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Dynamic analysis of a thermal–induced stress in an elastic circular plate

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Abstract

Stress is a phenomenon that could cause a lot of destruction to engineering structures if there are no adequate in-built absorbers in such structures. Buildings, bridges and such other structures must therefore be protected from excessive stress in other to maintain their shapes and forms and hence to guarantee the life span of these structures because of the negative effects this phenomenon may have on them if left unchecked. In this work we study the magnitude of a thermal-induced stress in a circular elastic plate of radius b with an indented circular hole of radius a at the center. The magnitudes of the normal and tangential components for various span ratio $\frac{a}{b}$ ***are computed. The results of this analysis show a definitive relationship between the stress profile and variability in span ratio.***

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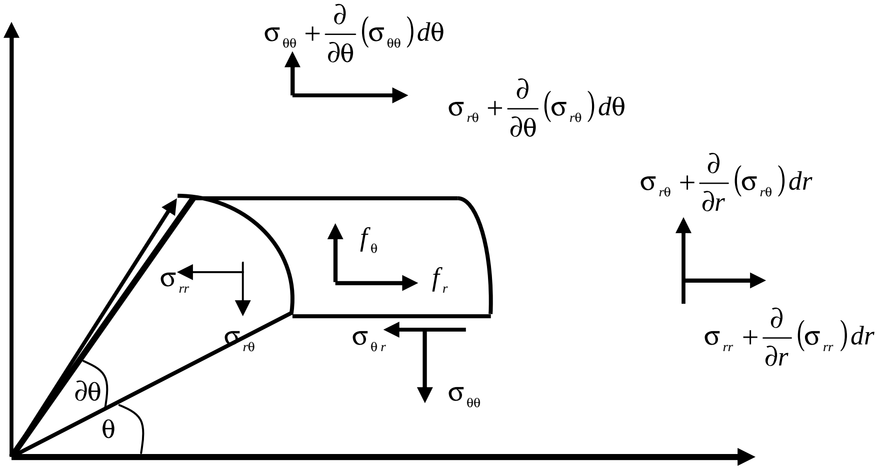
1.0 Introduction

Elasticity is a special branch of mathematics that pertains to the study of the relationship between the external forces acting on a body and the resultant deformation on the body. A body is in general said to be elastic if it regains its form completely when the external force is removed. Several problems of physical importance emanate from elasticity and hence this constitutes a very versatile area of research interest and therefore there exists a very considerable literature in this area. Aiyesimi [1,2] had earlier studied the effect of imposition of external forces on the dynamics state of some elastic materials. Under some specific physical constraints vibrations result as a bye-product of imposition of external forces on such structures. Thermodynamics on the other hand can simply be put as the study involving the effect of heat transfer on bodies. One of the commonest effect of applying heat to a structure is expansion as a direct of increase in the thermal a and external radius b as a result of the variation in the thermal state are examined for various span ratio $\frac{a}{b}$

2.0 Mathematical formulation

We consider a circular elastic plate of radius b with an indented circular hole of radius a in the centre ($b > a$)



From the free- body diagram above the equilibrium of forces in the radial direction requires that;

$$-\sigma_{\theta\theta} dr\, sin\left(\frac{d\theta}{2}\right)+\left[\sigma_{r\theta}+\frac{\partial}{\partial\theta}(\sigma_{r\theta})d\theta\right]dr\, cos\left(\frac{d\theta}{2}\right)-\sigma_{r\theta} dr\, cos\left(\frac{d\theta}{2}\right)-r\sigma_{rr} d\,\theta$$

$$+\left[\sigma_{rr}+\frac{\partial}{\partial r}(\sigma_{rr} d_r)\right](r+dr)d\,\theta-\left[\sigma_{\theta\theta}+\frac{\partial}{\partial\theta}(\sigma_{\theta\theta} d\theta)\right]d\,r\, sin\left(\frac{d\theta}{2}\right)+rf_r drd\theta=0 \qquad (2.1)$$

Recalling that both dr and $d\,\theta$ are infinitesimally small elements we observe that

$$cos\left(\frac{d\theta}{2}\right)=1 \text{ and } sin\left(\frac{d\theta}{2}\right)=\frac{d\theta}{2} \qquad (2.2)$$

and hence (2.1) above reduces to;

$$-\sigma_{\theta\theta} dr\left(\frac{d\theta}{2}\right)+\left[\sigma_{r\theta}+\frac{\partial}{\partial\theta}(\sigma_{r\theta})d\theta\right]dr-\sigma_{rr} dr-r\sigma_{rr} d\theta$$
$$+\left[\sigma_{rr}+\frac{\partial}{\partial r}(\sigma_{rr} dr)\right](r+dr)d\theta-\left[\sigma_{\theta\theta}+\frac{\partial}{\partial\theta}(\sigma_{\theta\theta} d\theta)\right]dr\left(\frac{d\theta}{2}\right)+rf_r drd\theta=0 \qquad (2.3)$$

