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Gravitational fields of prolate spheroidal bodies extension of gravitational fields of spherical bodies.

[1]**E. F. Musongong** and [2]**S. X. K. Howusu**
[1]**Department of Physics, Nasarawa State University, Keffi, Nigeria**.
e-mail>musongong @yahoo.co.uk
[2]**Department of Physics, University of Jos, Nigeria** .
e-mail>howusus@yahoo.co.uk

Abstract

The expressions for the gravitational fields of spherical bodies are well known. In this paper we derive the exact expressions for a homogenous massive prolate spheroidal, an extension of the gravitational fields of spherical body for investigations and applications.

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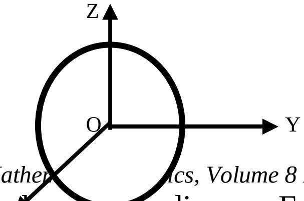
1.0 Introduction.

Some five and a half decades of years ago, the theoretical study of gravitational fields was a matter almost exclusively treated to fields of massive bodies of perfectly spherical geometry, simply because of mathematical convenience. An example is seen in the applications of Newton's Dynamical Theory of Universal Gravitation (NDTUG) in the treatment of the motion of particles (such as projectiles, satellites, penduli and even gas molecules) and the earth is treated under the general assumptions that the earth is a perfect sphere. Similarly, in the solar system the motion of bodies (such as comets, planets, asteroids and stars) is treated exclusively under the general assumption that the sun and these bodies are perfectly spherical in shape. In the same light, Einstein's Theory of Gravitation called General Relativity Theory (GRT), the motion of bodies (such as planets) and particles (such as photon) is treated under the assumption that the sun is exclusively a perfect sphere (the Schwarz child's space-time). But the real fact of nature is that all rotating planets, stars and galaxies in the universe are either oblate spheroidal or prolate spheroidal in shape.

Hence, we hereby prepare the way for the study of motion of all particles or bodies in the gravitational fields of prolate spheroidal geometry by deriving the expressions of the gravitational potentials. It is known that satellites orbits around the earth are governed by NDTUG and the second harmonics (pole of order 3), as well as fourth harmonics (pole of order 5) of gravitational scalar potential due to imperfect geometry. In 1952, Jeffreys [1] suggested the fourth harmonics, which yielded amplitude of only 86% of the value obtained by King-Hele and Merson [2] from the analysis of data on satellite orbits. In 1959, O'Keefe, Eckels, and Squires [3] improved on Hele and merson result using equatorial asymmetry for spherical shape. A year latter, Vinti [4] got a very good approximation of the second harmonics which reduced the problem of Sterne [5] and Garfinkel [6] in quadratures by applying oblate spheroidal coordinates to investigate the motion of an earth satellite. Yet there are still natural occurring particles whose geometrical shapes are prolate spheroidal (such as rain drops) and their geometry will have corresponding consequences and effects in the motion of all particles in their gravitational fields as pointed out in a paper [7].

2.0 Theoretical analysis

Consider a homogenous prolate spheroidal body of rest mass M_0. OZ points along the polar axis and OX towards the vernal equinox, the prolate spheroidal coordinates (η,ξ,ϕ) defined as shown in the figure by



x
and

$$x = a\left[\left(1-\eta^2\right)\left(\xi^2-1\right)\right]^{\frac{1}{2}} \cos\phi \tag{2.1}$$

$$y = a\left[\left(1-\eta^2\right)\left(\xi^2-1\right)\right]^{\frac{1}{2}} \sin\phi \tag{2.2}$$

$$z = a\eta\xi \tag{2.3}$$

where a is a constant and $\{-1 \le \eta \le 1;\ 0 \le \xi < \infty;\ 0 \le \phi \le 2\pi\}$ (2.4)

Since the body is homogenous, the density of active mass, ρ, is given by

$$\rho(r) = \begin{cases} \rho_0, & \xi \le \xi_0 \\ 0 & \xi > \xi_0 \end{cases} \tag{2.5}$$

where ρ_0 is the constant density of rest mass and the gravitational scalar potential of the body $\Phi_g(\eta,\xi,\phi)$ is static and hence satisfies the field equation

