Journal of the Nigerian Association of Mathematical Physics,
Volume 8 (*November 2004*)

Combination methods for numerical inclusion of the zeros of a polynomial

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Abstract

In the numerical inclusion and isolation of the zeros of a polynomial in an interval on the plane, hybrid combination methods have been found quite useful for their virtue of easy construction and reduced computational cost with respect to interval arithmetic operations, while still providing restrictive inclusion for the respective zeros simultaneously. In what now follows, consider a collection of combination methods arising from efficient enhancement of a class of basic simultaneous numerical inclusion methods under two different updating procedures of the generated iterates. The accuracy of the methods will be illustrated by insightful numerical experiments.

Keywords: combination methods, zeros of a polynomial, correction, *R*-order of convergence, interval methods, efficiency index

C.R. Category: G1.5

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1.0 Introduction

1.1 The hybrid combination of iteration methods for zero inclusion

Consider the numerical inclusion of the zeros $\{\lambda_j\}_{j=1}^N$ with respective multiplicities $\{\mu_j\}_{j=1}^N$ of the polynomial

$$P_n(z)=\sum_{j=1}^{n}a_j z^j=\prod_{j=1}^{N}(z-\lambda_j)^{\mu_j};\quad z,a_j\in\mathbb{C},\ \sum_{j=1}^{N}\mu_j=n, a_n=1, n>2 \qquad (1.1)$$

of degree *n*, by combination of point and interval inclusion arithmetic. Indeed, an approach of reduced computational cost is obtained by this hybrid combination of a point arithmetic method with an interval inclusion method applied once and only in the final stage of the iteration, see [14]. The hybridisation is such that the interval method provides the needed inclusion for the zeros. This is an efficient method that saves cost with respect to disk arithmetic operations on real time implementation. In this regard, is the class of combination methods

$$z_j^{(s+1)}=z_j^{(s)}-\frac{\Delta_{l-1,j}\left(z_j^{(s)}\right)}{\Delta_{l,j}\left(z_j^{(s)}\right)-B_l\left(SC_{1,j}^{(s)}(z),SC_{2,j}^{(s)}(z),\ldots,SC_{l,j}^{(s)}(z)\right)};\ \begin{matrix}s=0(1)m-1\\ l=1,2,\ldots\end{matrix} \qquad (1.2)$$

$$Z_j^{(m,1)}=z_j^{(m)}-\frac{\Delta_{k-1,j}\left(z_j^{(m)}\right)}{\Delta_{k,j}\left(z_j^{(m)}\right)-B_k\left(S_{1,j}^{(0)}(Z),S_{2,j}^{(0)}(Z),\ldots,S_{k,j}^{(0)}(Z)\right)};\ k=1,2,\ldots$$

$j=1(1)N$, for a fixed *l* and *k*, in which $l\le k$, $l=k$ or $l>k$, where

$$SC_{v,j}^{(s)}(z)=\frac{1}{\mu_j}\sum_{\substack{i=1\\ j\neq i}}^{N}\mu_i\left(\frac{1}{z_j^{(s)}-z_i^{(s)}}\right)^v;v=1(1)l \tag{1.3}$$

$$S_{v,j}^{(0)}(Z)=\frac{1}{\mu_j}\sum_{\substack{i=1\\ j\neq i}}^{N}\mu_i\left(\frac{1}{z_j^{(m)}-Z_i^{(0)}}\right)^v;v=1(1)k \tag{1.4}$$

and

$$\Delta_{k,j}\left(z_j^{(s)}\right)=\sum_{v=1}^{k}(-1)^{k-v}\frac{1}{\mu_j}\left(\frac{1}{\mu_j}+1\right)\cdots\left(\frac{1}{\mu_j}+v-1\right)\sum\prod_{\lambda=1}^{k}\frac{1}{q_\lambda!}\left(\frac{P_n^{(\lambda)}\left(z_j^{(s)}\right)}{\lambda!P_n\left(z_j^{(s)}\right)}\right)^{q\lambda};j=1(1)N$$

The summation without index runs through all nonnegative integers $\{q_l\}_{l=1(1)k}$ for which

$$\sum_{l=1}^{k}q_l=v;\ (v=1(1)k)\text{ and }\sum_{l=1}^{k}lq_l=k;\ (k=1,2,3,...)$$

are satisfied simultaneously. A few of these $\Delta_{k,j}\left(z_j^{(s)}\right)$, are

$$\Delta_{1,j}\left(z_j^{(s)}\right)=\frac{1}{\mu_j}\frac{P_n'}{P_n};\ P_n=P_n(z_j^{(s)}),\quad \Delta_{2,j}\left(z_j^{(s)}\right)=\frac{1}{2\mu_j}\left(\frac{1}{\mu_j}+1\right)\left(\frac{P_n'}{P_n}\right)^2-\frac{1}{2\mu_j}\frac{P_n''}{P_n}$$

$$\Delta_{3,j}\left(z_j^{(s)}\right)=\frac{1}{6\mu_j}\frac{P_n'''}{P_n}-\frac{1}{2\mu_j}\left(\frac{1}{\mu_j}+1\right)\frac{P_n'}{P_n}\frac{P_n''}{P_n}+\frac{1}{6\mu_j}\left(\frac{1}{\mu_j}+1\right)\left(\frac{1}{\mu_j}+2\right)\left(\frac{P_n'}{P_n}\right)^3$$

$$\Delta_{4,j}\left(z_j^{(s)}\right)=-\frac{1}{24\mu_j}\frac{P_n''''}{P_n}+\frac{1}{\mu_j}\left(\frac{1}{\mu_j}+1\right)\left(\frac{P_n'}{6P_n}\frac{P_n'''}{P_n}+\frac{1}{4}\left(\frac{P_n''}{2P_n}\right)^2\right)-\frac{1}{4\mu_j}\left(\frac{1}{\mu_j}+1\right)\left(\frac{1}{\mu_j}+2\right)\left(\frac{P_n'}{P_n}\right)^2\frac{P_n''}{P_n}$$
$$+\frac{1}{24\mu_j}\left(\frac{1}{\mu_j}+1\right)\left(\frac{1}{\mu_j}+2\right)\left(\frac{1}{\mu_j}+3\right)\left(\frac{P_n'}{P_n}\right)^4$$

For simple zeros, $\Delta_{k,j}\left(z_j^{(s)}\right)$ is easily computed from

$$\Delta_{k,j}\left(z_j^{(s)}\right)=\sum_{v=1}^{k}\frac{(-)^{v+1}}{v!}\frac{P_n^{(v)}\left(z_j^{(s)}\right)}{P_n\left(z_j^{(s)}\right)}\Delta_{k-v,j}\left(z_j^{(s)}\right);j=1(1)n \tag{1.5}$$

The $B_k;\ v=1(1)k$ are the Bell's polynomials in the points $\{z_1,z_2,z_3,...,z_v\}$ given by the recursion

$$B_k(z_1,z_2,z_3,...,z_k)=\begin{cases}1;k=0\\ z_1;k=1\\ \frac{1}{2}\left(z_2+z_1^2\right);k=2\\ \frac{1}{2}\left(3z_1z_2+z_1^3+2z_3\right);k=3\\ \frac{1}{4}z_4+\frac{1}{3}z_3z_1+\frac{1}{8}z_2^2+\frac{1}{4}z_2z_1^2+\frac{1}{24}z_1^4;k=4\\ \frac{1}{4}z_1z_4+\frac{1}{6}z_3z_1^2+\frac{1}{8}z_2^2z_1+\frac{1}{12}z_2z_1^3+\frac{1}{120}z_1^5+\frac{1}{6}z_2z_3+\frac{1}{5}z_5;k=5\\ \frac{1}{k}\sum_{v=1}^{k}z_vB_{k-v}(z_1,z_2,z_3,...,z_k);k=6,7,8,...\end{cases} \tag{1.6}$$

