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Deformation fields due to sheared semicircular edge notch in a non-homogeneous elastic material

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Abstract

A non-homogeneous semi-infinite elastic material containing a semicircular edge notch of radius a, is studied for determination of deformation fields and maximum anti-plane shear concentration. The mode of loading on variable intervals $[a_i, b_i]$, i =1,2, leads to expression for the maximal stress, $\sigma_{\theta z}$ (a, 0) as a product of two terms; the first is analogous to a known anti-plane stress concentration term for a circular hole in an infinite body while the other term is a measure of the contribution of material constants and changes at load site to the high stress concentration. The special case of our result for $\sigma_{i\theta z}$ (r, 0) when the notch is absent (a = 0) is in agreement with known results. The variations $\sigma_{\theta z}$ (a, 0) with $\frac{a}{b_1}\left(\text{or}\,\frac{b_1}{a}\right)$ are displayed on graphs

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1.0 Introduction

Stress analysis of homogeneous and isotropic elastic materials containing notches of various geometries have been carried out by various authors (see for example [1-5]). Mitchell [1] used truncated mapping function technique to analyse a homogeneous material whose geometry is similar to the one studied here but subjected to remote uniaxial tension and obtained results that indicate maximum stress concentration factor of 3.08. Rice [3] considered an elliptical hole, of semiaxes *a* in the *x* direction and *b* in the *y* direction in an remote biaxial inplane tensions $(\sigma_{xx})\infty, (\sigma_{yy})\infty$ and anti-plane shear $(\sigma_{yz})\ \infty$. stresses at the end of the semiaxis of length a are

$$\sigma_{yy}(a,0) = (\sigma_{yy})\infty\left[1+2\frac{a}{b}\right] - (\sigma_{xx})\ \infty \qquad (1.1)$$

$$\sigma_{yz}(a,0) = (\sigma_{yz})\infty\left[\ 1+\frac{a}{b}\ \right] \qquad (1.2)$$

The uniaxial tensile result may be obtained from (1.1) in the absence of $(\sigma_{xx})\infty$ as

$$\sigma_{yy}(a,0) = (\sigma_{yy})\infty\left[\ 1+2\frac{a}{b}\ \right] \qquad (1.3)$$

Anti–plane results often obtainable from simple calculations, closely predict tensile results, as (1.2) does to (1.3).

In this paper, we study states in a non-homogenous linearly elastic semi-infinite material weakened by a semicircular edge notch of radius a. the material is made of two quarter planes perfectly bonded along their interface in the *x* direction which terminates at the notch. The materials have elastic constants μ_1 for the upper quarter plane and μ_2 for the lower quarter plane opposite shear loads of aggregate magnitudes T_1 and T_2, which need not be equal, are prescribed on variable straight line segments of the feel surface, in the y direction. The notch surface is stress free (see Figure 1). The loaded line segments are intervals $[a_i, b_i]$ whose lengths $L_i = b_i - a_i$, $i = 1, 2$, need not be equal nor symmetric about the origin but whose alterations cause the changes in T_i, $i = 1, 2$. We adopt the convention of attaching the subscript 1 to items associated with the upper quarter plane and relate the subscript 2 to items concerning the lower quarter plane.

Our method of analysis and loading direr from those applied to the homogeneous cases cited and has been used [6,7] in studying problems with finite boundaries whose sub-segment are loaded.

2.0 Governing boundary value problem

The non-vanishing stresses satisfy the relations:

$$\sigma_{ixz}(x,y)=\mu_i\frac{\partial w_i}{\partial x}(x,y),\sigma_{iyz}(x,y)=\mu_i\frac{\partial w_i}{\partial y}(x,y), i=1,2 \qquad (2.1)$$

The following conditions are therefore satisfied at the load sites:

$$\frac{\partial w_1}{\partial \theta}(0,y)=-\frac{T_1}{\mu_1},\ a_1\le y\le b_1;\ \frac{\partial w_2}{\partial x}(0,y)=\frac{T_2}{\mu_2},\ -a_2\le y\le -b_2$$

In terms of polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$ the conditions at load sites become

$$\frac{\partial w_i}{\partial\theta}\left(r,\pm\frac{\pi}{2}\right)=\frac{rT_i}{\mu_i},\ i=1,2$$

Thus the problem is that of finding $w_i(r, \theta)$, $i = 1,2$ in the boundary value problem

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)w_i(r,\theta)=0,\ \frac{-\pi}{2}\le\theta\le\frac{\pi}{2},\ r\ge a,\ i=1,2 \qquad (2.2)$$

$$w_1(r,0)=w_2(r,0);\ \ \mu_1\frac{\partial w_2}{\partial\theta}(r,0)=\mu_2\frac{\partial w_2}{\partial\theta}(r,0),\ r\ge a \qquad (2.3a)$$

$$\frac{\partial w_i}{\partial x}\left(r,\pm\frac{\pi}{2}\right)=\frac{rT_i}{\mu}=\frac{rT_i}{\mu_i},\ a_i\le r\le b_i,\ \ \vartheta=(-)^{i-1}\frac{\pi}{2}, i=1,2\ \ a\le a_i<b_i \qquad (2.3b)$$

$$\frac{\partial w_i}{\partial\theta}(a,\theta)=0,\ \ \frac{-\pi}{2}\le\theta\le\frac{\pi}{2} \qquad (2.3c)$$

Utilizing the conformal mapping function defined by

$$\xi(z)=\frac{1}{2}\left(\frac{z}{a}-\frac{a}{z}\right),\qquad z=x+iy \qquad (2.4)$$

The original notched half plane is transformed into a plane with a cut along its entire left real line, Figure 11. Let (p,ϕ) denote polar coordinates in the ξ-plane such that $\xi(z) = pe^{i\phi}, z = re^{i\theta}$

$$-\pi\le\phi\le\pi,\ \frac{-\pi}{2}\le\theta\le\frac{\pi}{2},\ \ p\ge 0, r\ge a$$