$$\nabla^2 \Phi_g(\eta,\xi,\phi) = 4\pi G\rho(\eta,\xi,\phi) \tag{2.6}$$

The interior and exterior scalar potentials are given by

$$\frac{1}{a^2\left(\xi^2-\eta^2\right)}\left\{\frac{\partial}{\partial\eta}\left(1-\eta^2\right)\frac{\partial}{\partial\eta}+\frac{\partial}{\partial\xi}\left(\xi^2-1\right)\frac{\partial}{\partial\xi}+\frac{\xi^2-\eta^2}{\left(\xi^2-1\right)\left(1-\eta^2\right)}\frac{\partial^2}{\partial\phi^2}\right\}\Phi_g^-(\eta,\xi,\phi) = 4\pi G\rho_0 \tag{2.7}$$

and

$$\frac{1}{a^2\left(\xi^2-\eta^2\right)}\left\{\frac{\partial}{\partial\eta}\left(1-\eta^2\right)\frac{\partial}{\partial\eta}+\frac{\partial}{\partial\xi}\left(\xi^2-1\right)\frac{\partial}{\partial\xi}+\frac{\xi^2-\eta^2}{\left(\xi^2-1\right)\left(1-\eta^2\right)}\frac{\partial^2}{\partial\phi^2}\right\}\Phi_g^+(\eta,\xi,\phi) = 0 \tag{2.8}$$

A solution of variable separable, complementary independent of the azimulthal angle ϕ is given by

$$\Phi_g(\eta,\xi) = \Omega(\eta)T(\xi) \tag{2.9}$$

Hence

$$\left\{\begin{aligned}&\frac{\partial}{\partial\eta}\left(1-\eta^2\right)\frac{\partial}{\partial\eta}\Phi^-(\eta,\xi)\\&+\frac{\partial}{\partial\xi}\left(\xi^2-1\right)\frac{\partial}{\partial\xi}\Phi^-(\eta-\xi)\end{aligned}\right\} = \left\{\begin{aligned}&\frac{1}{\Omega(\eta)}\frac{\partial}{\partial\eta}\left(1-\eta^2\right)\frac{\partial}{\partial\eta}\Omega(\eta)\\&+\frac{1}{T(\xi)}\frac{\partial}{\partial\xi}\frac{\partial}{\partial\xi}\left(\xi^2-1\right)\frac{\partial}{\partial\xi}T(\xi)\end{aligned}\right\} \tag{2.10}$$

Rearranging and introducing a separation constant to the lap lace equation, we have

$$\frac{\partial}{\partial\eta}\left[\left(1-\eta^2\right)\frac{\partial}{\partial\eta}\Omega(\eta)\right]+\lambda\Omega(\eta) = 0 \tag{2.11}$$

and

$$\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)\frac{\partial}{\partial\xi}T(\xi)\right]-\lambda T(\xi) = 0 \tag{2.12}$$

where $\lambda = l(l+1);\ l = 0,1,2,\Lambda$. The solution of equation (2.11) and (2.12) of the legendre's differential equation are

$$\Omega(\eta) = \begin{cases} p_1(\eta) \\ Q_1(\eta) \end{cases} \tag{2.13}$$

and

$$T(\xi) = \begin{cases} p_1(\xi) \\ Q_1(\xi) \end{cases} \tag{2.14}$$

Consequently,

$$\Phi_g(\eta,\xi)=\left\{\frac{2}{3}\pi\, p_0 a^2\left(\eta^2+\xi^2\right)+\sum_{1=0}^{\infty}\left[A_l^- p_1(\xi)+B_l^- Q_1(\xi)\right]\left[C_l^- P_1(\eta)+D_l^- Q_1(\eta)\right]\right\} \qquad (2.15)$$

Where $A_1^-, B_1^-, C_1^-, D_1^-$ constants and P_1 Q_1 are the Legendre's functions of order 1. Similarly the exterior homogeneous part has a solution given by

$$\Phi_g(\eta,\xi)=\sum_{l=0}^{\infty}\left[A_l^+ p_l(\xi)+B_1^+ Q_1(\xi)\right]\left[C_l^+(\eta)+D_l^+ Q_1(\eta)\right] \qquad (2.16)$$