It should be noticed that the point method in the combination (1.2) is applied m number of times, while the inclusion method once only and in the final stage of the algorithmic process. The order of the point and interval arithmetic method in (1.2) is $K_p = l+2$ and $K_I = k+2$ respectively. The circular interval given as $Z = \{w : |w-z| \le r\} = \{z;r\}; z \in \mathbb{C}, r > 0$ is a disk or circular region of centre z and radius r. The parametric representation, $Z = \{z;r\}$ is usually adopted with the definitions that r = Rad(Z) and z = Mid(Z), see [3]. It degenerates to the centre point, z if however, $r = 0$, but this is without hindrance to our analysis, more interestingly interval arithmetic is consistent and compatible with point arithmetic. There is improvement of the convergence of (1.2) if the correction $C_{w,j}^{(s)}$, is judiciously introduced, so that instead,

$$SC_{v,j}^{(s)}(z) = \frac{1}{\mu_j}\sum_{\substack{i=1\\ j\neq i}}^{N}\mu_i\left(\frac{1}{z_j^{(s)} - z_i^{(s)} + C_{w,i}^{(s)}}\right)^v ; v = 1(1)l, w \le l \qquad (1.7)$$

where for example the correction could be taken as

$$C_{w,j}^{(s)} = \frac{\Delta_{w-1,j}(z_j^{(s)})}{\Delta_{w,j}(z_j^{(s)})}; (p = w+1), \text{ for a chosen } w \le l \qquad (1.8)$$

amongst other useful alternative choices of a corrector, this choice is efficient since $C_{w,j}^{(s)}$ is readily available. Note

that $z_{w,j}^{(s+1)} = z_{w,j}^{(s)} - C_{w,j}^{(s)}$ for a corrector $C_{w,j}^{(s)}$ is a method of its own right and therefore say that it is a corrector of order p. The instances $l = 1, k = 1$ and $l = 1, k = 2$ where the correction is $C_{1,j}^{(s)}$, form the cases considered in [14]. A further practicable example of the above is when $l = k = 2$ with the correction taken as $C_{1,i}^{(s)}$ or $C_{2,i}^{(s)}$. The inclusion intervals $\{Z_i^{(0)}\}_{i=1(1)N}$ are the non-overlapping initial disks isolating the respective zeros. It is noted that the root iterations, [3], are inefficient for combination purposes because of the difficulties in the computation of radicals and the delusion therefrom on the appropriate k-th root to choose. In general, for a basic circular interval process with order K_I (integer), under a correction of order p, it is deduced, see the appendix herein and with respect to the notations therein, that the error $\varepsilon_j = z_j - \lambda_j$ in the interval process of the improved Wang and Zheng's method,

$$Z_j^{(s+1)} = z_j^{(s)} - \frac{\Delta_{k-1,j}\left(z_j^{(s)}\right)}{\Delta_{k,j}\left(z_j^{(s)}\right) - B_k\left(SI_{1,j}^{(s)}(Z), SI_{2,j}^{(s)}(Z), \ldots, SI_{k,j}^{(s)}(Z)\right)}; k = 1,2,3,\Lambda \qquad (1.9)$$

with
$$SI_{v,j}^{(s)}(Z) = \frac{1}{\mu_j}\sum_{\substack{i=1\\ j\neq i}}^{N}\mu_i\left(INV\left(z_j^{(s)} - Z_i^{(s)} + C_i^{(s)}\right)\right)^v ; v = 1(1)k \qquad (1.10)$$

for a fixed k, propagates its effect in the way that $\hat{r}_j = o\left(r\varepsilon^{K_I-1}\right)$;

$$\hat{\varepsilon}_j = \begin{cases} \left[o\left(r^2\right) + o\left(\varepsilon^p\right)\right]o\left(\varepsilon^{K_I-1}\right); \beta = 1 \\ \left[o\left(r^2\right) + o\left(\varepsilon^{p-K_I+2}\right)\right]o\left(\varepsilon^{2K_I-3}\right); \beta = 0 \end{cases}; p \ge 2 \qquad (1.11)$$

where $C_i^{(s)}$ is the correction term which enhances convergence, is taken from the point method $z_i^{(s)} = z_i^{(s)} - C_i^{(s)}$ of order p and

$z_j = Mid(Z_j); r_j = Rad(Z_j); \varepsilon_j = z_j - \lambda_j; \varepsilon = \underset{j=1(1)n}{Max}|\varepsilon_j|; r = \underset{j=1(1)n}{Max}\{r_j\}$ also $Z_j, \hat{Z}_j, r_j, \hat{r}_j, z_j, \hat{z}_j$ denote $Z_j^{(s)}, Z_j^{(s+1)}, r_j^{(s)}, r_j^{(s+1)}, z_j^{(s)}, z_j^{(s+1)}$ and $\hat{r} = \underset{j=1(1)n}{Max}\left\{\hat{r}_j\right\}; \hat{\rho} = \underset{\substack{i,j=1(1)n\\ i\neq j}}{Min}\left\{\left|\hat{z}_i - \hat{z}_j\right| - \hat{r}_j\right\}$. The inversion $INV(Z)$ from [16] is given by any one of the choices in

$$INV(Z) = \begin{cases} \left\{\dfrac{\bar{z}}{|z|^2 - r^2}; \dfrac{r}{|z|^2 - r^2}\right\} & ;(\ )^{-1}, \quad \beta = 1 \\ \left\{\dfrac{1}{z}; \dfrac{r}{|z|(|z| - r)}\right\} & ;(\ )^{I_1}, \quad \beta = 0 \\ \left\{\dfrac{1}{z}; \dfrac{2r}{|z|^2 - r^2}\right\} & ;(\ )^{I_2}, \quad \beta = 0 \end{cases} \tag{1.12}$$

The number $\bar{z}$ is the conjugate of the complex point z and $Z^{-1} \subseteq Z^{I_1} \subseteq Z^{I_2}$. The inclusion relation

$$\sum_{j=1;j\neq i}^{n} Rad\left(\left(INV\left(z_i - Z_j\right)\right)^k\right) \leq \sum_{j=1;j\neq i}^{n} Rad\left(INV\left(\left(z_i - Z_j\right)^k\right)\right); 0 \notin Z_j = \{z_j; r_j\} \tag{1.13}$$

instructively implies that the methods need be implemented as it is presented. Similarly, the equivalent point method

$$Z_j^{(s+1)} = z_j^{(s)} - \frac{\Delta_{k-1,j}\left(z_j^{(s)}\right)}{\Delta_{k,j}\left(z_j^{(s)}\right) - B_k\left(SC_{1,j}^{(s)}(z), SC_{2,j}^{(s)}(z), \ldots, SC_{k,j}^{(s)}(z)\right)}; k = 1,2,3,\ldots \tag{1.14}$$

of (1.9) with

$$SC_{v,j}^{(s)}(z) = \frac{1}{\mu_j}\sum_{\substack{i=1\\ j\neq i}}^{N}\mu_i\left(\frac{1}{z_j^{(s)} - z_i^{(s)} + C_i^{(s)}}\right)^v; v = 1(1)k \tag{1.15}$$

for a fixed *k*, propagates its error effects according as

$$\hat{\varepsilon}_j = o\left(\varepsilon^{K_p + p - 1}\right); p = 2,3,\ldots,K_p - 1; K_p = k + 2 \tag{1.16}$$

so that its order is $K_p + p - 1$. In (1.9), we may as well choose that $C_i^{(s)} = C_{w,i}^{(s)};\ w \leq k$. The convergence requirement of the independent methods in (1.2) especially with respect to initial point/disk separation is similarly obtained as in [**9] for k=1,2. Thus for a point method of order K_P (integer), here the radius is set to zero from (1.11), the error is deduced to propagate as

$$\varepsilon^{(m+1)} = o\left((\varepsilon^{(m)})^{K_p}\right); \varepsilon^{(m)} = {}_p\varepsilon^{(m)} = o\left((\varepsilon^{(0)})^{K_p^m}\right) \tag{1.17}$$

where ${}_p\varepsilon^{(s)} = \underset{j}{Max}\left\{\left|{}_p\varepsilon_j^{(s)}\right|\right\}$ and the pre-subscript p is by this notation, to indicate that the sequences $\left\{\varepsilon_j^{(m)}\right\}_{m=0,j=1}^{\infty,n}$ are generated by the point arithmetic process, and with a correction of order p_1 introduced in this point process, then

$$\varepsilon^{(m+1)} = o\left(\left(\varepsilon^{(m)}\right)^{K_p + p_1 - 1}\right), \varepsilon^{(m)} = {}_p\varepsilon^{(m)} = o\left(\left(\varepsilon^{(0)}\right)^{(K_p + p_1 - 1)^m}\right); 2 \leq p_1 \leq K_p - 1 \quad (1.18)$$

from (1.16). For inclusion method of order K_I without correction,

$$r^{(s+1)} = o\left((\varepsilon^{(s)})^{K_I-1}\right)o\left(r^{(s)}\right), \varepsilon^{(s+1)} = o\left((\varepsilon^{(s)})^{K_I}\right) \tag{1.19}$$