Suppose $\xi(r,\theta)=u(r,\theta)+iv(r,\theta)$ then $u(r,\theta)=\rho\cos\phi, v(r,\theta)=p\sin\phi$ implies $\rho(r,\theta) = \{u^2(r, \theta) + v^2(r, \theta)\}^{1/2}$ and $\cot\phi=\frac{u(r,\theta)}{v(r,\theta)}$ where $u(r,\theta)=\left(\frac{1}{2}\frac{r}{a}-\frac{a}{r}\right)\cos\theta$ and $v(r,\theta)=\left(\frac{1}{2}\frac{r}{a}-\frac{a}{r}\right)\sin\theta$. Therefore

$$\frac{\partial p}{\partial r}(a,\theta)=0,\ \frac{-\pi}{2}\le\theta\le\frac{\pi}{2};\ \frac{\partial p}{\partial\theta}\left(r,\pm\frac{\pi}{2}\right)=0,\ \frac{\partial p}{\partial\theta}(r,0)=0,\ r\ge a$$

$$\frac{\partial\phi}{\partial r}(a,\theta)=\frac{-1}{a}\cot\theta,\ \ \frac{-\pi}{2}\le\theta\le\frac{\pi}{2},\ \ \theta\ne 0,\ \ \frac{\partial\phi}{\partial\theta}\left(r,\pm\frac{\pi}{2}\right)=\frac{\left(\frac{r}{a}-\frac{a}{r}\right)}{\left(\frac{r}{a}+\frac{a}{r}\right)} \qquad (2.5)$$

Since $v\left(r,\pm\frac{\pi}{2}\right)=\pm p\left(r,\pm\frac{\pi}{2}\right)=\pm\left[\frac{1}{2}\left(\frac{r}{a}+\frac{a}{r}\right)\right]^2$, $r\geq a$, it follows that $\frac{r}{a}=\rho+(\rho^2-1)^{1/2}$ and so $\frac{\partial\theta}{\partial\theta}\left(r,\pm\frac{\pi}{2}\right) = (\rho^2-1)\rho^{-1}$, $\rho \phi 1$. Using (2.5) together with the fact that $w_i(r,\theta)$, $i=1,2$, we get

$$\frac{\partial w_i}{\partial\theta}\left(r,\pm\frac{\pi}{2}\right)=\frac{\partial w_i}{\partial\phi}\left(p,\pm\frac{\pi}{2}\right)\frac{\partial\phi}{\partial\theta}\left(r,\pm\frac{\pi}{2}\right),\ i=1,2,\ p>1 \qquad (2.6a)$$

$$\frac{\partial w_i}{\partial r}(a,\theta)=\frac{\partial w_i}{\partial\theta}\left(p,\pm\frac{\pi}{2}\right)\frac{\partial w}{\partial\phi}(a,\theta),\ i=1,2,\ p<1 \qquad (2.6b)$$

The transformation converts the problem to that of solving the boundary value problem for $w_i(\rho,\phi)$, i = 1, 2 given by:

$$\left(\frac{\partial^2}{\partial p^2}+\frac{1}{p}\frac{\partial}{\partial p}+\frac{1}{p^2}\frac{\partial^2}{\partial\phi^2}\right)w_i(p,\phi)=0,\ p\geq 0,\ -\pi\leq\phi\leq\pi,\ i=1,2 \qquad (2.7)$$

$$w_1(\rho,0)=w_2(\rho,0),\ \mu_1\frac{\partial w_1}{\partial\phi}(\rho,0)=\mu_2\frac{\partial w_2}{\partial\phi}(\rho,0),\ \rho\geq 0 \qquad (2.8a)$$

$$\frac{\partial w_1}{\partial\phi}\left(\rho,\pm\frac{\pi}{2}\right)=\frac{aT_1}{\mu_1}\left[\rho+\rho^2(\rho^2-1)^{\frac{-1}{2}}\right],\ \alpha_i\leq\rho\leq\beta_i,\ i=1,2$$
$$=0,\ \rho\ \pi\ \alpha_i,\ \rho\ \phi\ \beta_i,\ i=1,2 \qquad (2.8b.c)$$

$$\alpha_i=\frac{1}{2}\left(\frac{a_i}{a}+\frac{a}{a_i}\right),\ \beta_o=\frac{1}{2}\left(\frac{b_1}{a}+\frac{a}{b_i}\right),\ \beta_i=\alpha_i\ \phi\ 0,\ i=1,2 \qquad (2.8d,e)$$

The asymptotic behaviours of $w_i(\rho,\phi)$ i = 1,2 as $\rho\to 0$ and as $\rho\to\infty$ are determined from (2.8c). The results are $w_i(p,\phi)=C_3p^{\frac{1}{2}}\sin\frac{\phi}{2}$ as $p\to 0$, $w_i(p,\phi)=C_4p^{-\frac{1}{2}}\sin\frac{\phi}{2}$ as $p\to\infty$.

3.0 Solution of the boundary value problems

The Mellin transform of $w_i(\rho,\phi)$ is denoted by $\overline{w}_i(s,\phi)=\int_0^\infty w(\rho,\phi)\,\rho^{s-1}d\rho,\ -\frac{1}{2}\ \pi\ Re\,s\ \pi\ \frac{1}{2}$.

Taking the Mellin transform of (2.7) and (2.8) gives

$$\left(\frac{d^2}{d\phi^2}+S^2\right)\overline{W}_i(s,\phi)=0,\ -\frac{1}{2}<Re\,s<\frac{1}{2},\ i=1,2 \qquad (3.1)$$

$$\overline{W}_1(S,0)=\overline{W}_2(s,0),\ \mu_1\frac{\partial\overline{W}_1}{\partial\phi}(S,0)=\mu_2\frac{\partial\overline{W}}{\partial\phi}(s,0) \qquad (3.2a)$$

$$\frac{\partial\overline{W}_i}{\partial\phi}\left(\rho,\pm\frac{\pi}{2}\right)=\frac{aT_i}{2\mu_i}f_i(s),i=1,2 \qquad (3.2b)$$