Now since the interior and exterior regions both contain the coordinates $\eta = 0$ which is a singularity of Q_1, we

choose $D_l^- = D_l^+ = 0$ for $l = 0,1,2\ldots$ (2.17)

in the general solutions (2.15) and (2.16). Also since $\xi = 0$ is singularity of Q_1, we choose

$$B_l^- = 0; \text{ for } l = 0,1,2,\ldots \qquad (2.18)$$

Also since p_1 is not defined $\xi \to \infty$ in the exterior region, we choose

$$A_l^+ = 0; \text{ for } l=0,1,2\ldots. \qquad (2.19)$$

Next, the conditions of the continuity of the potentials and their normal derivatives at $\xi = \xi_0$ (boundary of the spheroid), it follow that

$$B_0^+ = \frac{4\pi G\, p_0 a^2 \xi_0^2}{3\left[\frac{d}{d\xi}Q_0(\xi)\right]_{\xi=\xi_0}} \qquad (2.20)$$

and

$$A_0^+ = -\frac{2}{3}\pi G\, p_0 a^2\left[\xi_0^2+\frac{1}{3}\right]+\frac{4\pi G\, p_0 a^2\, \xi_0^2}{3\left[\frac{d}{d\xi}Q_0(\xi)\right]_{\xi=\xi_0}} \qquad (2.21)$$

and $B_1^+ = A_1^- = 0;$ and

$$B_2^+ = \frac{4\pi G p_0 a^2\left(\frac{d}{d\xi}p_2(\xi)\right)_{\xi=\xi_0}}{9\left\{Q_2(\xi)\left[\frac{d}{d\xi}p_2(\xi)\right]_{\xi=\xi_0}-P_2(\xi)\left[\frac{d}{d\xi}Q_2(\xi)\right]_{\xi=\xi_0}\right\}} \qquad (2.22)$$

$$A_2^- = \frac{4\pi G p_0 a^2\left(\frac{d}{d\xi}p_2(\xi)\right)_{\xi=\xi_0}}{9\left\{Q_2(\xi)\left[\frac{d}{d\xi}p_2(\xi)\right]_{\xi=\xi_0}-P_2(\xi)\left[\frac{d}{d\xi}Q_2(\xi)\right]_{\xi=\xi_0}\right\}} \qquad (2.23)$$

$$A_l^- = B_l^+ = 0 \text{ for } l = 3,4,5,\Lambda \qquad (2.24)$$

It follows that from (2.15) and (2.16) and (2.20) - (2.23), the potentials are given by

$$\Phi_g^-(\eta,\xi)=\frac{2\pi G\rho_0 a^2}{3}\left\{\begin{array}{l}\xi^2+\eta^2-\left(\xi_0^2+\frac{1}{3}\right)+\frac{2\xi_0 Q_0(\xi)P_0(\xi)P_0(\eta)}{\left[\frac{d}{d\xi}Q_0[\xi]\right]_{\xi=\xi_0}}+\\ +\frac{2\left[\frac{d}{d\xi}Q_2(\xi)\right]_{\xi=\xi_0}P_2(\xi)P_2(\eta)}{3\left[Q_2(\xi)\left\{\frac{d}{d\xi}P_2(\xi)\right\}_{\xi=\xi_0}-P_2(\xi)\left\{\frac{d}{d\xi}Q_\xi(\xi)\right\}_{\xi=\xi_0}\right]}\end{array}\right\} \tag{2.25}$$

and

$$\Phi_g^+(\eta,\xi)=\frac{4\pi GP_0 a^2}{3}\left\{\begin{array}{l}\frac{\xi_0^2 Q_0(\xi)P_0(\xi)P_0(\eta)}{\left[\frac{d}{d\xi}Q_0[\xi]\right]_{\xi=\xi_0}}\\ +\frac{\left[\frac{d}{d\xi}p_2(\xi)\right]_{\xi=\xi_0}Q_2(\xi)P_2(\eta)}{3\left[Q_2(\xi)\left\{\frac{d}{d\xi}P_2(\xi)\right\}_{\xi=\xi_0}-P_2(\xi)\left\{\frac{d}{d\xi}Q_2(\xi)\right\}_{\xi=\xi_0}\right]}\end{array}\right\} \tag{2.26}$$

The gravitational scalar potentials (2.25) and (2.26) can be expressed in terms of the rest mass M_0 of the prolate spheroidal body is given by

$$M_0=\frac{4}{3}a^3P_0\xi_0\left(\xi_0^2+1\right) \tag{2.27}$$

3.0 Summary and Conclusion.

In this paper we formulated and solved Newton's universal gravitational potential field equations for a homogeneous prolate spheroidal body, with the exact and complete results given by equations (2.25) and (2.26). These results are available for application in physics. For example, raindrops are prolate spheroidal or oblate spheroidial in geometry and these imperfect geometry may have its corresponding consequences and effects. It is certain that its investigations and applications will yield plausible results.

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