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Compactness of cores of targets for nonlinear delay systems

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Abstract

The purpose of this study is to investigate the compactness of cores of targets for nonlinear delay systems. Our results are obtained by exploiting the non-singularity of the fundamental matrix for the homogeneous part of the system and its "conjugate" equation. Hajek's arguments in* [4] *of the notion of asymptotic direction and other concepts of convex set theory stand monumental in the development of this study. With a perturbation function, satisfying a smoothness condition – growth condition. A relationship is established between the boundedness of cores of targets and the Euclidean controllability of the nonlinear system. This relationship gives vent to the establishment of the compactness of cores of target for the system. We complement Ukwu* [9] *and Chukwu* [1] *by answering in the affirmative that under certain smoothness conditions, the compactness of cores of targets for a linear system guarantees the compactness of cores of target for the linear perturbation.

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1.0 Introduction

A core of target' is the set of initial states of a system that can be steered to the target using appropriate control. The subject of cores of targets has captured the attention of many authors in recent times as it is being understood as the seeds for our expected outcomes. Markus [8] and Hajek [4] have investigated the compactness of cores of targets for ordinary systems without delays. Chukwu [1] extended the results of Hajek and Markus to some non-linear systems. Ukwu [10] exploited the results of these earlier authors to establish the compactness of cores of targets for linear delay systems. Iheagwam [6] established a relationship between cores of targets and Euclidean controllability of linear systems. A computable criterion for the compactness of cores of targets for linear delay systems was also articulated by Ukwu in [9] and initiated effort at studying the cores of targets for perturbed linear delay systems. he present endeavor is an attempt to extend the results obtained by Ukwu to non-linear delay systems with varying arguments in the perturbation function. The system below is therefore presented for investigation.

$$\dot{x}(t) = Ax(t) + Bx(t-h) + cu(t) + f(t, x(t), x(t-h), u(t), u(t-h)) \quad (1.1)$$

$$x(t) = \phi(t),\ t \in [t_0 - h, t_0] \quad h > 0;\ u(t) = u_{t_0} \text{ for } t \in [t_0 - h, t_0]$$

where A and B are n x n constant matrices, C is an n x m constant matrix and ϕ is continuous. The control u is a measurable m-vector values u(t) constrained to lie in a compact, convex non-void subset, U of the Euclidean m-space, such a u is called admissible, x and f are n-vector functions. The target set H is a closed, convex, non-void subset of E^n. Let $W_2^{(1)}$ denote the Sobolev space $W_2^{(1)}$ $([t_0 - h, t_0], E^n)$ of continuous functions $\phi : [t_0 - h, t_0] \to E^n$ which are absolutely continuous and whose derivatives are square integrable on finite time intervals. If $x: [t_0 - h, t_1] \to E^n$, then for $t \in [t_0, t_1]$ the symbol x_t denotes the continuous function on $[t_0 - h, t_0]$ defined by $x_t(s) = x(t+s)$, $s \in [t_0 - h, t_0]$.

2.0 **Notations and preliminaries**

Definition 2.1

The system (1.1) *is said to be Euclidean controllable if for each* $\phi \in W_2^{(1)}$, $x_1 \in E^n$ *there exists a time* $t_1 > t_0$ *and an admissible control u such that the solution* $x(t, \phi, u)$ *of (1.1) satisfies* $x_{t_0}(\phi, u) = \phi$ *and* $x(t_1\ \phi u) = x_1$.

Definition 2.2

The core of the target set $H \subseteq E^n$ *denoted by core (H) consists of all initial points* $\phi(t_0) \in E^n$ *for which* $\phi \in W_2^{(1)}$ *such that there is an admissible control u for which the solution* $x = x(\phi, u)$ *of* (1.1) *satisfies* $x(t) \in G$ *for all* $t \geq$

t_0. *If* $\phi \in W_2^{(1)}$ *and u is an admissible control, then there exists a unique solution of system (1.1) for* $t > t_0$ *satisfying* $x(t) = \phi(t)$ *for* $t \in [t_0 - h, t_0]$. *This solution is given by*

$$\dot{x}(t, \phi, u) = x(t, \phi, o) + \int_{t_0}^{t} X(t-s)[cu(s) + f(t, x(s), x(s-h), u(s), u(s-h))ds] \quad (2.1)$$

where $X(t)$, the fundamental matrix of the homogeneous system given below

$$\dot{x}(t) = Ax(t) + Bx(t-h), \ t > 0 \quad a.\,e \quad (2.2)$$

satisfies

$$X(t) = \begin{cases} I & t = 0 \\ o & t < 0 \end{cases} \quad (2.3)$$

and where

$$X(t, \phi, o) = X(t)\,\phi(t_0) + B\int_{t_0-h}^{t_0} X(t-s-h)\phi(s)ds \quad (2.4)$$