Set $r^{(m,1)} = \underset{j}{Max}\{r_j^{(m,1)}\}$ and $\varepsilon^{(m,1)} = \underset{j}{Max}\{|\varepsilon_j^{(m,1)}|\}$ where $r_j^{(m,1)}$ and $\varepsilon_j^{(m,1)}$, are the radii and the absolute errors in the centres of the disks $Z_j^{(m,1)}; j = 1(1)n$ respectively, generated by the combination method (1.2) to approximate and include the respective zeros. The rate of convergence of (1.2) is seen from (1.19), which $r^{(m,1)} =$

$$o\left((\varepsilon^{(0)})^{(K_I-1)K_P^m}\right)o\left(r^{(0)}\right) = o\left((r^{(0)})^{(K_I-1)K_P^m+1}\right), \varepsilon^{(m,1)} = o\left((\varepsilon^{(0)})^{K_P^m K_I}\right); \beta = 0,1 \tag{1.20}$$

obtained by setting $\varepsilon^{(0)} = {}_p\varepsilon^{(m)}$ from (1.17) into (1.19), for any combination method of which K_P is the order of convergence of the basic point process and K_I is the order of convergence of the basic interval method. It is instructive that $r_j^{(0)}$ need be accurate to the order at least 10^{-1} and also ${}_p\varepsilon_j^{(0)}$ is of the same order, then to calculate $r^{(m,1)}$ to the order $10^{-t}; t >> 1$ of accuracy at least the number of point iterations

$$M_1 \geq 1 + Int\left(\frac{log\left(\dfrac{t-1}{K_I-1}\right)}{log\ K_p}\right) \tag{1.21}$$

are needed and to achieve same for $\varepsilon^{(m,1)}$,

$$M_2 \geq 1 + Int\left(\frac{log\left(\dfrac{t}{K_I}\right)}{log\ K_p}\right) \tag{1.22}$$

iterations need be carried out by the combination process with the meaning that, *Int(A) = integer part of A.* To have both $r^{(m,1)}$ and $\varepsilon^{(m,1)}$ to the minimum $o(10^{-t})$ of accuracy, a minimum of $M = \underset{j=1,2}{Max}\{M_j\}$ number of iterations are required by the hybrid inclusion method (1.2), in most instances M_1 and M_2 may not differ significantly in magnitude. The number M therefore, may conveniently be estimated by M_1. Generally, as implied in (1.21,1.22) the order of accuracy increases steadily with an increasing number of the iterations. We wish now consider some efficient ways to improve accuracy in a combination method of the type in (1.2) by application of corrections and updating of the iterates. To this end, lets start from considering when there is no correction in the point algorithm, as it is in (1.2), further improvement in results can be attained for (1.2) when we introduce an efficient corrector $C_{u,i}^{*(0)}$ of order p_2 in the root inclusion part of the algorithm, in that stead let $S_{v,j}^{(0)}(Z)$ in (1.4) become

$$S_{v,j}^{(0)}(Z) = \frac{1}{\mu_j}\sum_{\substack{i=1 \\ j\neq i}}^{N}\mu_i\left(INV\left(z_j^{(m)} - Z_i^{(0)} + C_{u,i}^{*(0)}\right)\right)^v; v = 1(1)k, u \leq k \tag{1.23}$$

For an inclusion method with correction of order p_2

$$r^{(s+1)} = o\left((\varepsilon^{(s)})^{K_I-1}\right) o\left(r^{(s)}\right), \varepsilon^{(s+1)} = o\left((\varepsilon^{(s)})^{K_I-1}\right) o\left((r^{(s)})^2\right); \beta = 1,\ p_2 \geq 2 \quad (1.24)$$

and with $\beta = 0$;

$$r^{(s+1)} = o\left((\varepsilon^{(s)})^{K_I-1}\right) o\left(r^{(s)}\right), \varepsilon^{(s+1)} = \begin{cases} o\left((\varepsilon^{(s)})^{K_I+p_2-1}\right); p_2 = 2(1)K_I - 1 \\ o\left((\varepsilon^{(s)})^{2K_I-3}\right) o\left((r^{(s)})^2\right); p_2 \geq K_I \end{cases} \quad (1.25)$$

Then the rate of convergence, if $\beta = 1$ is seen from

$$r^{(m,1)} = o\left((\varepsilon^{(0)})^{(K_I-1)K_P^m}\right) o\left(r^{(0)}\right), \varepsilon^{(m,1)} = o\left((\varepsilon^{(0)})^{K_P^m(K_I-1)}\right) o\left((r^{(0)})^2\right); p_2 \geq 2 \quad (1.26)$$

by setting $\varepsilon^{(0)} =_p \varepsilon^{(m)}$ from (1.17) into (1.24). Now the least number of iterations to the desired accuracy for $r^{(m,1)}$ is as in (1.14) and that of $\varepsilon^{(m,1)}$ is

$$M_2 \geq 1 + Int\left(\frac{log\left(\frac{t-2}{K_I - 1}\right)}{log K_P} \right) \quad (1.27)$$

which is derived from the second expression of (1.26). If $\beta = 0$, then

$$r^{(m,1)} = o\left((\varepsilon^{(0)})^{(K_I-1)K_P^m}\right) o\left((r^{(0)})\right), \varepsilon^{(m,1)} = \begin{cases} o\left((\varepsilon^{(0)})^{K_P^m(K_I+p_2-1)}\right); p_2 = 2(1)K_I - 1 \\ o\left((\varepsilon^{(0)})^{K_P^m(2K_I-3)}\right) o\left((r^{(0)})^2\right); p_2 \geq K_I \end{cases} \quad (1.28)$$

where $\varepsilon^{(0)} =_p \varepsilon^{(m)}$ from (1.17) is inserted into (1.25), estimate in this case that

$$M_{2,1} \geq 1 + Int\left(\frac{log\left(\frac{t}{K_I + p_2 - 1}\right)}{log K_P} \right); p_2 = 2(1)K_I - 1$$

$$M_{2,2} \geq 1 + Int\left(\frac{log\left(\frac{t-2}{2K_I - 3}\right)}{log K_P} \right); p_2 \geq K_I \quad (1.29)$$

for $\varepsilon^{(m,1)}$ and that of $r^{(m,1)}$ is given by (1.25) and if the point process is corrected as well, then

$$r^{(m,1)} = o\left((\varepsilon^{(0)})^{(K_I-1)(K_P+p_1-1)^m}\right) o\left((r^{(0)})\right),$$

$$\varepsilon^{(m,1)} = \begin{cases} o\left((\varepsilon^{(0)})^{(K_P+p_1-1)^m(K_I+p_2-1)}\right); \begin{matrix} p_2 = 2(1)K_I - 1 \\ p_1 = 2(1)K_p - 1 \end{matrix} \\ o\left((\varepsilon^{(0)})^{(K_P+p_1-1)^m(2K_I-3)}\right) o\left((r^{(0)})^2\right); \begin{matrix} p_2 \geq K_I \\ p_1 = 2(1)K_p - 1 \end{matrix} \end{cases} \quad (1.30)$$

by setting $\varepsilon^{(0)} =_p \varepsilon^{(m)}$ from (1.17) into (1.21). Here, we can estimate that

$$M_{2,1} \geq 1 + Int\left(log \frac{\left(\frac{t}{K_I + p_2 - 1} \right)}{log(Y_P)} \right); \begin{matrix} p_1 = 2(1)K_p - 1 \\ p_2 = 2(1)K_I - 1 \end{matrix}$$

$$M_{2,2} \geq 1 + Int\left(log \frac{\left(\frac{t-2}{2K_I - 3} \right)}{log(Y_P)} \right); \begin{matrix} p_1 = 2(1)K_I - 1 \\ p_2 \geq K_I \end{matrix} \tag{1.31}$$

for $\varepsilon^{(m,1)}$ and for $r^{(m,1)}$,

$$M_1 \geq 1 + Int\left(log \frac{\left(\frac{t-1}{K_I - 1} \right)}{log\,(Y_P)} \right); Y_P = K_P + p_1 - 1 \tag{1.32}$$

It can therefore be seen that the method introduced in (1.23) with correction and the centred inversion $(\beta = 0)$, will give improved order of convergence of the iterates than (1.2) especially with respect to the centres of the disks, when $p_1, p_2 \geq 2,\ K_I \geq 3,\ K_p \geq 3$ with the point iteration at least $m = 3$ number of steps; and if without correction in the interval method, then $K_p \geq 2,\ \ p_1 = 1; \beta = 0{,}1$.