Where the Mellin transform of (2.8b) yields

$$f_i(s)=\int_{\alpha_i}^{\beta_i}\rho^{S+1}(\rho^2-1)^{-\frac{1}{2}}d\rho+\frac{\beta_i^{S+1}-\alpha_i^{S+1}}{s+1},\ i=1,2 \qquad (3.3)$$

The Taylor series expansion of $(1-t)^{-\frac{1}{2}}$, $|t|<1$ will be written as

$$(1-t)^{-\frac{1}{2}}=\sum_{k=0}^{\infty}a_kt^k \qquad (3.3b)$$

Where the coefficients are given by $a_k = \frac{(2k)!}{2^{2k}(k!)^2}$. In view of (3.3b) the integrand in (3.3a) contains $\rho^s\left(1-\rho^{-2}\right)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} a_k \rho^{s-2k}$. Thus $f_i(s) = \sum_{k=1}^{\infty} a_k \left(\frac{\beta_i^{S-2k+1} - \alpha_i^{S-2k+1}}{s-2k+1}\right) + 2\left(\frac{\beta_i^{S+1} - \alpha_i^{S+1}}{s+1}\right),\ i = 1,2$ (3.4)

Assuming the solution of (3.1) in the form

$$\overline{W}_i(s,\phi) = A_i(s)\sin s\phi + B_i(s)\cos\phi \quad i = 1,2 \qquad (3.5)$$

The continuity conditions (3.2a) yield

$$B_1(s) = B_2(s),\ \mu_1 A_1(s) = \mu_2 A_2(s) \qquad (3.6)$$

Using (3.2b), (3.5) and (3.6) we get

$$A_i(s) = \frac{a}{2\mu_i}[(1+\gamma)T_1 f_1(s) + (1-\gamma)T_2 f_2(s)]\frac{1}{s\cos\frac{\pi}{2}s} \qquad (3.7a)$$

$$B_i(s) = \frac{a}{2}\left[(1+\gamma)\frac{T_2}{\mu_2} f_2(S) - (1-\gamma)\frac{T_1 f_1(s)}{\mu_i}\right]\frac{1}{s\ \ \sin\frac{\pi}{2}s} \qquad (3.7b)$$

$$\gamma = \frac{\mu_2 - \mu_1}{\mu_2 + \mu_1}$$

The displacement sought for is given by the inverse Mellin transform denoted by

$$w(\rho,\phi) = \frac{1}{2\pi_i}\int_{c-i\infty}^{c+i\infty} \overline{W}(s,\phi)\rho^{-s} ds,\ -\frac{1}{2} < Re\, s < \frac{1}{2}$$

Using (3.5) and (3.7) we see that

$$w_j(\rho,\phi) = \frac{a}{2\mu_j}\left\{\frac{1}{2\pi\ i}\int_{c-i\infty}^{c+i\infty}\left[(1+\gamma)T_1 f_1(s) + (1-\gamma)T_2 f_2(s)\right]\rho^{-s}\frac{\sin s\phi}{s\cos\frac{\pi}{2}s} ds\right.$$

$$\left. + \frac{\mu_j}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[(1+\gamma)\frac{T_2}{\mu_2} f_2(s) - (1-\gamma)\frac{T_1}{\mu_1} f_1(s)\right]\rho^{-s}\frac{\cos s\phi}{s\sin\frac{\pi}{2}s} ds\right\},\ j = 1,2 \qquad (3.8)$$

The singularities that enter the evaluation of (3.8) by residue technique are better understood by expressing $f_i(s)\rho^{-s}$ in the form $f_i(s)\rho^{-s} = g(s,\beta_i)\left(\frac{\rho}{\beta_i}\right)^{-s} - g(s,\alpha_i)\left(\frac{\rho}{\alpha_i}\right)$ where $g(s,t) = \sum_{k=1}^{\infty}\frac{a_k t^{1-2k}}{s-2k+1} + \frac{2t}{s+1}$. To obtain $w_i(\rho,\phi)$ when $\alpha_i \le \rho \le \beta_i, i = 1,2$ we note that $\frac{\rho}{\beta_i} < 1$ *and* $\frac{\rho}{\alpha_i} > 1$ jointly. Jordan's lemma then implies closure of contours in the left half plane Res < 0 for $\rho < \beta_i$ and in the right half plane Res >0 for $\rho > \alpha_i$, i = 1,2. When $\rho > \beta_i$, the first integrand in (3.8) involves $g(s, \beta_i)\ \frac{\sin s\phi}{s\cos\frac{\pi}{2}s}$ which has simple poles located at s = - (2n - 1), n = 1,2,3, … and a pole of order 2 at s = -1. Consequently we derive

$$1_{\beta j}^{(1)}(\rho,\phi) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g(S,\beta_j)\left(\frac{p}{\beta_j}\right)^{-s}\frac{\sin s\phi}{s\cos\frac{\pi}{2}} ds = \frac{4}{\pi}[-in\left(\frac{p}{\beta_j}\right)\sin\phi - \phi\cos\phi + \sin\phi\}p$$

$$+\frac{2}{\pi}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}a_k\,\frac{\beta_J^{\,2-2n-2k}}{2-2n-2k}\,p^{2n-1}\,sin(2n-1)\phi,\quad j=1,2$$

in the second integrand of (3.8) is found $g\,(s, \beta_j)\,\dfrac{\cos s\phi}{s\sin\frac{\pi}{2}s}$ which has simple at $s = 2n$, n = 1,2,3, … and at s = -1. Therefore

$$1_{\beta j}^{(2)}(\rho,\phi)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}g(s,\beta_j)\left(\frac{\rho}{\beta_j}\right)^{-s}\frac{cos\,s\phi}{s\,sin\frac{\pi}{2}s}\,ds=2p\,cos\,\phi-\frac{1}{\pi}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^n}{n}a_k\,\frac{\beta_j^{\,1-2n-2k}}{1-2n-2k}\,p^{2n}\,cos\,2n\phi,\ j=1,2$$

$$when\ \frac{p}{\alpha_i}>1,\ the\ functions\ to\ be\ considered\ are-g(s,\alpha_i)\frac{sin\,s\phi}{s\,cos\frac{\pi}{2}s}\ and-g(s,\alpha_i)\frac{cos\,ss\phi}{s\,sin\frac{\pi}{2}s},\ i=1,2$$

