $(\tilde{A})$ *denote the linear space of all continuous endomorphisms of* $\tilde{A}$. *Then the above definition enables us to exhibit a class of Lipschitzian maps as follows.*

Theorem 2.2

Let A: $IR_+ \mapsto L(\tilde{A})L$ *and* $F : IR_+ \times \tilde{A} \mapsto \tilde{A}$. *be given by F{t, x) = A (t)x, for* $x \in \tilde{A}, t \in IR_+$, *then F is Lipschitzian.*

Proof

Let $x, y \in \tilde{A}, t \in R_+$, *then*

$$\| F(t,x) - F(t,y) \|\eta\xi = \|A(t)\ x - A(t)y\|\eta\xi = \| A(t)(x - y) \|\eta\xi \le C^A_{\eta\hat{\imath}}(t) \| x - y \|_{\Theta A}\ (\eta\ \xi),$$

where $C^A_{\eta\hat{\imath}}(t)$ is a positive function depending on *A, t*, ξ, and Θ_A .is a map from $(ID \underline{\otimes} IE)^2$ into *F in* $(ID \underline{\otimes} IE)^2$.

Remark

(a) *Theorem 2.2 demonstrates that all continuous linear maps of* $\tilde{A}$ *into it self are automatically Lipschitzian in the sense of this paper.*

(b) *Since* Θ *is a finite set, we see that* $\| x \|_\Theta = \| x \|_{\eta'\ \xi'}$ *for some* $(\eta'\ \xi') \in \Theta$. *Using the foregoing fact, we employ in the proof our main results below the fact that a map* $\Phi : I \times \tilde{A} \to \tilde{A}$ *is Lipschitzian if given any* $(\eta\ \xi')$ $\in (ID \underline{\otimes} IE)^2$, *there corresponds* $(\eta', \xi') \in (ID \underline{\otimes} IE)^2$ *such that* $\| \Phi(t,x) - \Phi(t,y) \|_{\eta\xi} \le K^{\Phi}_{\eta\ \xi}(t) \| x - y \|_{\eta'\ \xi'}$ *for all* $x, y \in \tilde{A}$ *and almost all* $t \in I$.

(c) *By employing the definition of the map P(t, x)(η ξ) that appears in (1.2), and by the remark above, it is easy to show that if the coefficients of (1.1) are Lipschitzian, then the map P(t, x)(η ξ') is also Lipschitzian. That is, there exits* $(\eta', \xi) \in (ID \underline{\otimes} IE)^2 I$ *such that for all* $x, y \in \tilde{A}$, $|P(t,x)(\xi) - P(t,y)(\eta,\xi)| \le K^p_{n\ \xi}(t) \|x - y\| \eta'_{\xi'}$, *where the map* $K^P_{\eta\xi} : [0,T] \to IR+$ *lies in* $L^1[0,T]$.

(d) *Using the definition in (b), we see that if P:* $IR_+ \mapsto \tilde{A}$ *and* $(\eta_0, \xi_0) \in (ID \underline{\otimes} IE)^2$ *is a fixed point, then the map F defined by* $F(t,x) = <\eta_0, x^{\xi}{}_0> P(t)$ *is Lipschitzian, since for any* $t \in IR_+, x, y, \in \tilde{A}$,

$$\|F(t,x) - F(t,y)\|_{\eta\xi} \le \|P(t)\|_{\eta\xi} \|x - y\|_{\eta_0\xi_0}$$

In what follows, we shall frequently employ the first fundamental theorem of Hudson and Parthasarathy (see [1,2,5]).

3.0 Existence, uniqueness and stability of solution

The main results of this paper are established in this section. We recall here that by a solution of equation (1.1), we mean a weakly absolutely continuous stochastic process $\phi \in L^2_{loc}(\widetilde{A})$ satisfying equation (1.1). First, we present the following notations.

***Definition* 3.1.**

Let $K^p_{\eta\xi}(t)$ be the Lipschitz functions appearing in (1.3.). Then, we define the following constants

$$K_{\eta j\xi j} = ess\ sup\, K^p_{\eta j\xi j}(s),\quad j = 0,1,\cdots 1,\quad (\eta_0,\xi_0) = (\eta,\xi)$$
$$L_{\eta\xi,n} = max\{K_{\eta j\xi j},\ j = 0,1.2\cdots,n-1\},\ L_{\eta\xi} = \sup_{n\in N}\{L_{\eta\xi,n}\}$$

***Theorem* 3.2**

Suppose that the coefficients E, F, G, H appearing in equation (1.1) satisfy estimate of the form (2.1) and belong to $L^2_{loc}(I\times\widetilde{A})$. Then for any fixed point X_0 of $\widetilde{A}$ there exists a unique adapted and weakly absolutely continuous solution Φ of quantum stochastic differential equation (1.1) satisfying $\Phi(0) = X_0$.