2.0 The application of updating procedures to the generated iterates

In the combination method the desire is to apply interval arithmetic only once, to save considerable cost of disk arithmetic, for improved inclusion of the zeros, the Gauss-Seidel type updating procedure is appropriate in the inclusion part of the combination, by this

$$Z_j^{(m,1)} = z_j^{(m)} - \frac{\Delta_{k-1,j}\left(z_j^{(m)}\right)}{\Delta_{k,j}\left(z_j^{(m)}\right) - B_k\left(S_{1,j}^{(1)}(Z), S_{2,j}^{(1)}(Z), \ldots, S_{k,j}^{(1)}(Z)\right)}; k = 1,2,.\Lambda \tag{2.1}$$

with $S_{v,j}^{(1)}(Z) = \frac{1}{\mu_j}\left[\sum_{i=1}^{j-1} \mu_i \left(INV\left(z_j^{(m)} - Z_i^{(m,1)}\right)\right)^v + \sum_{i=j+1}^{N} \mu_i \left(INV\left(z_j^{(m)} - Z_i^{(0)} + C_{u,i}^{*(0)}\right)\right)^v \right]; v = 1(1)k$ replaces the inclusion part of (1.2) with respect to (1.23). The error in the above propagates according to see the appendix, as $r_j^{(m,1)} = o(\left(\varepsilon_j^{(0)}\right)^{K_I - 1})\left(\sum_{i=1}^{j-1} o\left(r_i^{(m,1)}\right) + \sum_{i=j+1}^{N} o\left(r_i^{(0)}\right) \right)$

$$\varepsilon_j^{(m,1)} = o(\left(\varepsilon_j^{(0)}\right)^{K_I - 1}) \begin{cases} \sum_{i=1}^{j-1} o\left(\varepsilon_i^{(m,1)}\right) + \sum_{i=j+1}^{N} o(\left(\varepsilon_i^{(0)}\right)^{p2}); p_2 = 2,3,\ldots,K_I - 1; \beta = 0 \\ \sum_{i=1}^{j-1} o\left(\varepsilon_i^{(m,1)}\right) + \sum_{i=j+1}^{N} o(\left(\varepsilon_i^{(0)}\right)^{p2}) + \sum_{i=1}^{j-1} o(\left(r_i^{(m,1)}\right)^2) + \sum_{i=j+1}^{N} o(\left(r_i^{(0)}\right)^2); \\ \beta = 1, p_2 \geq 2 \end{cases} \tag{2.2}$$

Considering (2.2) where of course p_2 is the order of the corrector $C_{u,i}^{*(0)}$ in the inclusion method and if there is no need for correction set this to be $p_2 = 1$, and with $\beta = 1$the middle terms in the above brackets of the second expression are to be ignored since its order of magnitude is smaller compared to the other terms; we know $|\varepsilon_i| < \varepsilon \le r_i \le r < 1$ when convergence sets in, in the long run on m, therefrom,

$$r_j^{(m,1)} = o(\left(\varepsilon_j^{(0)}\right)^{K_I - 1})\left(\sum_{i=1}^{j-1} o\left(r_i^{(m,1)}\right) + \sum_{i=j+1}^{N} o\left(r_i^{(0)}\right)\right);$$

$$\varepsilon_j^{(m,1)} = o(\left(\varepsilon_j^{(0)}\right)^{K_I - 1})\begin{cases}\left(\sum_{i=1}^{j-1} o\left(\varepsilon_i^{(m,1)}\right) + \sum_{i=j+1}^{N} o(\left(\varepsilon_i^{(0)}\right)^{p2})\right); p_2 = 2,3,\ldots,K_I - 1; \beta = 0 \\ \left(\sum_{i=1}^{j-1} o\left(\varepsilon_i^{(m,1)}\right) + \sum_{i=j+1}^{N} o(\left(r_i^{(0)}\right)^{2})\right); p_2 \ge 2, \beta = 1\end{cases} \tag{2.3 }$$

refer to the appendix. Therefore, if no correction in the point process as it is in (1.2),

$$r_j^{(m,1)} = o(\left(\varepsilon_j^{(0)}\right)^{K_p^m(K_I - 1)})\left(\sum_{i=1}^{j-1} o\left(r_i^{(m,1)}\right) + \sum_{i=j+1}^{N} o\left(r_i^{(0)}\right)\right)$$

$$\varepsilon_j^{(m,1)} = o(\left(\varepsilon_j^{(0)}\right)^{K_p^m(K_I - 1)})\begin{cases}\sum_{i=1}^{j-1} o\left(\varepsilon_i^{(m,1)}\right) + \sum_{i=j+1}^{N} o(\left(\varepsilon_i^{(0)}\right)^{p2}); p_2 = 2,3,\ldots,K_I - 1; \beta = 0 \\ \sum_{i=1}^{j-1} o\left(\varepsilon_i^{(m,1)}\right) + \sum_{i=j+1}^{N} o(\left(r_i^{(0)}\right)^{2}); p_2 \ge 2, \beta = 1\end{cases} \tag{2.4}$$

as inferred by inserting (1.17) into (2.3), and as for a corrected point process as well, then

$$r_j^{(m,1)} = o(\left(\varepsilon_j^{(0)}\right)^{(K_p + p_1 - 1)^m (K_I - 1)})\left(\sum_{i=1}^{j-1} o\left(r_i^{(m,1)}\right) + \sum_{i=j+1}^{N} o\left(r_i^{(0)}\right)\right)$$

$$\varepsilon_j^{(m,1)} = o(\left(\varepsilon_j^{(0)}\right)^{(K_p + p_1 - 1)^m (K_I - 1)})\left(\sum_{i=1}^{j-1} o\left(\varepsilon_i^{(m,1)}\right) + \begin{cases}\sum_{i=j+1}^{N} o(\left(r_i^{(0)}\right)^{p2})\right); \beta = 0 \\ \sum_{i=j+1}^{N} o(\left(r_i^{(0)}\right)^{2})\right); \beta = 1\end{cases} \tag{2.5}$$

using (1.18). However, the least number of iterations to calculate $r^{(m,1)}$ to the accuracy of order $10^{-t}; t >> 1$ can be estimated similarly from

$$r_j^{(m,1)} = o(\left(\varepsilon_j^{(0)}\right)^{(K_p + p_1 - 1)^m (K_I - 1)})\left(\sum_{i=1}^{j-1} o\left(r_i^{(m,1)}\right) + \sum_{i=j+1}^{N} o\left(r_i^{(0)}\right)\right) \cong o(\left(\varepsilon^{(0)}\right)^{(K_p + p_1 - 1)^m (K_I - 1)}) o\left(r^{(0)}\right) \tag{2.6 }$$

Now since $r_i^{(m,1)} < r_i^{(0)}$ and noting that $\varepsilon_i^{(0)} = o(r_i^{(0)})$ then,

$$r^{(m,1)} = o(\left(\varepsilon^{(0)}\right)^{(K_p + p_1 - 1)^m (K_I - 1)}) o\left(r^{(0)}\right) = o\left(\left(\varepsilon^{(0)}\right)^{(K_p + p_1 - 1)^m (K_I - 1) + 1}\right) \tag{2.7}$$

to have that for the method (1.2) with $SC_{v,j}^{(s)}(z)$ as defined in (1.23),

$$M_1 \ge 1 + Int\left(log\left(\frac{t-1}{K_I - 1}\right) / log\left(K_P + p_1 - 1\right)\right); \; 2 \le p_1 \le K_p - 1 \tag{2.8}$$

This expression could also be obtained from writing (2.6) as

$$r^{(m,1)} = o(\left(\varepsilon^{(0)}\right)^{(K_p + p_1 - 1)^m (K_I - 1)})\left(o\left(r^{(m,1)}\right) + o\left(r^{(0)}\right)\right) \tag{2.9}$$

and again for $\varepsilon^{(m,1)}$, by using

$$\varepsilon^{(m,1)} = o(\left(\varepsilon^{(0)}\right)^{(K_p + p_1 - 1)^m (K_I - 1)})\left(o\left(\varepsilon^{(m,1)}\right) + o(\left(\varepsilon^{(0)}\right)^{p2})\right); \beta = 0$$

$$\varepsilon^{(m,1)} = o(\left(\varepsilon^{(0)}\right)^{(K_p + p_1 - 1)^m (K_I - 1)})\left(o\left(\varepsilon^{(m,1)}\right) + o(\left(r^{(0)}\right)^{2})\right); \beta = 1$$

from (2.5), then

$$M_{2,1} \geq 1+Int\left(log\left(\frac{t-p_2}{K_I-1}\right)/\,log\,\left(K_p+p_1-1\right)\right); \qquad \begin{array}{c} 2\leq p_1 \leq K_p-1, \beta=0 \\ 2 \leq p_2 \leq K_I-1 \end{array} \tag{2.10}$$

$$M_{2,2} \geq 1+Int\left(log\left(\frac{t-2}{K_I-1}\right)/\,log\,\left(K_p+p_1-1\right)\right);\; 2\leq p_1 \leq K_p-1, p_2 \geq 2, \beta=1 \tag{2.11}$$

respectively. Now, if Gauss-Jacobi updating is applied several times $(\delta = 0(1)q-1;$ $q = 2,3,\Lambda\,)$ to the point method under correction, we have

$$z_j^{(s+\frac{\delta+1}{q})} = z_j^{(s)} - \frac{\Delta_{l-1,j}\left(z_j^{(s)}\right)}{\Delta_{l,j}\left(z_j^{(s)}\right)-B_l\left(SC_{1,j}^{(s+\frac{\delta}{q})}(z), SC_{2,j}^{(s+\frac{\delta}{q})}(z),\ldots,SC_{l,j}^{(s+\frac{\delta}{q})}(z)\right)} \tag{2.12}$$

for a chosen k, where

$$SC_{v,j}^{(s+\frac{\delta+1}{q})}(z) = \frac{1}{\mu_j}\sum_{\substack{i=1\\ i\neq j}}^{N}\mu_i\left(\frac{1}{z_j^{(s)}-z_i^{(s+\frac{\delta}{q})}+C_{w,i}^{(s+\frac{\delta}{q})}}\right)^v;\; v=1(1)l, w\leq l \tag{2.13}$$