in the first and second integrands respectively. The first integrand has poles of orders 2 at s =2n - 1, n = 1, 2, 3,… . While the second has simple poles at s = 2n - 1, s = 2n, n = 1, 2, 3,… . Hence the integrals are evaluated as follows:

$$1_{\alpha j}^{(1)}(\rho,\phi)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}g(s,\alpha_j)\left(\frac{\rho}{\alpha_j}\right)^{-s}\frac{sin\,s\,\phi}{s\,cos\frac{\pi}{2}s}\,ds=\frac{1}{\pi}\sum_{n=1}^{\infty}\sum_{\substack{k=1\\k\neq n}}^{\infty}\frac{(-1)^{n-1}}{2n-1}ak\,\frac{\alpha_j^{2(n-k)}}{n-k}\,\rho^{-(2n-1)}\,sin(n-1)\phi$$

$$-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}a_n\left[-ln\frac{\rho}{\alpha_j}sin\,(2n-1)\phi+\phi\,cos(2n-1)\phi-\frac{sin(2n-1)\phi}{2n-1}\right]\rho-(2n-1),\ j=1,2$$

$$1_{\alpha j}^{(2)}(\rho,\phi)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}g(s,\alpha_j)\left(\frac{\rho}{\alpha_j}\right)^{-s}\frac{cos\,s\phi}{s\,sin\frac{\pi}{2}s}\,ds=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}a_n\rho^{-(2n-1)}\,cos(2n-1)\phi$$

$$+\frac{1}{\pi}\sum_{n-1}^{\infty}\frac{(-1)^n}{n}g(2n,\alpha_j)\rho^{-2n}\,cos\,2n\,\phi,\quad j=1,2$$

The form of the solution $w_i(\rho,\phi)$ when $\alpha_i\le\rho\le\beta_1, i=1,2$ is then written as

$$w_i(\rho,\phi)=\frac{1}{2\mu i}\Big\{(1+\gamma)T_1\big[1_{\beta 1}^{(1)}(\rho,\phi)+1_{\alpha 1}^{(1)}(\rho,\phi)\big]+(1-\gamma)T_2\big[1_{\beta 2}^{(1)}(\rho,\phi)+1_{\alpha 2}^{(1)}(\rho,\phi)\big]$$

$$+\mu_i\left[(1+\gamma)\frac{T_2}{\mu_2}\{1_{\beta 2}^{(2)}(\rho,\phi)\}-(1-\gamma)\frac{T_1}{\mu_1}\{1_{\beta 1}^{(2)}(\rho,\phi)+1_{\alpha 1}^{(2)}(\rho,\phi)\}\right]\Big\},i=1,2 \qquad (3.9)$$

Next we derive the form of $w_i(\rho,\phi)$ for $0 < \rho < \alpha_i$, i = 1, 2,… . By first noting that $\frac{\rho}{\alpha_i}<1$ and $\frac{\rho}{\beta_i}<1$ all at once. This leads to the integrals in (3.8) being evaluated with p<1 and therefore with f_i(s) given in (3.4). Jordan's lemma indicates closure of contours in the left half plane Res < 0. All poles there are simple and contributed by cos $\frac{\pi}{2}$s in the first integrand and by sin $\frac{\pi}{2}$s in the second integrand. Hence

$$1_j^{(1)}(\upsilon,\phi)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f_j(s)\rho^{-s}\ \frac{\sin s\phi}{s\cos\frac{\pi}{2}s}\,ds=\frac{2}{\pi}\sum_{n-1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\ f_j(1-2n)p^{2n-1}\ \sin(2n-1)\phi,\ j=1,2\ \text{and}$$

$$1_j^{(2)}(\rho,\phi)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f_j(s)\rho^{-s}\ \frac{\cos s\phi}{s\sin\frac{\pi}{2}s}=\frac{1}{\pi}\sum_{n-1}^{\infty}\frac{(-1)^{n}}{n}\ f_j(-2n)\rho^{2n}\ \cos 2n\phi,\ j=1,2$$

which then imply that for $0<\rho<\alpha_i i=1,2$, the solution is

$$w_i(\rho,\phi)=\frac{a}{2\mu_i}\Big\{\big[(1+\gamma)T_1 1_1^{(1)}(\rho,\phi)+(1-\gamma)T_2 1_2^{(1)}(\rho,\phi)\big]$$
$$+\mu_i\left[(1+\gamma)\frac{T_2}{\mu_2}1_2^{(2)}(\rho,\phi)-(1-\gamma)\frac{T_1}{\mu_1}1_1^{(2)}(\rho,\phi)\right]\Big\},\ i=1,2 \qquad (3.10)$$

4.0 Notch surface-interface junction fields

The notch surface- interface junction is approached as $\rho\to 0$ sequel to which (3.10) yields the displacement fields there as

$$W_i(\rho,\phi)=\frac{a}{\pi\varpi_i}\{(1-\gamma)T_1 f_1(-1)+(1-\gamma)T_2 f_2(-1)\}\rho\sin\phi\ \text{as}\ \rho\to 0, i=1,2$$

From (3.4), $f_1(-1)=\sum_{k=1}^{\infty}a_k\ \frac{\left(\beta_i^{-2k}-\alpha_i^{-2k}\right)}{-2k}+2\ In\left(\frac{\beta_i}{\alpha_i}\right)\ i=1,2$. Using the relationship

$$\sum_{k=1}^{\infty}\frac{a_k}{k}t^k=-2\ In\ t+2In\left(1-\sqrt{1-t}\right)-2In\,2,|t|\le 1$$

it is readily seen that

$$f_i(-1)=In\left(\frac{\beta_i}{\alpha_i}\right)+In\left(\frac{\alpha_i-\sqrt{\alpha_i^2-1}}{\beta_i-\sqrt{\beta_i^2-1}}\right), i=1,2 \qquad (4.1a)$$