On dividing through (2.2) by $r\,drd\,\theta$ and taking the first-order terms (neglecting higher orders) then we have the following partial differential equations in the axial direction:

$$\frac{\partial}{\partial r}(\sigma_{rr})+\frac{1}{r}\frac{\partial}{\partial r}(\sigma_{r\theta})+\frac{1}{r}(\sigma_{rr}-\sigma_{\theta\theta})+f_r=0 \qquad (2.4)$$

Similarly, the resultant forces acting in the tangential direction are governed by the differential equation;

$$\frac{1}{r}\frac{\partial}{\partial\theta}(\sigma_{\theta\theta})+\frac{\partial}{\partial\theta}(\sigma_{rr})+\frac{2}{r}\sigma_{r\theta}+f_\theta \qquad (2.5)$$

In view of the axial symmetry of the system and the fact that both the radial force and the shearing stresses vanish the two governing differential equations above reduce to the single partial differential equation;

$$\frac{\partial}{\partial r}(\sigma_{rr})+\frac{1}{r}(\sigma_{rr}-\sigma_{\theta\theta})=0 \qquad (2.6)$$

From the stress- strain relationship Hooke's law [3] we have $\varepsilon_{ii}=\frac{\sigma_{ii}-\gamma\sigma_{jj}}{E}\quad i,j=1,2,3$ (2.7)

where ε is the strain, E is the Young modulus of the material of the plate and γ is the Poisons ratio. In view of the thermal effect accompanied with expansion [4] we therefore have a modified form of (2.7) given as;

$$\varepsilon_{ii}-\alpha T=\frac{\sigma_{ii}-\gamma\sigma_{jj}}{E} \qquad (2.8)$$

The above equation (2.8) gives a tensor representation and so in our present 2-*D* polar we have the following equivalent representation: $\varepsilon_{rr}=\alpha T=\frac{\sigma_{rr}-\gamma\sigma_{\theta\theta}}{E}$ *and* $\varepsilon_{\theta\theta}-\alpha T=\frac{\sigma_{\theta\theta}-\gamma\sigma_{rr}}{E}$ (2.9)

In the above equation α and T are respectively the coefficient of thermal expansion and the variable temperature of the plate. This in effect results in the following expressions for the stress components:

$$\sigma_{rr}=\frac{E[\varepsilon_{rr}+\gamma\varepsilon_{\theta\theta}-(1+\gamma)\alpha T}{1-\gamma^2},\ \sigma_{\theta\theta}=\frac{E[\varepsilon_{\theta\theta}+\gamma\varepsilon_{rr}-(1+\gamma)\alpha T}{1-\gamma^2} \qquad (2.10)$$

We also recall from [3] that the radial displacement function u satisfies the following relationship:

$$\varepsilon_{rr}=\frac{du}{dr} \quad and \quad \varepsilon_{\theta\theta}=\frac{u}{r} \qquad (2.11)$$

By virtue of (2.10) above (2.6) therefore becomes; $\frac{\partial}{\partial r}(\varepsilon_{rr}+\varepsilon_{v\theta})+(1-\gamma)(\varepsilon_{\theta\theta}-\varepsilon_{\theta\theta})=(1+\gamma)\alpha r\frac{dT}{dr}$, which on substitution of (2.11) results in the following $\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}=(1+\gamma)\alpha T\frac{dT}{dr}$, that is,

$$\frac{d}{dr}\left[\frac{1}{d}\frac{d}{dr}(u\,r)\right]=(1+\gamma)\alpha T\frac{dT}{dr} \qquad (2.12)$$

On integrating the differential equation in (2.12) we obtain

$$\left.\begin{aligned} &u\,r=(1+\gamma)\alpha\int_a^r \xi T d\,\xi+\frac{1}{2}\beta r^2+\delta,\quad ur=(1+\gamma)\alpha\int_a^r \xi T\,d\,\xi+\frac{1}{2}\beta r^2+\delta \\ &u(r)=\frac{1}{r}(1+\gamma)\alpha\int_a^r \xi T\,d\,\xi+\frac{1}{2}\beta r+\frac{\delta}{r} \end{aligned}\right\} \qquad (2.13)$$

