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Gravitational time dilation and spectral shift in the field of a massive oblate spheroidal body.

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Abstract

In this paper, we derive expressions for the time dilation and spectral shift in terms of proper time and proper frequency in the field of a massive oblate spheroidal body using an approximate value of $g_{\mu\nu}$

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1.0 Introduction.

In a certain sense, all measurements give dimensionless numbers since the measured values are compared with units defined by other measurement. Time for instance is measured by comparing an interval of time with the natural oscillation time of some physical system, e.g.; the period of light emitted by a well-defined transition in an atom or molecule. The results of such measurements can be compared by two observers in different modes: i.e they, may both compared with local standard frequencies or they may transmit the signal emitted by local clocks to each other such that direct comparison is enhanced. Experiments show that if all external fields are made equal in two systems that are equally accelerated, e.g falling freely without rotation, the measurements will show the same dimensionless numbers as observed by Lars Falk [1]. This shows that if physically different clocks in the same place and time, their measurements will be consistent. A direct comparison between clocks at different places, there is very little reason to assume that they will give the same results. Since clocks are physical objects, they are expected to be sensitive to external conditions, such as the presence of massive bodies. When universal time was introduced by Newton, it was consequently criticized as being too abstracts as edited in the Principia [2] as "absolute true and mathematical time of itself, and from its own nature flows equally without relation to anything external, …." The contest between homogeneity of time and space as Landau [3] in 1976 puts it " it would, however, be much more natural to say that the motion of a point particle test the homogeneity of space and time'. This statement prompted other physicist to carryout experiments and such experiments show that clocks move at different rates at different distances from a massive body, e.g; the star. That is time is in fact inhomogeneous. A well- known example is provided by the red shift of spectral lines from massive stars. These measurements are rather uncertain and have been improved by the well-known Mossbauer experiments on earth. All these experiments suggest, that the flow of time is slowed down close to a massive body.

2.0 Analysis

According to G. Arfken and Hildebrand [4, 5] for an oblate spheroidal coordinates of space (η,ξ,ϕ) are defined in terms of the Cartesian coordinates (x,y,z) as

$$x = a\left[\left(1-\eta^2\right)\left(1+\xi^2\right)\right]^{\frac{1}{2}}\cos\phi \tag{2.1}$$

$$y = a\left[\left(1-\eta^2\right)\left(1+\xi^2\right)\right]^{\frac{1}{2}}\sin\phi \tag{2.2}$$

$$z = a\eta\xi \tag{2.3}$$

where a is a constant parameter and $-1 \le \eta \le 1,\ 0 \le \xi < \infty,\ 0 \le \phi \le 2\pi$ (2.4)

and the surface is given by the equation $\xi = \xi_0$ (2.5) for some constant ξ_0.

It follows that, for a flat space of the invariance of the line element ds^2, the covariant metric tensor in oblate spheroidal coordinates is given by $g_{\infty} = 1$ (2.6)

$$g_{11} = -\frac{a^2\left(\eta^2 + \xi^2\right)}{\left(1-\eta^2\right)} \quad (2.7)$$

$$g_{22} = \frac{a^2\left(\eta^2 + \xi^2\right)}{\left(1+\xi^2\right)} \quad (2.8)$$

$$g_{33} = -a^2\left(1-\eta^2\right)\left(1+\xi^2\right) \quad (2.9)$$

$$g_{\mu\nu} = 0, \quad otherwise \quad (2.10)$$

In the paper [6] the covariant metric tensor for a massive oblate spheroidal body is derived to be

$$g_{00} = e^{-F} \quad (2.11)$$

$$g_{11} = -\ e^{-G} \quad (2.12)$$

$$g_{11} = -e^{-H} \quad (2.13)$$

$$g_{33} = -a^2\left(1-\eta^2\right)\left(1+\xi^2\right) \quad (2.14)$$

where F,G and H are functions of η *and* ξ only. It follows that for a massive body of invariance of line element ds^2 is given by $ds^2 = g_{00}dt^2 + g_{11}d\eta^2 + g_{22}d\xi^2 + g_{33}d\phi^2$ (2.15)

By definition, the proper time element $d\tau$ is given in terms of the metric tensor exterior to the oblate spheroidal body as $c^2 d\tau^2 = g_{00}c^2(dt)^2 + g_{11}(d\eta)^2 + g_{22}(d\xi)^2 + g_{33}(d\phi)^2$
(2.16)

where (t,η,ξ,ϕ) are the spheroidal coordinates of space time. Consequently, for a fixed position, i.e., (η,ξ,ϕ) constant, $0 = d\eta = d\xi = d\phi$ (2.17)

Now, it is well-known that a good approximation for g_{00} in any gravitational filed is given by moller and Anderson [7, 8] as $g_{00} = \left(1 + \frac{2}{c^2}\Phi_g\right)$ (2.18)

where Φ_g is the corresponding universal gravitational scalar potential. From the paper [6] Φ_g s derived to be

$$\Phi_g\left(\eta\xi\phi\right) = B_0^+ Q_{0(-i\xi)}P_{0(\eta)} + B_2^+ Q_{2(-i\xi)}P_{2(\eta)} \quad (2.19)$$

where P_i and Q_i are the two linearly independent Legendre functions of the order

$$i = 0, 1,2 \quad (2.20)$$

Consequently, by the condition of continuity of the potentials and their normal derivatives at the boundary of the spheroid, it follows [9] that at $\xi = \xi_0,$ $B_0^+ = \dfrac{-4\pi G\, p_0 a^2 \xi_0}{3\left[\dfrac{d}{d\xi}Q_{0(-i\xi)}\right]_{\xi=\xi_0}}$ (2.21)

and

$$B_2^+ = \frac{4\pi G p_0 a^2 \left[\dfrac{d}{d\xi}p_{2(-i\xi)}\right]_{\xi=\xi_0}}{9\left\{Q_{2(-i\xi)}\left[\dfrac{d}{d\xi}p_{2(-i\xi)}\right]_{\xi=\xi_0} - p_{2(-i\xi)}\left[\dfrac{d}{d\xi}Q_{2(-i\xi)}\right]_{\xi=\xi_0}\right\}} \quad (2.22)$$