Inserting (2.8d.e) into (4.1a) and using the fact that $\frac{1}{4}\left(q+\frac{1}{q}\right)^2-1=\frac{1}{4}\left(q-\frac{1}{q}\right)^2$, we get

$$f_i(-1)=\left\{In\left[\frac{\frac{b_i}{a}+\frac{a}{b_i}}{\frac{a_i}{a}+\frac{a}{a_i}}\right]+In\left(\frac{b_i}{a_i}\right)\right\}, b_i>a_i\ge a, i=1,2 \qquad (4.1b)$$

From (2.4) we obtain the relationship $\frac{1}{2}\left(\frac{r}{a}+\frac{a}{r}\right)\sin\theta=\rho\sin\phi$ as $\rho\to 0$ *and* $r\to a$. Hence

$$w_i(r,\theta)=\frac{a}{2\pi\mu_i}\{(1+\gamma)T_1 f_1(-1)+(1-\gamma)T_2 f_2(-1)\}\left(\frac{r}{a}+\frac{a}{r}\right)\sin\theta,\ r\to a,\ i=1,2 \qquad (4.2)$$

The stress fields are obtained from polar equivalents of (2.1). The results are

$$\sigma_{iz}(r,\theta)=\frac{1}{2\pi}\{(1+\gamma)T_1 f_1(-1)+(1-\gamma)T_2 f_2(-1)\}\left(1-\frac{a^2}{r^2}\right)\sin\theta,\ r\to a,\ i=1,2 \qquad (4.3)$$

$$\sigma_{i\theta z}(r,\theta)=\frac{1}{2\pi}\{(1+\gamma)T_1 f_1(-1)+(1-\gamma)T_2 f_2(-1)\}\left(1+\frac{a^2}{r^2}\right)\cos\theta,\ r\to a,\ i=1,2 \qquad (4.4)$$

The result in (4.4) agrees with the case given in (4.2) of [8] when there is no notch ($a = 0$) under concentrated shear forces. The results indicate absence of stress singularities even when a is small enough to approximate a narrow notch.

5.0 Conclusion

To understand the location with highest stress concentration, we investigate the fields near the point (a, 0) and as $r \to \infty$. Let q be a ration umber close to but greater than 1. At all locations (r,θ) within the material with $r = aq$, we see that $\sigma_{irz}(qa,\theta) \le \sigma_{i\theta z}(a,0)$ and $\sigma_{i\theta z}(qa,\theta) \le \sigma_{i\theta z}(a,0)$, $i = 1,2$. The form of $w_i(r,\theta)$, as $r \to \infty$, $i = 1,2$ is deduced from that of $w_i(\rho,\phi)$, when $\rho \phi \beta_i, i = 1,2$. Since for this case $\rho \phi \beta_i$ and $\rho \phi \alpha_i$ at the same time, (3.8) is evaluated with $f_i(s)$ given in (3.4) and $\rho \phi 1$. The contours are closed in the right half place Res > 0 where the integrands in (3.8) have simple poles. The dominant term of $w_i(\rho,\phi)$ is obtained as

$$w_i(\rho,\phi) = \frac{-a}{2\pi\mu_i}\{(1+\gamma)T_1 f_1(1) + (1-\gamma)T_2 f_2(1)\}\rho^{-1} \sin\frac{\phi}{2}, \text{ as } \rho \to \infty,\ i = 1,2 .$$

Application of the relation $\rho^{-1}\sin\phi = 2ar^{-1}\sin\theta$ obtained from the mapping (2.4) as $\rho \to \infty$, and $r \to \infty$ yields

$$w_i(r,\phi) = \frac{-2a^2}{\pi\mu_i}\{(1+\gamma)T_1 f_1(1) + (1-\gamma)T_2 f_2(1)\}r^{-1}\sin\theta, \text{ as } r \to \infty,\ i = 1,2 \qquad (5.1)$$

$$w_{rzi}(r,\phi) = \frac{-2a^2}{\pi}\{(1+\gamma)T_1 f_1(1) + (1-\gamma)T_2 f_2(1)\}r^{-1}\sin\theta, \text{ as } r \to \infty,\ i = 1,2 \qquad (5.2)$$

$$w_{\theta zi}(r,\phi) = \frac{-2a^2}{\pi}\{(1+\gamma)T_1 f_1(1) + (1-\gamma)T_2 f_2(1)\}r^{-1}\cos\theta, \text{ as } r \to \infty,\ i = 1,2 \qquad (5.3)$$

From (3.4) $f_1(1)$ contains $\sum_{k=2}^{\infty} a_k \frac{t^{2(1-k)}}{1-k}$ whose sum is found by noting that integrating both sides of $\sum_{k=2}^{\infty} a_k t^{-k} = -t - \frac{1}{2}t^{-1} + (1-t^{-1})^{\frac{-1}{2}}$, |$t$| > 1 and changing variables through $\varepsilon^2 = t^{-1}$ yields

$$\sum_{k=2}^{\infty} a_k \frac{t^{1-k}}{1-k} = -t - \frac{1}{2}\ln t + 2\int (1+\varepsilon^2)^{\frac{1}{2}} d\varepsilon + c, \ |t| \ge 1$$

from which we get

$$\sum_{k=2}^{\infty} a_k \frac{t^{1-k}}{1-k} = -t - \frac{1}{2}\ln t + t^{\frac{1}{2}}(t-1)^{\frac{1}{2}} + \ln\left(\sqrt{t} + \sqrt{t-1}\right) + c, \ |t| \ge 1 \text{ then}$$

$$f_1(1) = \sum_{k=2}^{\infty} a_k \frac{\left(\beta_i^{2(1-k)} - \alpha_i^{2(1-k)}\right)}{2(1-k)} + \left(\beta_i^2 - \alpha_i^2\right) = \beta_i^2 - \alpha_i^2 + \beta_i^2\left(\beta_i^2 - 1\right)^{\frac{1}{2}} - \alpha_i^2\left(\alpha_i^2 - 1\right)^{\frac{1}{2}}$$
$$+ \ln\left[\frac{\beta_i + \sqrt{\beta_i^2 - 1}}{\alpha_i + \sqrt{\alpha_i^2 - 1}}\right], \ i = 1,2 \qquad (5.4)$$