In view of (2.9) and (2.13) we therefore have the following expressions for the two strain components:

$$\varepsilon_{rr}=-\frac{\alpha}{r^2}(1+\gamma)\int_a^r \xi T\,d\,\xi+\frac{\beta}{2}-\frac{\delta}{r^2}+(1+\gamma)\alpha T \;\; and \;\; \varepsilon_{\theta\theta}=\frac{\alpha}{r^2}(1+\gamma)\int_a^r \xi T\,d\,\xi+\frac{\beta}{2}-\frac{\delta}{r^2}$$

Finally we have the following expressions for the stress components:

$$\sigma_{rr}=-\frac{\alpha E}{r^2}\int_a^r \xi T\,d\,\xi+\frac{E}{1-\gamma}\left[\frac{\beta}{2}-\frac{\delta}{r^2}\right] \;\text{and}\; \sigma_{\theta\theta}=\frac{\alpha E}{r^2}\int_a^r \xi T\,d\,\xi\frac{E}{1-\gamma}\left[\frac{\lambda}{2}-\frac{\chi}{r^2}\right]-\alpha E T$$

where β,δ,λ and χ are all arbitrary constants of integration. Boundary condition. The temperature of this plate is maintained at zero at the outer boundary and kept at T_o at the inner boundary. This temperature model is satisfied by; $T=\frac{\left(1-\frac{r^2}{b^2}\right)b^2T_0}{b^2-a^2}$, so that

$$a_{rr}=-\frac{\alpha E T_o}{(b^2-a^2)r^2}\left[\frac{r^2-a^2}{2}-\frac{r^4-a^4}{4b^2}\right]+\frac{E(1+\gamma)\left[\frac{\beta}{2}-\frac{\delta}{r^2}\right]}{1-\gamma^2}.$$

Recalling that the normal axial stress vanish at the boundary ($r = a,b$) we thus have

$$\beta=\frac{\alpha(1-\gamma)T_0}{2}+\delta=\frac{\alpha(1-\gamma)a^2T_0}{4} \qquad (2.17)$$

On substitution and rearranging we therefore have the final expression defining σ_{rr} given by

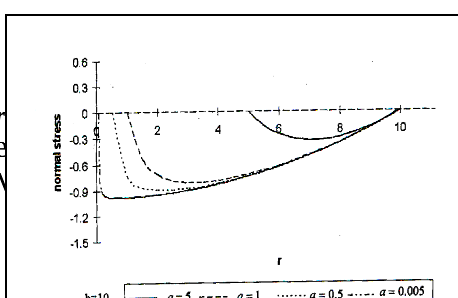
$$\sigma_{rr}=\frac{E\alpha T_0}{4}\left[\frac{\left(1-\frac{a^2}{r^2}\right)(r^2-b^2)}{b^2-a^2}\right] \qquad (2.18)$$

Finally, applying appropriate boundary conditions the tangential stress $\sigma_{\theta\theta}$ is given by the expression

$$\sigma_{\theta\theta}=\frac{E\alpha T_0}{4(b^2-a^2)}\left[\frac{\left(1-\frac{a^2}{r^2}\right)(r^2-2b^2+a^2)}{a^2-r^2}+(r^2-a^2)\right] \qquad (2.19)$$

3.0 Numerical examples

In what follows the stress profiles (both normal and tangential stress components) on the plate are computer for various span ratio $\frac{b}{a}$ thus comparing the magnitudes of the profiles with changing span ratio



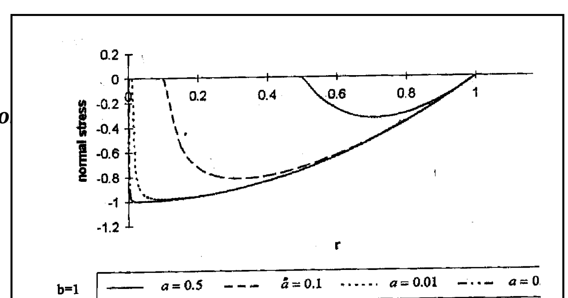
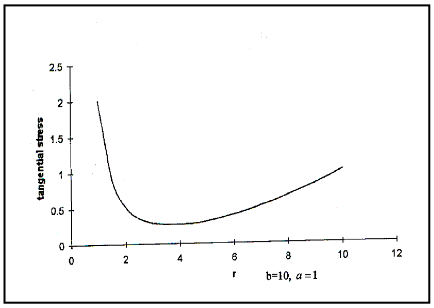