Proof

As in [1], we will establish the theorem by constructing a Cauchy sequence $\{\Phi_n\}_n \geq 0$ of successive approximations of Φ in $\widetilde{A}$. In what follows, let $\eta,\xi \in ID \underline{\otimes} IE$ be arbitrary. Let $t \in [0,T]$ and define $\Phi_0(t) = X_0$, and for $n \geq 0$.

$$\Phi_{n+1}(t) = X_0 + \int_0^t \left(E(s,\Phi_n(s))d\Lambda_\pi(s) + F\ (s,\Phi_n(s))dA_f(s) + G\ (s,\Phi_n(s))dA^+_g(s) + H\ (s,\Phi_n(s))ds\right).$$

We have established in [1, p. 53] that each $\Phi_n(t), n \geq 1$ defines an adapted weakly absolutely continuous process in $L^2_{loc}(\widetilde{A})$. Next, we consider the convergence of the successive approximations. We have by using the definition of the map P that appears in (1.2)

$$\|\Phi_{n+1}(t) - \Phi_n(t)\|_{\eta\xi} = 1 < \eta, (\Phi_{n+1}(t) - \Phi_n(t))\xi > | = |\int_0^t (P(s,\Phi_n(s))(\eta,\xi) - P(s,\Phi_{n-1}(s))(\eta,\xi))ds|.$$

Since the coefficients E, F, G, H are Lipschitzian, then P is also Lipschitzian; in the sense that corresponding to any pair of elements $\eta,\xi \in ID \underline{\otimes} IE$, there exists another pair $\eta_1,\xi_1 \in ID \underline{\otimes} IE$ such that

$$|P(t,x)(\eta,\xi) - P(t,y)(\eta,\xi)| \leq K^p_{\eta\xi}(t)\,\|x-y\|_{\eta_1\xi_1 1\xi 1,}\ \forall x,y \in \widetilde{A},\ t \in [0,T],$$

with Lipschitz functions $K^p_{\eta\xi}:[0,T] \to (0,\infty)$ lying in $L'_{loc}([0,T])$.

Hence, $\quad \Phi_{n+1}(t) - \Phi_n(t)\|_{\eta\xi} \leq \int_0^t K^p_{\eta\xi}(s')\,\|\Phi_n(s') - \Phi_{n-1}(s')\|_{\eta 1\xi 1}\, ds'.$

Again, $\quad \|\Phi_n(s') - \Phi_{n-1}(s')\|_{\eta 1\xi 1} \leq \int_0^{s'} K^P_{\eta 1\xi 1}(s'')\,\|\Phi_{n-1}(s'') - \Phi_{n-2}(s'')\|_{\eta 2\xi 2}\, ds''.$

Therefore, $\|\Phi_{n+1}(t) - \Phi_n(t)\|_{\eta\xi} \leq \int_0^t K^p_{\eta\xi}(s') \int_0^{s'} K^p_{\eta 1\xi 1}(s'')\|\Phi_{n-1}(s'') - \Phi_{n-2}(s'')\|_{\eta 2\xi 2}\, ds'' ds'$

$$\leq K_{\eta\xi} K_{\eta 1\xi 1} \int_0^t\int_0^{s'} \|\Phi_{n-1}(s'') - \Phi_{n-2}(s'')\|_{\eta_2\xi_2}\, ds''\, ds'.$$

Consequently, at the nth iterate, we obtain

$$\|\Phi_{n+1}(t) - \Phi_n(t)\|_{\eta\xi} \leq \int_0^t K^P_{\eta\xi}(s_1)\int_0^{s1} K^P_{\eta 1\xi 1}(s_2)\cdots\int_0^{sn-1} K^P_{\eta n-1\xi n-1}(s_n)\,\|\Phi_1(s_n) - \Phi_0(s_n)\|_{\eta n\xi n}\, ds_1 ds_2\cdots ds_n.$$