In view of (3.4) $f_i(1)$, $i = 1,2$ is defined at all finite values of $\beta_i \phi \alpha_i \phi 1$. It follows from (5.2) and (5.3) that the stresses vanish as $r \to \infty$. On the other hand when $r = a$ and $\theta = 0$, (4.2) indicates that the junction of the notch and the interface is not displaced but that the stress concentrated there is deduced from (4.4) as

$$\sigma_{\theta z}(a,0) = \frac{1}{\pi}\left[(1+\gamma)T_1 f_1(-1) + (1-\gamma)T_2 f_2(-1)\right] \qquad (5.5)$$

The notch tip stress (5.5) therefore experiences the maximum stress concentration. This implies that cracking induced by loads will commence at the notch tip. The response of $\sigma_{\theta z}(a,0)$ to variations of the applied loads achieved by changing the load site lengths is displayed in Figure 3 for the case when

$T_2 = 0,\ a_1 = \lambda b_1,\ \lambda \phi 0$, so that the variable load site is the interval $[\lambda b_1, b_1]$ of length $L = (1-\lambda) b_1,\ 0 \pi \lambda \pi 1$. From (4.1b) $f_i(-1)$ takes the form

$$f_1(-1) = \begin{cases} 2\, ln\left(\dfrac{1}{\lambda}\right) + ln\left[\dfrac{x^2+1}{\left(\dfrac{x}{\lambda}\right)^2 + 1}\right], & x = \dfrac{a}{b} \pi \sqrt{\lambda} \\ ln\left[\dfrac{x^2+1}{\left[(\lambda x)^2+1\right]}\right], & x = \dfrac{b_1}{a} \phi \dfrac{1}{\sqrt{\lambda}} \end{cases} \qquad (5.6a,b)$$

The load site length L may be varied by selecting λ from terms of a sequence that converge to 1. From (5.5) we see that $\sigma_{\theta z}(a,0)$ depends on materials constants except when $T_1 f_1(-1) = T_2 f_2(-1)$ which arises from application of equal and opposite loads on segments of equal length for which

$$\sigma_{\theta z}(a,0) = \frac{2}{\pi} T_1 \left\{ ln\left[\frac{\dfrac{b_1}{a} + \dfrac{a}{b_1}}{\dfrac{a_1}{a} + \dfrac{a}{a_1}}\right] + ln\left(\frac{b_1}{a_1}\right) \right\}$$

The stress concentration for a hole in an infinite plane under remote anti-plane shear may be deduced from (1.2) when $a = b$ as $\sigma_{yz}(a,0) = 2(\sigma_{yz})\infty$ (5.7)

The infinite plane with a circular hole is equivalent to a semicircular notch in a semi-finite plane blended with its mirror image. Now substituting $f_i(-1),\ i = 1{,}2$ of (4.1b) into (5.5) yields the maximal stress concentration as

$$\sigma_{\theta z}(a,0) = 2T_1 \left\{ ln\left[\frac{\dfrac{b_1}{a} + \dfrac{a}{b_1}}{\dfrac{a_1}{a} + \dfrac{a}{a_1}}\right] + ln\left(\frac{b_1}{a_1}\right) \right\}\left(\frac{1+\gamma}{2\pi}\right) + 2T_2 \left\{ ln\left[\frac{\dfrac{b_2}{a} + \dfrac{a}{b_2}}{\dfrac{a_2}{a} + \dfrac{a}{a_2}}\right] + ln\left(\frac{b_2}{a_2}\right) \right\}\left(\frac{1+\gamma}{2\pi}\right) \quad (5.8)$$

In (5.8) each term on the right hand side is composed of $2T_i,\ i = 1{,}2$ comparable to $2(\sigma_{yz})\infty$, of a circular hole in an infinite plane, and $\dfrac{(1 \pm \gamma)}{\pi}\dfrac{1}{2} f_i(-1),\ i = 1{,}2$ that contains material constants and is a facilitator of estimates of effects of load site perturbations on the stress concentration.

5.1 **Concentrated shear force**

The relation [9] $ln\left(\dfrac{p}{q}\right) = \dfrac{p}{q} - 1 + 0\left[\left(\dfrac{p}{q} - 1\right)^2\right], \left|\dfrac{p}{q} - 1\right| \pi 1$ implies $ln\left(\dfrac{p}{q}\right) = \dfrac{p}{q} - 1$ as $q \to p$ and may
be applied to (14.1b) to get

$$f_i(-1) = \left\{\left[\frac{b_i - a_i}{a} - \frac{a}{a_i b_i}(b_i - a_i)\right]\left(\frac{a_i}{a} + \frac{a}{a_i}\right)^{-1} + \frac{b_i - a_i}{a}\right\} \text{ as } a_i \to b_i,\ i = 1{,}2 \qquad (5.1.1)$$

Shear force τ_i, concentrated at a distance b_i from the origin is obtained if $T_i \to \infty$ and $(b_i - a_i) T_i \to \tau_i$ as $a_i \to b_i$. With such consideration, (5.9) leads to $T_i f_i(-1) = 2\dfrac{\tau_i}{b_i}\left[\dfrac{b_i}{a}\left(\dfrac{b_i}{a} + \dfrac{a}{b_i}\right)^{-1}\right]$ as $a_i \to b_i\ T_i(b_i - a_i) \to$

τ_i, $i=1,2$ which substituted into (5.5) gives the state of the stress concentration at the tip of the notch due to prescribed concentrated shear force. Division by b_i to get $\frac{\tau_i}{b_i}$ in the expression for $T_1 f_1(-1)$ was introduced for dimensional consistency (see for example equation (4.2) in [8]). Here, $T_1 f_1(-1) = 2\frac{\tau_i}{b_i}$, when a = 0. The case T_2 = 0 gives stress $\sigma_{\theta z}(a,0)$ due to concentrated share force τ_i as

$$\frac{b_1}{\tau_1}\sigma_{\theta z}(a,0) = \frac{2}{\pi}(1-\gamma)\left(x^2+1\right)^{-1},\; x=\frac{a}{b_1}\,\pi\,1 = \frac{2}{\pi}(1-\gamma)x^2\left(x^2+1\right)^{-1}, x=\frac{b_1}{a}\,\phi\,1$$

and is used to study the variation of $\sigma_{\theta z}(a,0)$ relative to $\frac{a}{b_1}$ or $\frac{b_1}{a}$ under the prescribed concentrated loads as shown in Figure 4.