Then using (19) in the appendix the error in (2.12) propagates as

$${}_p\varepsilon_j^{(s+\frac{\delta+1}{q})} = o\left(\left({}_p\varepsilon_j^{(s)}\right)^{K_p-1}\right)o\left(\left({}_p\varepsilon_j^{(s+\frac{\delta}{q})}\right)^{p_1}\right); \;\begin{array}{c} p_1 = 2(1)K_p-1 \\ s=0(1)m-1 \end{array} \tag{2.14}$$

see [9, equation (6.6)]. But by the Gauss-Seidel type updating, which now

$$z_j^{(s+\frac{\delta+1}{q})} = z_j^{(s)} - \frac{\Delta_{l-1,j}\left(z_j^{(s)}\right)}{\Delta_{l,j}\left(z_j^{(s)}\right)-B_l\left(SC_{1,j}^{(s+\frac{\delta+1}{q})}(z), SC_{2,j}^{(s+\frac{\delta+1}{q})}(z),\ldots,SC_{l,j}^{(s+\frac{\delta+1}{q})}(z)\right)} \tag{2.15}$$

where

$$SC_{v,j}^{(s+\frac{\delta+1}{q})}(Z) = \frac{1}{\mu_j}\left[\sum_{i=1}^{j-1}\mu_i\left(\frac{1}{z_j^{(s)}-z_i^{(s+\frac{\delta+1}{q})}}\right)^v + \sum_{i=j+1}^{N}\mu_i\left(\frac{1}{z_j^{(s)}-z_i^{(s+\frac{\delta}{q})}+C_{w,i}^{(s+\frac{\delta}{q})}}\right)^v\right];\, v=1(1)l \tag{2.16}$$

for a chosen l and efficient $C_{w,l}^{(s)}$,then again using (21) in the appendix, the propagation of error is such that

$${}_p\varepsilon_j^{(s+\frac{\delta+1}{q})} = o\left(\left({}_p\varepsilon_j^{(s)}\right)^{(K_p-1)}\right)\left(\sum_{i=1}^{j-1}o\left({}_p\varepsilon_i^{(s+\frac{\delta+1}{q})}\right)+\sum_{i=j+1}^{N}o\left(\left({}_p\varepsilon_i^{(s+\frac{\delta}{q})}\right)^{p_1}\right)\right);\begin{array}{c} p_1 = 2(1)K_p-1 \\ s=1(1)m-1 \end{array} \tag{2.17}$$

here $\delta = 0(1)q-1; q = 1,2,3,\ldots$ In this case the error propagation with once application of Gauss-Seidel type updating in the inclusion part of the combination method, is somewhat complicated, but this is obtained by setting ${}_p\varepsilon_j^{(m)}$ derived from (2.14) or (2.17) accordingly to $\varepsilon_j^{(0)}$ in (2.3). Like before, the least number of iterations to calculate $r^{(m,1)}$ to the accuracy $o(10^{-t})$ in this circumstance, can then similarly be estimated from

$$r_j^{(m,1)} = o\left(\left({}_p\varepsilon_j^{(m)}\right)^{K_I-1}\right)\left(\sum_{i=1}^{j-1}o\left(r_i^{(m,1)}\right)+\sum_{i=j+1}^{N}o\left(r_i^{(0)}\right)\right) = o\left(\left({}_p\varepsilon_j^{(m)}\right)^{K_I-1}\right)o\left(r_i^{(0)}\right) \tag{2.18}$$

and for $\varepsilon^{(m,1)}$ is to use

$$\varepsilon_j^{(m,1)} = o\left(\left({}_p\varepsilon_j^{(m)} \right)^{K_I - 1} \right) \begin{cases} \left(\sum_{i=1}^{j-1} o\left(\varepsilon_i^{(m,1)} \right) + \sum_{i=j+1}^{N} o\left(\left(\varepsilon_i^{(0)} \right)^{p_2} \right) \right); p_2 = 2,3,\ldots,K_I - 1; \beta = 0 \\ \left(\sum_{i=1}^{j-1} o\left(\varepsilon_i^{(m,1)} \right) + \sum_{i=j+1}^{N} o\left(\left(r_i^{(0)} \right)^2 \right) \right); p_2 \geq 2, \beta = 1 \end{cases} \tag{2.19}$$

where ${}_p\varepsilon_j^{(m)}$ is derived from (2.14) or (2.17) accordingly. In particular

$$ {}_p\varepsilon_j^{(m)} = o\left(\left(\varepsilon_j^{(0)} \right)^{\left(p_1^q + \left(\frac{1-p_1^q}{1-p_1} \right)(K_p - 1) \right)^m} \right); 2 \leq p_1 \leq K_p - 1, m = 1,2,\ldots \tag{2.20}$$

is the case from (2.14). Thus

$$M_1 \geq 1 + Int\left(log\left(\frac{t-1}{K_I - 1} \right) / log\left(\left(\frac{1-p_1^q}{1-p_1} \right)(K_P - 1) + p_1^q \right) \right); \; 2 \leq p_1 \leq K_p - 1 \tag{2.21}$$

for $r^{(m,1)}$ using (2.18) or (2.19) and for $\varepsilon^{(m,1)}$

$$M_{2,1} \geq 1 + Int\left(log\left(\frac{t-p_2}{K_I - 1} \right) / log\left(\left(\frac{1-p_1^q}{1-p_1} \right)(K_P - 1) + p_1^q \right) \right); \; \begin{matrix} 2 \leq p_1 \leq K_p - 1, \beta = 0 \\ 2 \leq p_2 \leq K_I - 1 \end{matrix} \tag{2.22}$$

$$M_{2,2} \geq 1 + Int\left(log\left(\frac{t-2}{K_I - 1} \right) / log\left(\left(\frac{1-p_1^q}{1-p_1} \right)(K_P - 1) + p_1^q \right) \right); \; \begin{matrix} p_2 \geq 2, \beta = 1 \\ 2 \leq p_1 \leq K_p - 1, \end{matrix} \tag{2.23}$$

from (2.19) using (2.20). The error behaviour in (2.17) is complicated, but can be approximated by (2.14) in the sense of writing (2.17) as

$$ {}_p\varepsilon^{(s+\frac{\delta+1}{q})} = o\left(\left({}_p\varepsilon^{(s)} \right)^{(K_p - 1)} \right) \left(o\left({}_p\varepsilon^{(s+\frac{\delta+1}{q})} \right) + o\left(\left({}_p\varepsilon^{(s+\frac{\delta}{q})} \right)^{p_1} \right) \right); \begin{matrix} p_1 = 2(1)K_p - 1 \\ s = 1(1)m - 1 \end{matrix} \tag{2.24}$$

It could be expressed that

$$ {}_p\varepsilon^{(s+\frac{\delta+1}{q})} = \frac{1}{1 - o\left(\left({}_p\varepsilon^{(s)} \right)^{K_p - 1} \right)} o\left(\left({}_p\varepsilon^{(s)} \right)^{K_p - 1} \right) o\left(\left({}_p\varepsilon^{(s+\frac{\delta}{q})} \right)^{p_1} \right) \tag{2.25}$$

from which,

$$ {}_p\varepsilon_j^{(s+1)} = \frac{1}{\left(1 + o\left(\left({}_p\varepsilon_j^{(s)} \right)^{K_p - 1} \right) \right)^{\frac{1-p_1^q}{1-p_1}}} o\left(\left({}_p\varepsilon_j^{(s)} \right)^{p_1^q + \left(\frac{1-p_1^q}{1-p_1} \right)(K_p - 1)} \right)$$

$$\cong \left(1 + o\left(\left({}_p\varepsilon_j^{(s)} \right)^{K_p - 1} \right) \right)^{\frac{1-p_1^q}{1-p_1}} o\left(\left({}_p\varepsilon_j^{(s)} \right)^{p_1^q + \left(\frac{1-p_1^q}{1-p_1} \right)(K_p - 1)} \right) \cong o\left(\left({}_p\varepsilon_j^{(s)} \right)^{p_1^q + \left(\frac{1-p_1^q}{1-p_1} \right)(K_p - 1)} \right);$$

$$2 \leq p_1 \leq K_p - 1, s = 0(1)m - 1 \tag{2.26}$$

by assuming that K_p is reasonably large enough to ensure that $o\left(\left({}_p\varepsilon^{(s)} \right)^{K_p - 1} \right)$ is insignificant in magnitude when convergence sets in, in the long run on s. Then (2.14) in some sense approximates (2.17) in error terms and the conclusions (2.21, 2.22, 2.23) therefrom reasonably apply in this case as well. The application of these updating procedures gives lesser number of iterations to the desired accuracy than the other approaches considered above as would be noticed from (2.21), (2.22) and (2.23). Based on the experience of [9], we have thus to remark that

* When updating is needless in the point method in (1.2) then the order of the correction can be such that $2 \le p_1 \le K_p - 1$. But when Gauss-Jacobi updating $(q \ge 2)$ is to be applied on the point method then the order of the correction need not exceed $p_1 = 2$, higher order correction may present only marginal or not any advantage in convergence, see [9, equation (6.9a)]. Similarly, if Gauss-Seidel type updating is to be employed, then the order of correction has the flexibility of lying in the range $2 \le p_1 \le K_p - 1$.