Since the map $s \to \|\Phi_1(s) - X_0\|_{\eta n\xi n}$ is continuous on [0,T], we put $R_{\eta n\xi n} = \sup_{s\in[0,T]} \|\Phi_1(s) - X_0\|_{\eta n\xi n}$. Then the last inequality satisfies

$$\|\Phi_{n+1}(t) - \Phi_n(t)\|_{\eta\xi} \leq K_{\eta\varepsilon}K_{\eta 1\varepsilon 1}\cdots K_{\eta n-1\ \varepsilon\ n-1} R_{\eta n\ \varepsilon n}\int_0^t\int_0^{s1}\cdots\int_0^{sn-1} ds_1 ds_2\cdots ds_n \leq (L_{\eta\xi,n})^n R_{\eta n\ \xi n}\frac{t^n}{n!} \leq R_{\eta\ \xi}\frac{(L_{\eta\varepsilon}T)^n}{n!}$$

$$\textit{therefore, for any } n > k, \|\Phi_{n+1}(t) - \Phi_{k+1}(t)\|_{\eta\ \xi} \leq \sum_{m=k+1}^{n}\|\Phi_{m+1}(t) - \Phi_m(t)\|_{\eta\xi} \leq R_{\eta\xi}\sum_{m=k+1}^{n}\frac{(L_{\eta\xi}T)}{m!} \leq R_{\eta\xi}\, exp(L_{\eta\xi}T) < \infty$$

It follows that $\{\Phi_n(t)\}$ is a Cauchy sequence in $\widetilde{A}$ and converges to some adapted, weakly absolutely continuous process $\Phi(t)$. We conclude that the limit satisfies equation (1.1) exactly in the same way as in [1].

3.1 **Uniqueness**

Suppose that $Y(t)$, $t \in [0,T]$ is another adapted absolutely continuous solution of (1.1) with $Y(0) = X_0$. Then, in the same way as in the proof of existence of solution, we obtain the estimate

$$\|\Phi(t)-Y(t)\|_\eta \le \frac{(L_{\eta\xi}T)^n}{n!}\sup_{[0,T]}\|\Phi(t)-Y(t)\|_{\eta_n\xi_n}\ .$$

Since the right hand side of the inequality is finite for each $n \in N$, the sequence converges to zero as n $\to \infty$. Consequently $\|\Phi(t)-Y(t)\|_\varepsilon = 0, \forall \eta,\varepsilon \in D \underline{\otimes} E,$ and so $\Phi(t)=Y(t)$ on $D \underline{\otimes} E,\ t \in [0.,T]$

3.2 **Stability**

As in [1], we show under our present Lipschitz condition, that the solutions to the stochastic differential equation (1.1) are stable. By this stability, we mean that small changes in the initial condition lead to small changes in the solution over a given finite time interval and for arbitrary pair of elements $\eta,\varepsilon \in D \underline{\otimes} E$. to this end, we make the following statements. The coefficients E, F, G, H and the integrators $\wedge_\pi, A_f, A_g^+$ and the Lebesgue measure remain as in Theorem 3.2 above. Let $X(t)$, $Y(t)$, $t \in [0,T]$ be the solutions to the QSDE (1.1) corresponding to the initial conditions $X(t_0) = X_0$ and $Y(t_0) = Y_0$, respectively where $X_0, Y_0 \in \widetilde{A}$.

***Theorem* 3.3**

Given $\varepsilon > 0, \exists\ \partial > 0$ such that if $\|X_0 - Y_0\|_{\eta\varepsilon} < \partial$, for all $\eta,\varepsilon \in D \underline{\otimes} E$, then $\|X(t)-Y(t)\|_{\eta\varepsilon} < \in$ fo*r all t* $\in$ *[0,T].*

Proof

As in the proof of Theorem 3.2, let $X_n(t)$, for $n = 0, 1,\dots$ and $Y_n(t)$, for $n = 0, 1\dots$ be the iterates corresponding to initial conditions X_0 and Y_0 respectively , so that $X_0(t) = X_0$ and $Y_0(t) = Y_0$ for all $0 \le t \le T$. then we obtain the following inequalities.