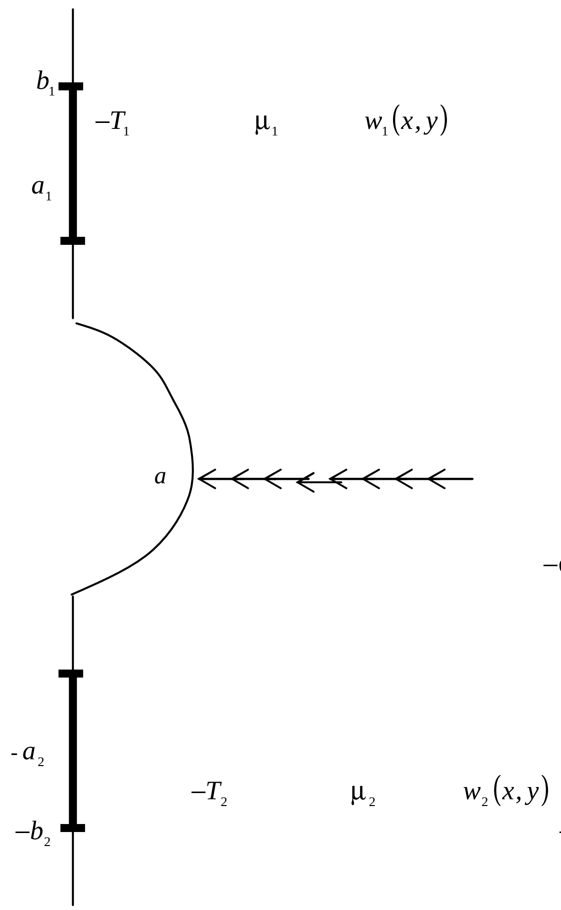

Figure 1: The semicircular notch and load sites (not necessarily Symmetric): $[a_1, b_1]$ for T_1 and $[a_2, b_2]$ for T_2

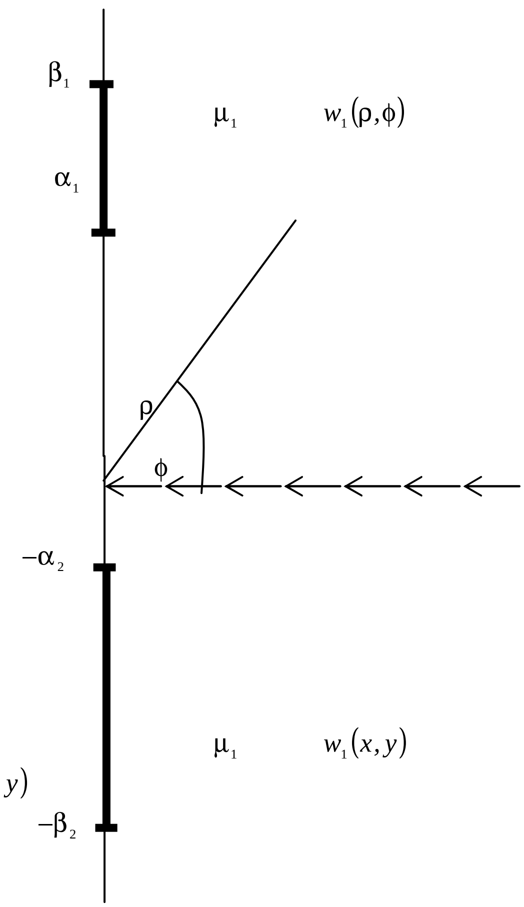

Figure 2: Corresponding load site in the $\rho,\phi-$ plane (not necessarily symmetric)