Remark

As a consequence of (1.1) being autonomous $X(t, s) = X(t - s, 0) = X(t - s)$ (2.5)

For further definition of the solution matrix see ref [5] page 145. By the transformation $X(t, \phi\ 0) = T(t)\,\phi(t_0)$ (2.2) becomes

$$X(t, \phi, u) = T(t)\,\phi(t_0) + \int_{t_0}^{t} X(t-s)[cu(s) + f(s, x(s), x(s-h), u(s)), u(s-h)]ds \quad (2.6)$$

where the family $\{T(t): t \geq t_0\}$ is a semi–group of linear transformations with properties spelt out in ref [5] and [10]

Lemma 2.1 [ref [10] see proof in Section 3.0]

For any $a \in E^n$, $X(t - s)a = T(t - s)a$

3.0 **Main results**

Proposition 3.1

With the following conditions imposed on f:

(i) $f(t\ x(t), x(t-h), 0, 0) = 0$

(ii) The set $[f(t\ x(t), x(t-h), u(t)\ u(t-h): u \in U]$ is convex for all $t \geq t_0$, $x \in E^n$

(iii) f is continuous and bounded locally in u, if $0 \in H$ and $0 \in U$ then $0 \in$ core (H) and hence core (H) is non-empty with respect to system (1.1)

Proof

Since $f(.,.,.,0) = 0$, we can choose $\phi(t_0) = 0$ and $u = 0$ in (2.7) to get $x(t, \phi, 0) = 0$ for all $t > t_0$. This implies that 0 is in the target H; and so $0 \in$ core (H) and core (H) is non-empty. Sequel to the convexity of U and H and the conditions imposed on f in proposition (3.1), the convexity of core (*H*) with respect to system (*H*) becomes immediate. The closedness was proved using a weak compactness argument.

Theorem 3.1:

Under the standing assumption on the control system (1.1) core (H) is convex and closed.

Proof **(Convexity)**

Let $\phi_1(t_0)$, $\phi_2(t_0) \in$ Core (*H*). Then there correspond two admissible controls u_1 and u_2 and two solutions $x(t,\phi_1\ u_1)$ and $x(t,\phi_2\ u_2)$ of (1.1) such that $x(t, \phi_k, u_k) \in H$ for all $t \geq t_0$, $k = 1, 2$. Let $\alpha, \geq 0$ $\beta \geq 0 \ni \alpha + \beta = 1$. Then $\alpha x\ (t, \phi_1,u_1) + \beta x\ (t_1\ \phi_2\ u_2) \in H$ for all $t \geq t_0$ since H is convex. From (2.6)

$$T(t)\ (\alpha\phi_1 + \beta\phi_2)\ (t_0) + \int_{t_0}^{t} X(t-s)[cv(s) + f(s,x(s),x(s-h),v(s),v(s-h))]ds \in H,\ \forall\ t \geq t_0 \quad (3.1)$$

and $v = \alpha u_1 + \beta u_2$ is admissible. Hence $(\alpha\phi_1 + \beta\phi_2)(t_0) \in$ core (*H*). This shows that core (*H*) is convex.

(Closedness)

The set $G = \{u$: $u \in L_2^{loc}$ [L(t_0, ∞), U] is a closed, convex and bounded subset of L_2^{loc} ([t_0 ∞) E^m). Since L_2 is reflexive, we conclude that G is weakly compact. Let $\{\phi_k(t_0)\}$, $k = 1, 2, \ldots$ be a sequence of points in core (*H*) such that $\lim_{j\to\infty} \phi_k(t_0 = \phi(t_0))$. Let u_k, $k = 1, 2, \ldots$, be the appropriate corresponding admissible controls such that $X(t, \phi_k, u_k) \in H$ for all $t \geq t_0$. Since G is weakly compact there is a subsequence $\{u_{kj}\}$, $k = 1, 2, \ldots$ which converges weakly to an admissible control function $u \in$ G on $[t_0, t_1]$, $t_1 < \infty$. That is

$$\lim_{j\to\infty} \int_{t_0}^{t} X(t-s)\left[cu_{k_j} + f\left(x(s),x(s-h),u_{k_j}\right)\right] ds =$$

$$\int_{t_0}^{t} X(t-s)[cu(s) + f(x(s),x(s-h),u(s)u(s-h))]\ ds \quad (3.2)$$

Let $\{\phi_{k_j}(t_0)\}_1^\infty$ be a subsequences of $\{\phi_k(t_0)\}_1^\infty$ corresponding to $\{u_{k_j}\}_1^\infty$ then