* Applying the Gauss-Seidel type updating $(q = 1)$ in the inclusion part of the method of the hybrid combination (1.2; 1.23), with the choice of inversion as $\beta = 1$, then the order of correction in the inclusion method need not exceed $p_2 = 2$, because in this case there is only a marginal convergence advantage in higher order correction, in the otherwise of $\beta = 0$ then we can have $2 \le p_2 \le K_I - 1$. It is quite possible to pick $C_{u,i}^{*(0)}$ to have $p_2 \ge K_I$ or in fact, $2 \le p_2 \le Max\{p_1 + K_p - 1, K_I\}$ without hindering the efficiency of the process, such possibility is considered in what now will follow and examples of such methods are given in the sequel.

Interestingly, we could take the correction, $C_{u,i}^{*(0)}$ in the interval part of the algorithm to be

$$ {}_kC_{l,j}^{*(0)} = \frac{\Delta_{l-1,j}\left(z_j^{(0)}\right)}{\Delta_{l,j}\left(z_j^{(0)}\right) - B_l\left(SC_{1,j}^{(0)}(z), SC_{2,j}^{(0)}(z), \ldots, SC_{l,j}^{(0)}(z)\right)} \tag{2.27} $$

since it is known to be available in the beginning of the point iteration and still retain the efficiency of the hybridisation. When $SC_{l,j}^{(s)}(z)$ is as in (1.3), the order of the point arithmetic method in (1.2) is $K_p = l + 2,$ and if it is as in (1.7) then we may speak of order as the R-order of convergence of which lower bound is found to be

$$ O_R \ge \begin{cases} p_1 + l + 1; \ \ p_1 = 2, 3, \ldots, l + 1 \\ l + 1 + \sqrt{(l+1)^2 + 1}; \ p_1 \ge l + 2 \end{cases} \tag{2.28} $$

The interval arithmetic method in (1.2) with $S_{v,j}^{(0)}(Z)$ defined in (1.16) has the lower bound

$$ O_R \ge \begin{cases} \dfrac{k + 2 + \sqrt{(k+2)^2 + 4(k+1)}}{2}; p_2 \ge 2, \beta = 1 \\ p_2 + k + 1; \ \ p_2 = 2, 3, 4, \ldots, k + 1, \beta = 0 \\ k + 1 + \sqrt{(k+1)^2 + 1}; \ p_2 \ge k + 2, \beta = 0 \end{cases} \tag{2.29} $$

of *R*-order, obtained from (16) in the appendix, by a generalisation of the analysis in [9], [16] on the corrected Wang and Zheng method. It can now be seen in (2.29) with $\beta = 0$ the marginal advantage in convergence of having ${}_kC_{k,i}^{*(0)}$ in place of $C_{k,i}^{*(0)}$. In particular, suppose (1.2) has its components as defined in (1.3) and (1.23) let

$l = k$ then $p_2 = K_I$, this incidentally, gives maximum lower bound R-order of convergence for the inclusion method when $\beta = 0$ in the hybrid combination. This method will be illustrated by some particular examples. Finally, the idea of the application of the updating approach highlighted earlier under a correction process gives by far further improvement of results in all. The case in which the order of correction in the point arithmetic part is such that $p_1 \geq K_p$ often turn out to be inefficient and may therefore not be recommended, though like before, it leads to maximum lower bound R-order of convergence for the point iteration in the combination algorithm, see [9] and in general when there is no intention of applying correction at all in a constituent method of the combination, set $C_{u,i}^{(0)} = 0$ or $C_{u,i}^{*(0)} = 0$ and correspondingly $p_1 = 1$ or $p_2 = 1$ in all of the above analysis accordingly.

3.0 Examples of the hybrid combination methods and their efficiency index

The first, is the simplest member of the collections of (1.2, 1.7, 1.23)

CM(1,1):

$$z_i^{(s+1)} = z_i^{(s)} - \frac{\mu_i}{\dfrac{1}{N_i^{(s)}} - \sum_{\substack{j=1 \\ j\neq i}}^{N} \dfrac{\mu_j}{z_i^{(s)} - z_j^{(s)} + C_{1,j}^{(s)}}} ; i = 1(1)n; s = 0(1)m-1$$

$$Z_i^{(m,1)} = z_i^{(m)} - \frac{\mu_i}{\dfrac{1}{N_i^{(m)}} - \sum_{\substack{j=1 \\ j\neq i}}^{N} \mu_j . INV\left(z_i^{(m)} - Z_j^{(0)} + C_{1,j}^{*(0)}\right)} \qquad (3.1)$$

in the class of the hybrid combination of methods (1.2), with $C_{1,j}^{*(0)}$ chosen from any of the correctors in

$$C_{1,j}^{*(0)} = \begin{cases} C_{1,j}^{(0)} = \mu_j N_j^{(0)} ; N_j^{(0)} = \dfrac{P_n(z_j^{(0)})}{P_n^{/}(z_j^{(0)})} \text{ or} \\ {}_1C_{1,j}^{*(0)} = \dfrac{\mu_j}{\dfrac{1}{N_j^{(0)}} - \sum\limits_{i=1, j\neq i}^{N} \dfrac{\mu_i}{z_j^{(0)} - z_i^{(0)} + C_{1,i}^{(0)}}} \end{cases} \qquad (3.2)$$

However, a method of improved computational complexity to the above is

$$z_i^{(s+1)} = z_i^{(s)} - \mu_i N_i^{(s)} ; i = 1(1)n; s = 0(1)m-1$$

$$Z_i^{(m,1)} = z_i^{(m)} - \frac{\mu_i}{\dfrac{1}{N_i^{(m)}} - \sum_{\substack{j=1 \\ j\neq i}}^{N} \mu_j . INV\left(z_i^{(m)} - Z_j^{(0)} + C_{1,j}^{(0)}\right)} \qquad (3.3)$$

and another is the combination method

CM(2,k) :

$$z_i^{(s+1)} = z_i^{(s)} - \frac{\mu_i}{\dfrac{1}{N_i^{(s)}} - \sum_{\substack{j=1 \\ j\neq i}}^{N} \dfrac{\mu_j}{z_i^{(s)} - z_j^{(s)} + C_{1,j}^{(s)}}} ; i = 1(1)n; s = 0(1)m-1$$

$$Z_i^{(m,1)} = z_i^{(m)} - \frac{1}{\dfrac{1}{H_i^{(m)}} - \dfrac{N_i^{(m)}}{2}\left[\sum_{\substack{j=1 \\ j\neq i}}^{N} \mu_j . \left(INV\left(D_{i,j,1}^{*(m,0)}(Z)\right)\right)^2 + \dfrac{1}{\mu_i}\left(\sum_{\substack{j=1 \\ j\neq i}}^{N} \mu_j . INV\left(D_{i,j,1}^{*(m,0)}(Z)\right)\right)^2 \right]} \qquad (3.4)$$

with the abbreviation that $D_{i,j,1}^{*(m,0)}(Z) = z_i^{(m)} - Z_j^{(0)} + C_{1,j}^{*(0)} ; i = (1)n, s = 1(1)m-1$. Furthermore, is the hybrid