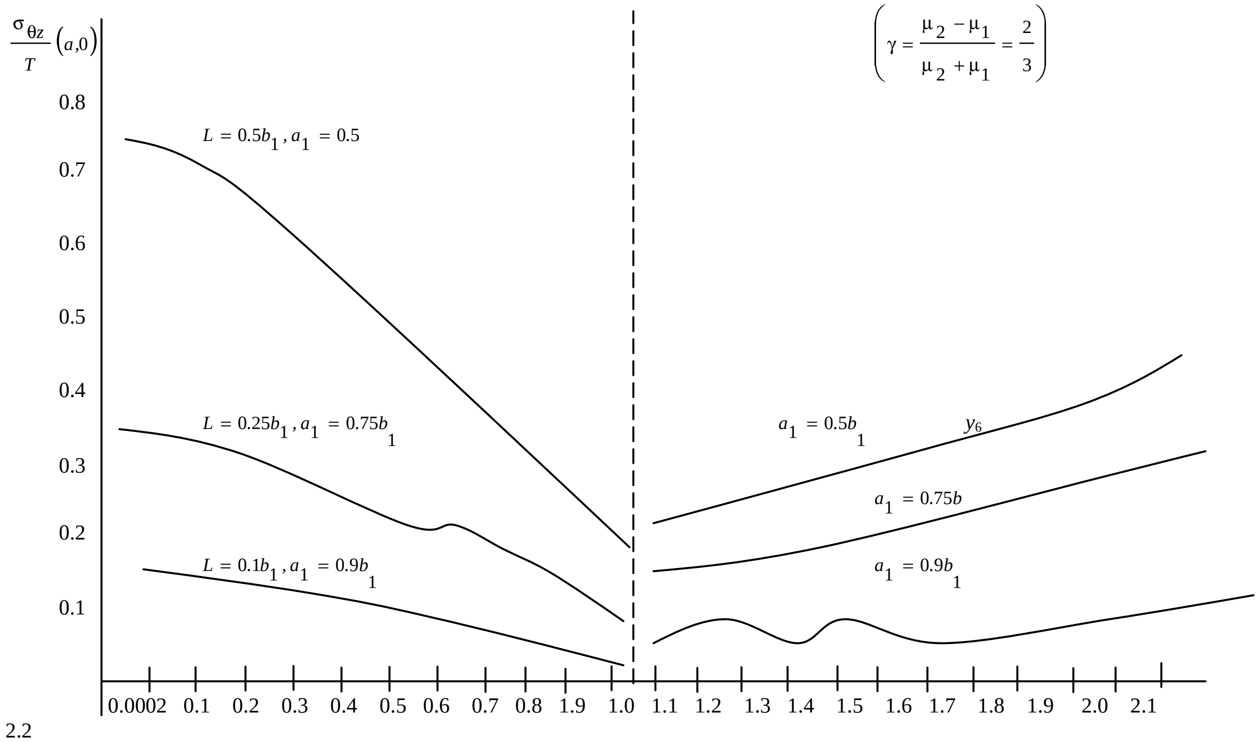

Figure 3: Variation $\frac{\sigma_{\theta z}}{T}(a,0)$ with $\frac{a}{b_1}\left(\text{or}\,\frac{b_1}{a}\right)$ for various values of L

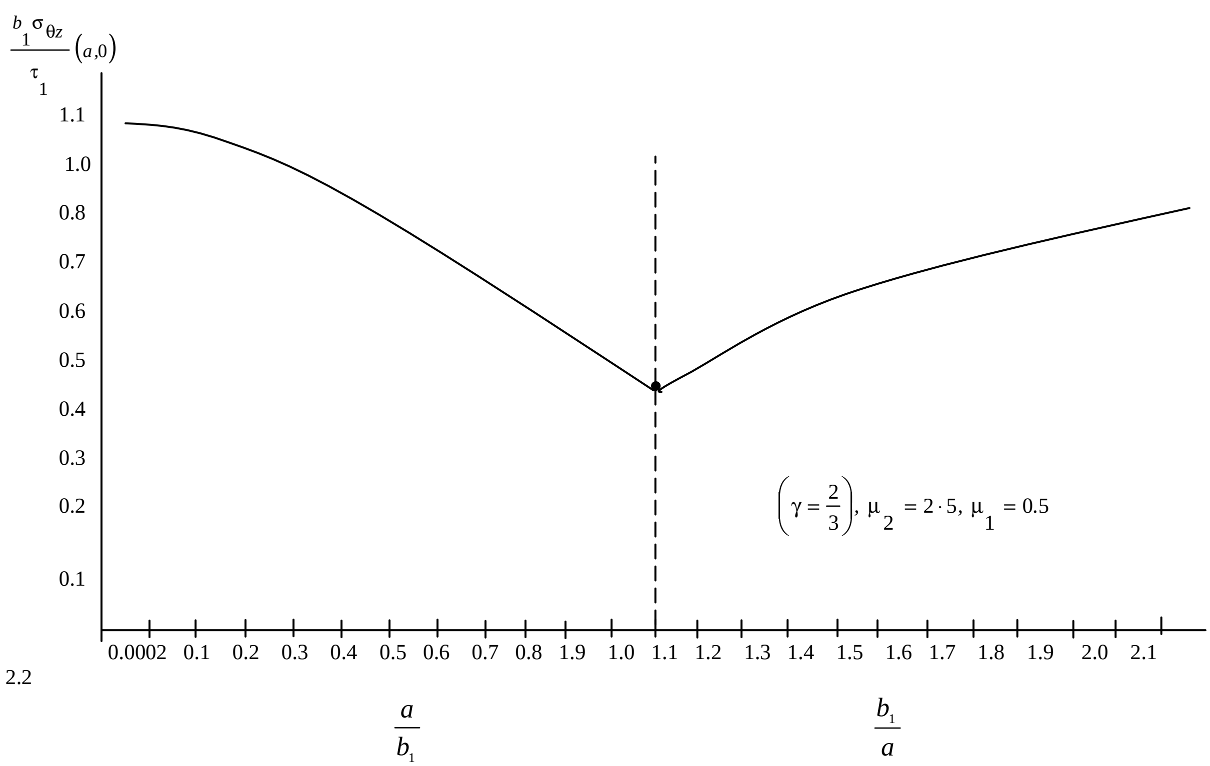

Figure 4: Variation of $\frac{b_1\sigma_{\theta z}}{\tau_1}(a,0)$ with $\frac{a}{b_1}\left(\text{or}\,\frac{b_1}{a}\right)$ under concentrated shear force

References

[1] L. H. Mitchel, Stress concentration at a semicircular notch. Journal of Applied Mechanics, Vol. 32, 1965, pp 938 – 939.

[2] J. R. Rice, Stresses due to a sharp notch in a work-hardening elastic-plastic material loaded by longitudinal shear, Journal of Applied Mechanics, Vol. 33, 1967, pp 287-298

[3] J. R. Rice, Mathematical Analysis in the mechanics of fracture: In fracture an Advanced Treatise (H. Liebowitz, Ed.) Vol. II, Academic Press, New York, 1968, pp 191-311

[4] J. F. Knott, Fundamentals of fracture mechanics, John Wiley and Sons, New York, 1973, pp 29-31

[5] U. B. C. O. Ejike, Symmetrically notched elastic bar. International Journal of England, 1973, Vol II, pp 1175 –1183.

[6] J. N. Nnadi, A Neuman Problem for an elastic cylinder under out-of-plane loading, Global Journal of Pure and Applied Sciences, Vol. 7, No. 3, 2001, pp 541-544.

[7] J. N. Nnadi, Longitudinal shear deformation of a composite cylinder sectionally loaded across interface edges, Journal of Nigerian Association of Mathematical Physics, Vol 6., 2002.

[8] E. H. Hwang, et al., Inclined edge crack in two bonded elastic quarter planes under out-of-plane loading. International Journal of Fracture, 1992, 56, pp R39-R49.

[9] R. V. Church et al., Complex variables and applications McGraw Hill Kogakusha Ltd. Tokyo, 1974, p 161.