$$X(t,\phi_{k_j} u_{k_j}) = T(t)\phi_{k_j}(t_0) + \int_{t_0}^{t} X(t-s)\left[cu_{k_j}(s) + f(x(s),x(s-h),u_{k_j}(s),u_{k_j}(s-h)\right] ds \in H \quad (3.3)$$

for all $t \geq t_0$ since H is closed. By continuity of the class of $\{T(t)\}$ we have

$$\lim_{j\to\infty} T(t)\phi_{k_j}(t_0) = T(t)\lim_{j\to\infty}\phi_{k_j}(t_0) = T(t)\phi(t_0) \quad (3.4)$$

We obtain from (3.2), (3.3) and (3.4) that

$$\lim_{j\to\infty} x(t,\phi_{k_j} u_{k_j}) = T(t)\phi(t_0) + \int_{t_0}^{t} X(t-s)[cu(s) + f(s,(s),x(s-h),u(s),u(s-h))]ds \in H \quad (3.5)$$

for all $t \geq t_0$. Therefore $\phi(t_0) \in$ core (*H*) and hence core (*H*) is closed

Definition 3.3

A point $a \in E^n$ is an asymptotic direction of a convex set $D \subseteq E^n$ if for some $X \in D$ and all $t \geq 0$ $x + ta \in D$.

Proposition 3.2

A non-void convex subset of E^n is bounded if and only if 0 is its only asymptotic direction

Proposition 3.3

If a non-void convex set D is of the form D = L + E where E is bounded and contains zero then L is a linear subspace of D and necessary coincides with the set of asymptotic directions of D.

Theorem 3.2

Under the standing hypotheses on system (1.1) $a \in E^n$ is an asymptotic direction of core (H) if and only if X (t -s)a is an asymptotic direction of H.

Proof:

From (2.7) it is deduced that

$$X(t-s, \phi, u) = T(t-s)\,\phi(t_0) + \int_{t_0}^{t+s} X(t-s-\tau)\left[cu(\tau) + f(s, x(\tau), x(\tau-h), u(\tau)), u(\tau-h)\right] d\tau \quad (3.6)$$

for fixed $t \geq t_0 + s$. Take any asymptotic direction a of core (H). Choose $\phi(t_0) \in$ core (H) so that $\phi(t_0) + a \in$ core (H) for each $\theta > a$ Choose an appropriate corresponding admissible control $u:[t_0\ \infty) \to U$ such that

$$X(t, \phi, u_\theta) = T(t-s)\left[\phi(t_0) + \theta a\right] + \int_{t_0}^{t+s} X(t-s-\tau)\left[cu(\tau) + f(\tau, x(\tau), x(\tau-h), u(\tau)), u(\tau-h)\right] d\tau \in H \qquad (3.7)$$

for $t \geq t_0 + s$. For $\theta = 0$ the proof is trivial. For $\theta \neq 0$, we divide through by θ and take limit as $\theta \to \infty$ to deduce that

$$X(t-s)a = \lim_{\theta\to\infty} \frac{1}{\theta} b_\theta \qquad (3.8)$$

for some $b_\theta \in H$. To show that X(t - s) a is an asymptotic direction of H, take any $c \in$ H, $\lambda \geq 0$. We must show that $c + \lambda x(t-s)\, a \in$ H given that (3.8) holds. Keeping λ fixed, if $\geq \lambda$ then $0 < \lambda/\theta \leq 1$ Thus

$$\left(1 - \frac{\lambda}{\theta}\right)c + \frac{\lambda}{\theta} b_\theta \in H \qquad (3.9)$$

by the convexity of H. Take limit of (3.9) as $\theta \to \infty$. Since H is closed it follows from (3.8) and (3.9) that

$$c + \lambda X(t-s)\, a \in H \qquad (3.10)$$

Therefore X(t – s)a is an asymptotic direction of H. Conversely let X(t – s)a be any asymptotic direction of H for $t \geq t_0$ + s. Then $H + \theta X(t-s)\, a \subset H$

(3.11)

for all $\theta \geq 0$. Take any $\phi(t_0) \in$ core (H) and an admissible control u_θ such that

$$X(t-s)\left[\phi(t_0) + \theta a\right] + \int_{t_0}^{t-s} X(t-s-\tau)\left[cu_0(\tau) + f(\tau, x(\tau) X(\tau-h) u_\theta(\tau) u_\theta(\tau-h))\right]$$

$$\in H + \theta X(t-s)a \quad for\ t \geq t_0 + s \qquad (3.12)$$

We therefore conclude that $\phi(t_0) + \theta a \in$ Core H since the same control holds the point $\phi(t_0) + \theta_a$ with H, showing that a is an asymptotic direction of core (H). This completes the proof.