CM(3,k):

$$z_i^{(s+1)} = z_i^{(s)} - \frac{1}{\dfrac{1}{H_i^{(s)}} - \dfrac{N_i^{(s)}}{2}\left[\sum_{\substack{j=1 \\ j\neq i}}^{N} \mu_j \left(\dfrac{1}{P_{i,j,k}^{(s)}(z)} \right)^2 + \dfrac{1}{\mu_i}\left(\sum_{\substack{j=1 \\ j\neq i}}^{N} \dfrac{\mu_j}{P_{i,j,k}^{(s)}(z)} \right)^2 \right]} \qquad (3.5)$$

$$Z_i^{(m,1)} = z_i^{(m)} - \frac{1}{\frac{1}{H_i^{(m)}} - \frac{N_i^{(m)}}{2}\left[\sum_{\substack{j=1\\j\neq i}}^{N}\mu_j.\left(INV\left(D_{i,j,k}^{*(m,0)}(Z)\right)\right)^2 + \frac{1}{\mu_i}\left(\sum_{\substack{j=1\\j\neq i}}^{N}\mu_j.INV\left(D_{i,j,k}^{*(m,0)}(Z)\right)\right)^2\right]}$$

with $C_{2,i}^{*(0)}$ any of the varieties

$$\text{s}\, C_{2,i}^{*(0)} = \begin{cases} C_{1,i}^{(0)} = \mu_i.N_i^{(0)} & \text{or} \\ C_{2,i}^{(0)} = H_i^{(0)} & \text{or} \\ {}_2C_{2,i}^{*(0)} = \dfrac{1}{\frac{1}{H_i^{(0)}} - \frac{N_i^{(0)}}{2}\left[\sum_{\substack{j=1\\j\neq i}}^{N}\mu_j\left(\frac{1}{P_{i,j,k}^{(0)}(z)}\right)^2 + \frac{1}{\mu_i}\left(\sum_{\substack{j=1\\j\neq i}}^{N}\frac{\mu_j}{P_{i,j,k}^{(0)}(z)}\right)^2\right]} & ;k=1,2 \end{cases} \tag{3.6}$$

with $$H_i^{(s)} = \frac{1}{\frac{P_n^{/}(z_i^{(s)})}{2P_n(z_i^{(s)})}\left(1+\frac{1}{\mu_i}\right) - \frac{P_n^{//}(z_i^{(s)})}{2P_n^{/}(z_i^{(s)})}};\ P_{i,j,k}^{(0)}(z) = z_i^{(0)} - z_j^{(0)} + C_{k,j}^{(0)}\ ;k=1,2 \tag{3.7}$$

It will be noticed that the cases of the methods for which $C_{k,i}^{*(0)} = {}_kC_{k,i}^{*(0)}; k=1,2,...$ are as efficient as the others. In any case, it is not mandatory to have $K_p \le K_I$ that is $l \le k$ and more so, the order of correction in the point method may be chosen not to differ from that in the interval algorithm as in some methods of the above. Finally, an example of method for which $K_p > K_I$ arise when $l > k$ in the hybrid,

CM(4,k): $$z_i^{(s+1)} = z_i^{(s)} - \frac{1}{\frac{1}{H_i^{(s)}} - \frac{N_i^{(s)}}{2}\left[\sum_{\substack{j=1\\j\neq i}}^{N}\mu_j\left(\frac{1}{P_{i,j,k}^{(s)}(z)}\right)^2 + \frac{1}{\mu_i}\left(\sum_{\substack{j=1\\j\neq i}}^{N}\frac{\mu_j}{P_{i,j,k}^{(s)}(z)}\right)^2\right]};\quad k=1,2$$

$$Z_i^{(m,1)} = z_i^{(m)} - \frac{\mu_i}{\frac{1}{N_i^{(m)}} - \sum_{\substack{j=1\\j\neq i}}^{N}\mu_j.INV\left(z_i^{(m)} - Z_j^{(0)} + {}_2C_{2,j}^{*(0)}\right)} \tag{3.8}$$

Furthermore, this and the class of methods which $K_p \ge K_I$ are the methods favoured by theory, because their error propagation decay far more rapidly compared to when $K_p < K_I$, see for example (1.20), (1.26) or (1.28). The updating procedures (2.1,2.12,2.15) can easily be applied accordingly to these highlighted methods. Several other similar hybridisations of methods by this can easily be constructed. Obviously in (1.2), for higher order k the methods becomes uncontrollably robust, unwieldy, complicated and with the implementation quite demanding. However, for the performance evaluation of an iteration method, are its computational complexity and efficiency index. In a point arithmetic process the complexity $\theta_p(n)$ takes into account the normalised total number of basic arithmetic operations per iteration, where the efficiency is given by

$$E_p(n) = \left(K_p\right)^{\frac{1}{\theta_p(n)}} \tag{3.9}$$

The efficiency of an interval method is similarly, $E_I(n) = (K_I)^{\frac{1}{\theta_I(n)}}$ (3.10)
whereas the efficiency of the combination method which employ both point and interval arithmetic, for example (1.2), is $E_{CM}(n) = \left[(K_I - 1)K_p^{M_1} + 1\right]^{\frac{1}{M_1\theta_p(n)+\theta_I(n)}}$ (3.11)
see [14], where M_1 is obtained from (1.21) with $\theta_p(n)$ and $\theta_I(n)$ as the normalised total cost of basic arithmetic operational count in the point and interval method constituting the hybrid combination method respectively. That of (2.12,2.15) is similarly given as

$$E_{CM}(n) = \left[\left(\frac{1-p_1^q}{1-p_1}(K_P - 1) + p_1^q\right)^{M_1}(K_I - 1) + 1\right]^{\frac{1}{M_1\theta_p(n)+\theta_I(n)}} ;\beta = 0,1 \qquad (3.12)$$

here M_1 given by (2.21), is the minimum number of iterations to achieve a minimum accuracy of $o(10^{-t})$ in $r^{(m,1)}$ and $\varepsilon^{(m,1)}$ simultaneously. However, we could in the estimate of $E_{CM}(n)$ use *R*-order of the constituent method instead of order of convergence. To compute the efficiency index of the methods above it is worthwhile

to adopt the optimised computational complexity count of real arithmetic operations in [3].

4.0 Results from numerical experiments and conclusion.

Consider the problem of isolating the eigenvalues of the Hessenberg's matrix ***H***, P[1]:

$$H = \begin{pmatrix} 8+12i & 1 & 0 & 0 \\ 0 & 6+9i & 1 & 0 \\ 0 & 0 & 4+6i & 1 \\ 1 & 0 & 0 & 2+3i \end{pmatrix}$$

$$Det(H - zI) = z^4 - (20+30i)z^3 + (175+420i)z^2 + (2300-450i)z - 2857 - 2880i$$

in the disks $\{Z_j^{(m,1)}\}_{j=1,m=1}^{4,t} ;t = 1,2$ by starting with the inclusion disks $Z_j^{(0)} = \{h_{jj}; R^{(0)}\}_{j=1(1)4}$; $R^{(0)} = 0.4(0.2)1.2$. We have considered (3.8) in the form

$$z_i^{(s+1)} = z_i^{(s)} - \frac{1}{\dfrac{1}{H_i^{(s)}} - \dfrac{N_i^{(s)}}{2}\left(\left(\sum_{\substack{j=1\\j\neq i}}^{n}\dfrac{1}{z_i^{(s)} - z_j^{(s)} + C_{t,j}^{(s)}}\right)^2 + \left(\sum_{\substack{j=1\\j\neq i}}^{n}\dfrac{1}{z_i^{(s)} - z_j^{(s)} + C_{t,j}^{(s)}}\right)^2\right)}$$

$$Z_i^{(m,1)} = z_i^{(m)} - \frac{1}{\dfrac{1}{N_i^{(m)}} - \sum_{\substack{j=1\\j\neq i}}^{n}\dfrac{1}{z_i^{(m)} - Z_j^{(0)}}} \qquad (4.1)$$

and with updating $Z_i^{(m,1)} = z_i^{(m)} - \frac{1}{\frac{1}{N_i^{(m)}} - \sum_{j=1}^{i-1}\frac{1}{z_i^{(m)} - Z_j^{(m,1)}} + \sum_{j=i+1}^{n}\frac{1}{z_i^{(m)} - Z_j^{(0)}}}$. The improved Maehly's method is

$$Z_j^{(s+1)} = z_j^{(s)} - \frac{1}{\dfrac{1}{N_j^{(s)}} - \sum_{\substack{i=1\\i\neq j}}^{n} INV\left(z_j^{(s)} - Z_i^{(s)} + C_{m,i}^{(s)}\right)} \qquad (4.2)$$

Tables (4.1- 4.6) show the results of the numerical computations using methods (4.1). The maximum radii $r^{(s)} = \underset{j=1(1)n}{Max}\{r_j^{(s)}\}$ are compared with results from [14] displayed in table (4.2). The exact eigenvalues of the Hessenberg's matrix ***H*,** which are indeed**,** the zeros of $P_4(z) = Det(H - zI)$ are in Table 4.1, but the last digit may be in error.