But

$$Q_{0(-i\xi)} = i\left(\xi^{-1} + \frac{1}{3}\xi^{-3} + \frac{1}{5}\xi^{-5} + ...\right) \quad (2.23)$$

and

$$Q_{2(-i\xi)} = -\frac{i}{2}\left\{2\,\xi^{-1} + \left(\frac{1}{3}+\frac{3}{5}\right)\xi^{-3} + \left(\frac{1}{5}+\frac{3}{7}\right)\xi^{-5}....\right\} \qquad (2.24)$$

The universal gravitational potential of an oblate spheroidal massive body contained expressions in terms of imaginary arguments are everywhere real values. It follows from equations (2.18) and (2.12) - (2.24) that for a clock in the gravitational field of an oblate spheroidal massive body

$$dt = \left[1 - \frac{2}{c^2}\left[\frac{4\pi\,\rho_0 a^2 \xi_0}{3i}\right]\left\{\frac{-Q_{0(-i\xi)}\,p_{0(\eta)}}{\frac{d}{d\xi}\left(\xi^{-1}+\frac{1}{3}\xi^{-3}+\xi^{-5}..\right)_{\xi=\xi_0}}\;\frac{-2\left[\frac{d}{d\xi}\left(3\cos^2(-i\xi)-1\right)\right]_{\xi=\xi_0} Q_{2(-i\xi)}\,p_{2(\eta)}}{3\left\{\begin{array}{l} 2\,\xi^{-1} + \left(\frac{1}{3}+\frac{3}{5}\right)\xi^{-3} + \left(\frac{1}{5}+\frac{3}{7}\right)\xi^{-5}... \\ x\left[\frac{d}{d\xi}\left(3\cos^2(-i\xi)-1\right)\right]_{\xi=\xi_0} \\ -\left[3\cos^2 -1\right]\left[\frac{d}{d\xi}\left(\begin{array}{l}2\xi^{-1}+\left[\frac{1}{3}+\frac{3}{7}\right]\xi^{-3} \\ +\left(\frac{1}{5}+\frac{3}{7}\right)\xi^{-5}....\end{array}\right)_{\xi=\xi_0}\right]\end{array}\right\}}\right\}\right]^{\frac{-1}{2}} \qquad (2.25)$$

Equation (2.25) is an approximate gravitational time dilation formula in the field exterior to an oblate spheroidal massive body. Also, it is well known that the frequency of a clock is inversely proportional to its period and hence its spectral shift in terms of proper frequency v_0 is given by

$$v = 1 - \frac{2}{c^2}\left[\frac{4\pi\,\rho_0 a^2 \xi_0}{3i}\right]\left\{\frac{\dfrac{-Q_{0(-i\xi)}\,p_{0(\eta)}}{\dfrac{d}{d\xi}\left(\xi^{-1}+\dfrac{1}{3}\xi^{-3}+\xi^{-5}\,..\right)_{\xi=\xi 0}} - 2\left[\dfrac{d}{d\xi}\left(3\cos^2(-i\xi)-1\right)\right]_{\xi=\xi_0} Q_{2(-i\xi)}\,p_{2(\eta)}}{3\left\{\begin{array}{l}\left[2\xi^{-1}+\left(\dfrac{1}{3}+\dfrac{3}{5}\right)\xi^{-3}+\left(\dfrac{1}{5}+\dfrac{3}{7}\right)\xi^{-5}\ldots\right]\\ x\left[\dfrac{d}{d\xi}\left(3\cos^2(-i\xi)-1\right)\right]_{\xi=\xi_0}\\ -\left[3\cos^2-1\right]\left[\dfrac{d}{d\xi}\left(\begin{array}{l}2\xi^{-1}+\left[\dfrac{1}{3}+\dfrac{3}{7}\right]\xi^{-3}\\ +\left(\dfrac{1}{5}+\dfrac{3}{7}\right)\xi^{-5}\ldots.\end{array}\right)_{\xi=\xi_o}\right]\end{array}\right\}}\right\}^{\frac{1}{2}} v_0 \quad (2.26)$$

The gravitational scalar potential of rest mass M_0 can be expressed as

$$M_0 = \frac{4}{9}\pi a^3\ \rho_0 \xi_0 \left(3+\xi_0^2\right) \quad (2.27)$$

3.0 Summary and Conclusion.

In this paper, we derived the expressions for the time dilation and spectral shift of light in the gravitational field of a massive oblate spheroidal body as (2.25) and (2.26). The expressions contained complex arguments, which are real values everywhere in terms of the new spheroidal coordinates (η,ξ,ϕ) with very

many corrections and need physical interpretations and investigations. Little attention has been given to other orthogonal coordinates such as oblate spheroidal, prolate spheroidal, toroidal whose applications may yield plausible results to most of our unsolved physical problems.

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