Figure 1a

Figure 1a



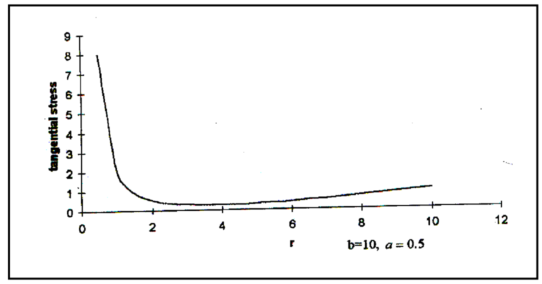
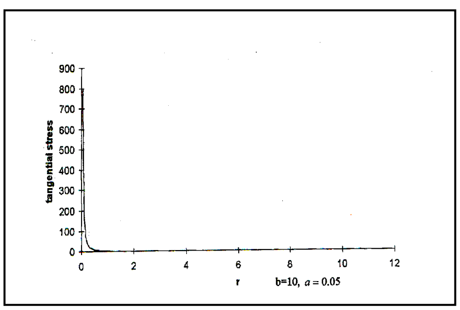


Figure 2a

Figure 2b



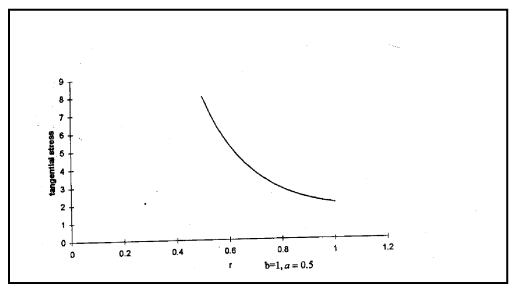
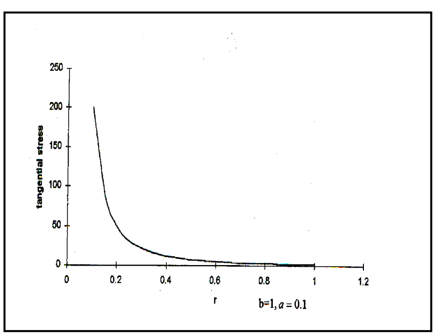


Figure 2c

Figure 3a



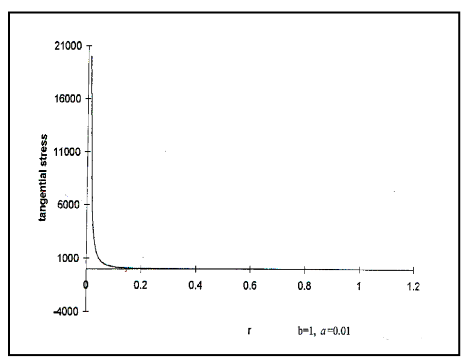


Figure 3b

Figure 3c

4.0 Discussions of results

In Figure 1a we have the profiles for normal stress when b = 10 for the following values of a: 5.0, 1.0, 0.50 and 0.05. On the other hand Figure 1b represents normal stress profiles for b = 1.0 for the values a = 0.05, 0.1, 0.01. Figures 2a, 2b, and 2c demonstrate the tangential stress profiles for the values of a 1.0, 0.50 and 0.05 corresponding to value b = 10 finally. The Figures in 3a, 3b, and 3c represent the tangential stress profiles for a = 0.0, 0.01 corresponding to the value b = 10

5.0 Conclusion

From the stress profiles demonstrate d in the figures above the following findings are immediate:

(1) The amplitude of stress increases with the increasing ratio $\frac{b}{a}$. In the case of tangential stress this rapidly with the ratio and tends to $\infty \; as \frac{b}{a} \to \infty$.

(2) The profile is fairly distributed within the annulus and is actually quadratic for normal stress with ratio $\frac{b}{a}$ is of the order 10^{-1}.

(3) The distribution is concentrated within the innermost part of the annulus when the ration $\frac{b}{a} \to 0$

(4) The amplitude of the stress increases as the values of $b \to 0$

(5) For similar ratio the amplitude of tangential stress ϕϕ amplitude of normal stress.

References

[1] Aiyesimi, Y. M. The dynamic analysis of an elastic plate with point–moving load journal of Nigerian Association of Mathematical physics. Vol 3, 249-257, 1999

[2] Aiyesimi, Y. M. The dynamic response of an elastic plate to a longitudinally moving concentrated load. The Nigeria Geographical Teacher's Journal. Vol. 2, 123-130, 1999

[3] Spiegel, M. R., M. R applied differential equations, Prentice Hall, 1985

[4] Welty, J. R. Fundamentals of momentum, heat and mass transfer, second edition. John Willey and Sons .NY.