$$\|X_{n+1}(t)-Y_{n+1}(t)\|_{\eta\varepsilon} \le \|(X_0 - Y_0\|_{\eta\varepsilon} + K_{\eta\varepsilon}\left[\int_0^t \left(\|X_0 - Y_0\|_{\eta1\varepsilon1} + K_{\eta1\varepsilon1}\int_0^{t1}\|X_{n-1}(t_2)-Y_{n-1}(t_2)\|_{\eta2\varepsilon2}\,dt_2\right)dt_1\right]$$

$$= \|X_0 - Y_0\|_{\eta\varepsilon} + K_{\eta\varepsilon}t\|X_0 - Y_0\|_{\eta1\varepsilon1} + K_{\eta\varepsilon}K_{\eta1\varepsilon1}\int_0^t\int_0^{t1}\|X_{n-1}(t_2)-Y_{n-1}(t_2)\|_{\eta2\varepsilon2}\,dt_2dt_1.$$

Again, there exits $(\eta_3,\varepsilon_3) \in (D \underline{\otimes} E)^2$ sucn that

$$\|X_{n+1}(t)-Y_{n+1}(t)\|_{\eta\varepsilon} \le \|(X_0 - Y_0\|_{\eta\varepsilon} + K_{\eta\varepsilon}t\|X_0 - Y_0\|_{\eta1\varepsilon1}$$

$$+ K_{\eta\varepsilon}K_{\eta1\varepsilon1}\int_0^t\int_0^{t1}\left(\|X_0 - Y_0\|_{\eta2\varepsilon2} + K_{\eta2\varepsilon2}\int_0^{t2}\|X_{n-2}(t_3)-Y_{n-2}(t_3)\|_{\eta3\varepsilon3}\,dt_3\right)dt_2dt_1$$

$$= \|(X_0 - Y_0\|_{\eta\varepsilon} + K_{\eta\varepsilon}t\|X_0 - Y_0\|_{\eta1\varepsilon1}\ K_{\eta\varepsilon}K_{n1\varepsilon1}\frac{t^2}{2}\|X_0 - Y_0\|_{\eta2\varepsilon2}$$

$$+ K_{\eta\varepsilon}K_{\eta1\varepsilon1}K_{\eta2\varepsilon2}\int_0^t\int_0^{t1}\int_0^{t2}\|X_{n-2}(t_3)-Y_{n-2}(t_3)\|_{\eta3\varepsilon3}\,dt_3dt_2dt_1.$$

At the nth iterate, setting $(\eta_0,\xi_0) = (\eta,\xi)$, we obtain the estimate

$$\|X_{n+1}(t)-Y_{n+1}(t)\|_{\eta\varepsilon} \le \|(X_0 - Y_0\|_{\eta\varepsilon} + L_{\eta\varepsilon}\|X_0 - Y_0\|_{\eta1\varepsilon1} + \cdots + \cdots + \frac{L_{\eta\varepsilon}^{n}\,t^n}{n!}\|X_0 - Y_0\|_{\eta n\varepsilon n}$$

$$+ L_{\eta\varepsilon}^{n+1}\int_0^t\int_0^{t1}\cdots\int_0^{tn}\|X_0(t_n+1)-Y_0(t_{n+1})\|_{\eta n+1\varepsilon n+1}\,dt_1dt_2\cdots dt_{n+1}. \le \sum_{m=0}^{n=1}\frac{L_{\eta\varepsilon}^m}{m!}t^m\|X_0 - Y_0\|_{\eta m\varepsilon m}$$

$$\le \sum_{m=0}^{n+1}\frac{L_{\eta\varepsilon}^m}{m!}T^m\|X_0 - Y_0\|_{\eta m\varepsilon m} \le \|X_0 - Y_0\|_{\eta1\varepsilon'}\ exp(L_{\eta\varepsilon}T)$$

where $\|X_0 - Y_0\|_{\eta'\varepsilon'} = max\{\|X_0 - Y_0\|_{\eta m\varepsilon m}, m = 0,1,2\cdots n+1\}$.

We now apply the condition that $\|X_0 - Y_0\|_{\eta'\varepsilon'} < \partial$ *for all* $(\eta',\varepsilon') \in (ID \underline{\otimes} IE)^2$, and conclude by letting $n \to \infty$, *that* $\|X(t)-Y(t)\|_{\eta\varepsilon} \le \varepsilon$, where $\partial = \dfrac{\in}{exp(L_{\eta\varepsilon}T)}$

Acknowledgement
The authors are grateful to Professor G. O. S Ekhaguere, for useful discussions and suggestions on several occasions

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