Ukwu [9] in conjunction with Iheagwam [6] have earlier provided a fairly long proof of the Euclidean controllability of the system

$$\dot{x}(t) = A^T x(t) + B^T x(t-h) + M^T u(t) + f(t, x(t), x(t-h), u(t), u(t-h)) \quad (3.13)$$

This result will here be used to conclude the boundedness of core (H) with respect to system (1.1)

Theorem 3.3

Consider system (1.1) with its basic assumptions. Let the target H be of the form $H = L + E$ where $L = \{x \in E^n: Mx = 0\}$ is a linear space and E, a compact, convex set of system (1) with $0 \in E$; M is an m x n *constant matrix. Let the continuous function $f = f(t, x(t), X(t-h), u(t)\, u(t-h))$ satisfy the condition.*

$$\lim_{|u|\to\infty} \frac{\left[\left| f\left(t, x(t), x(t-h), u(t)\right), u(t-h)\right|\right]}{|u|} = 0 \qquad (3.14)$$

uniformly in $(t, x(t), x(t-h)\, u(t), u(t-h)) \in E \times E^n \times E^n \times E^m \times E^m$. Assume that $0 \in U$ and $0 \in H$ then Core (H) is compact if and only if the system $\dot{x}(t) = A^t x(t) + B^T x(t-h) + M^T u(t) + f(t, x(t), x(t-h), u(t)\, u(t-h))$ is Euclidean controllable.

Proof

Let $\{\phi_n(t_0): n = 1, 2, 3\ldots\}$ denote the set of asymptotic directions of Core (H). Then by theorem 2, $\{X(t-s)\,\phi_n(t_0): n = 1, 2, 3\ldots\}$ is the set of asymptotic directions of H we invoke proposition (3) to deduce that

$$L = \{X(t-s)\,\phi_n(t_0): n = 1, 2, 3\} \qquad (3.15)$$

for all $t > t_0 + s$. It follows from the hypotheses of theorem 3.3 that

$$MX(t-s)\,\phi_n(t_0) = 0 \qquad (3.16)$$

for each n and $t \geq t_0 + s$. Taking the transpose of both sides of (3.16) yields $\phi_n^T(t_0) X^T(t-s) M^T = 0$ for each n and $t \geq t_0 + s$. By Lemma 2.1 and Theorem 3.1. Core (H) is a non-void, convex and closed subset of E^n. By Proposition 3.2, Core (H) is bounded if and only if 0 is its only asymptotic direction. By the Euclidean controllability of system (3.12) its linear part is also Euclidean controllable and this is equivalent to requiring that $\phi_n^T(t_0) X^T(t-s) M^T = 0$ implies $\phi_n(t_0) = 0$ for all $t \geq t_0 + s$ and for all n. This shows that 0 is the only asymptotic direction of Core (H). Hence Core H is bounded. This result and Theorem 3.1 yield the compactness of the system, since boundedness and closeness implies compactness of subset of finite dimensional spaces. Conversely let Core (H) be compact then 0 is its only asymptotic direction. This means that $\phi_n^T(t_0) X^T(t-s) M^T = 0$ implies $\phi_n(t_0) = 0$ for all n and $t \geq t_0 + s$. We then conclude that

$$\dot{x}(t) = A^T(t) + B^T x(t-h) + M^T u(t)$$

Is Euclidean controllable on $[t_0\ t_1]$ $t_1 > t_0$ (Ref [2]) and hence system (3.16) is Euclidean controllable. (Ref [3]).

4.0 Conclusion

A core of target for a system is the set of all initial points that can be steered to the target using appropriate control energy. With the growing significance of the subject; as it is becoming known as the set of seeds for our expected outcomes, the need to characterize cores of targets is now of great interest. The present study investigated the convexity, boundedness and closedness of cores of Targets for nonlinear systems, which have been established in the affirmative. Beyond these results is the realization of the relationship between compactness of cores of Targets for a system and the Euclidean controllability of a related system. The notion of asymptotic direction, weak compactness argument and the imposition of a growth condition on the perturbation function to make it smooth remarkably reinforced each other in providing the method of approval for this investigation. This research is an extension of the results in [10] and [5] and a complementation of the efforts in [9]. The success of this study is a happy augury as it not only characterizes cores of targets for non-linear systems but also provides governments and entrepreneurs with broad policy guidelines for the commencement and execution of projects to ensure their completion.

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