Table 4.1**:** Exact zeros of $P_4(z) = Det(H - zI)$

7.99650507021978 + 11.99932088106346*i*…
6.01045579118225 + 9.00205697329131*i*…
3.98954420881768 + 5.99794302670865*i*…
2.00349492978029 + 3.00067911893660*i*…

Table 4.2:Error in Point Maehly's Method (4.2): $\varepsilon^{(s)}$

	$C_j^{(s)} = N_j^{(s)}$	$C_j^{(s)} = 0$
s=1	1.1(-2)	1.1(-2)
s=2	9.4(-11)	1.2(-7)

Table 4.3: Maximum radii for P[1]

	Algorithm (4.1), $C_{2,i}^{(0)} = N_i^{(0)}$					*Algorithm* (4.1); $C_{2,i}^{(0)} = H_i^{(0)}$				
$R^{(0)}$	0.4	0.6	0.8	1.0	1.2	0.4	0.6	0.8	1.0	1.2
$r^{(1,1)}$	4.0(-26)	6.1(-26)	8.3(-26)	1.2(-25)	1.3(-25)	6.6(-29)	1.0(-28)	1.4(-28)	1.7(-28)	2.2(-28)
$r^{(2,1)}$	1.6(-28)	2.4(-28)	3.3(-28)	4.3(-28)	5.3(-28)	1.7(-28)	2.6(-28)	3.6(-28)	4.6(-28)	5.6(-28)

Table 4.4: Maximum radii for P[1]

	Algorithm (4.1), $C_{t,j}^{(s)} = 0$				
$R^{(0)}$	0.4	0.6	0.8	1.0	1.2
$r^{(1,1)}$	8.2(-21)	1.2(-20)	1.7(-20)	2.1(-20)	2.7(-20)
$r^{(2,1)}$	8.8(-29)	1.3(-28)	1.8(-28)	2.3(-28)	2.9(-28)

Table 4.5: Maximum radii for P[1***]; $C_{t,j} = N_j$***

	Algorithm (4.2); Z^{I_2}					*Algorithm* (4.2); Z^{I_1}				
$R^{(0)}$	0.4	0.6	0.8	1.0	1.2	0.4	0.6	0.8	1.0	1.2
$r^{(1)}$	1.6(-5)	2.4(-5)	3.3(-5)	4.2(-25)	5.2(-5)	8.8(-6)	1.4(-5)	2.0(-5)	2.7(-5)	3.5(-5)
$r^{(2)}$	1.5(-21)	2.4(-20)	5.6(-20)	2.0(-28)	5.8(-19)	3.7(-23)	3.9(-22)	2.3(-21)	9.8(-21)	3.6(-20)
$r^{(3)}$	9.8(-49)	1.8(-48)	2.3(-47)	1.4(-46)	1.1(-46)	1.0(-50)	2.9(-50)	7.5(-49)	8.8(-49)	1.5(-47)

Table 4.6: Maximum radii for P[1]

Petkovic [14; p. 51,52]

	Algorithm (3.7); p.51					*Algorithm* (3.8); p.52				
$R^{(0)}$	0.4	0.6	0.8	1.0	1.2	0.4	0.6	0.8	1.0	1.2
$r^{(1)}$	3.3(-3)	5.4(-3)	7.9(-3)	1.1(-2)	1.5(-2)	3.6(-3)	6.5(-3)	1.2(-2)	3.5(-2)	7.7(-2)
$r^{(2)}$	3.9(-7)	1.4(-6)	3.7(-6)	8.4(-6)	1.7(-5)	1.4(-6)	7.3(-6)	3.5(-6)	3.2(-4)	1.6(-3)
$r^{(3)}$	2.8(-15)	5.1(-14)	4.5(-13)	2.7(-12)	1.3(-11)	1.3(-13)	5.2(-12)	1.8(-10)	2.2(-8)	1.1(-6)

Compare these with results in table (4.6) from Petkovic [14].

Appendix

The error propagation in the Wang and Zheng's method and its enhancement

The Wang and Zheng's method, from [14] is the family of methods

$$Z_j^{(s+1)} = z_j^{(s)} - \frac{\Delta_{k-1,j}\left(z_j^{(s)}\right)}{\Delta_{k,j}\left(z_j^{(s)}\right) - B_k\left(S_{1,j}^{(s)}(Z), S_{2,j}^{(s)}(Z), \ldots, S_{k,j}^{(s)}(Z)\right)}; k = 1,2,3,\ldots \qquad (1)$$

where

$$S_{v,j}^{(s)}(Z) = \frac{1}{\mu_j}\sum_{\substack{i=1 \\ j\neq i}}^{N} \mu_i \left(\frac{1}{z_j^{(s)} - Z_i^{(s)}}\right)^v ; v = 1(1)k \qquad (2)$$

An improvement of this is the new class of correction methods

$$Z_j^{(s+1)} = z_j^{(s)} - \frac{\Delta_{k-1,j}\left(z_j^{(s)}\right)}{\Delta_{k,j}\left(z_j^{(s)}\right) - B_k\left(SI_{1,j}^{(s)}(Z), SI_{2,j}^{(s)}(Z), \ldots, SI_{k,j}^{(s)}(Z)\right)}; k = 1,2,3,\ldots \qquad (3)$$

where

$$SI_{v,j}^{(s)}(Z) = \frac{1}{\mu_j}\sum_{\substack{i=1 \\ j\neq i}}^{N} \mu_i \left(INV\left(z_j^{(s)} - Z_i^{(s)} + C_i^{(s)}\right)\right)^v ; v = 1(1)k \qquad (4)$$

for a fixed k. The term $C_i^{(s)}$ is as defined, a correction factor introduced from the point method $z_i^{(s+1)} = z_i^{(s)} - C_i^{(s)}$ of order p to enhance the convergence rate of the basic methods (1). The choice is influence

by efficiency consideration. To establish the error propagation of the new family of methods (3) and for a simplified analysis, is to assume that the zeros of the polynomial are simple, even then the conclusion is still valid for multiple zeros, we re-arrange (3) as

$$Z_j^{(s+1)} = z_j^{(s)} - \left(\frac{\Delta_{k-1,j}\left(z_j^{(s)}\right)}{\Delta_{k,j}\left(z_j^{(s)}\right)}\right) \frac{1}{1 - \frac{1}{\Delta_{k,j}\left(z_j^{(s)}\right)} B_k\left(SI_{1,j}^{(s)}(Z), SI_{2,j}^{(s)}(Z), \ldots, SI_{k,j}^{(s)}(Z)\right)} \qquad (5)$$

By this, this method can be written in a new notation of letting $Z_j, \hat{Z}_j, r_j, \hat{r}_j, z_j, \hat{z}_j$ represent $Z_j^{(s)}, Z_j^{(s+1)}, r_j^{(s)}, r_j^{(s+1)}, z_j^{(s)}, z_j^{(s+1)}$, as

$$\hat{Z}_j = z_j - \left(\frac{\Delta_{k-1,j}(z_j)}{\Delta_{k,j}(z_j)}\right)\left(H_{2,j}\right)^{-1} = z_j - \left(\frac{\Delta_{k-1,j}(z_j)}{\Delta_{k,j}(z_j)}\right)\left(\frac{1}{\{u_j; \rho_j\}}\right) \qquad (6)$$

where the definitions are that

$$H_{2,j}=1-\frac{1}{\Delta_{k,j}(z)}B_k\left(SI_{1,j}(Z),SI_{2,j}(Z),\Lambda\ ,SI_{k,j}(Z)\right)=\{u_j;\rho_j\}$$

$$u_j=1-T_{2,j};$$

$$\rho_j=\frac{1}{\left|\Delta_{k,j}\left(z_j^{(s)}\right)\right|}Rad\left(B_k\left(SI_{1,j}^{(s)}(Z),SI_{2,j}^{(s)}(Z),.\Lambda\ ,SI_{k,j}^{(s)}(Z)\right)\right) \qquad (7)$$

with

$$T_{2,j}=\frac{1}{\Delta_{k,j}\left(z_j^{(s)}\right)}B_k\left(SI_{1,j}^{(s)}(Z),\Lambda\ ,SI_{k,j}^{(s)}(Z)\right)=\frac{1}{\Delta_{k,j}\left(z_j^{(s)}\right)}B_k\left(SI_{1,j}^{(s)}(Z;\beta),\Lambda\ ,SI_{k,j}^{(s)}(Z;\beta)\right)$$

and

$$SI_{v,j}(Z;\beta)=\sum_{\substack{i=1\\i\neq j}}^{n}\left(\frac{1}{v_{ij}\left(1-\frac{\beta r_i^2}{|v_{ij}|^2}\right)}\right)^v;\beta=0,1;v=1(1)k; \qquad (8)$$

Method (3) is equivalently,

$$\hat{z}_j=Mid\left(\hat{Z}_j\right)=z_j-\left(\frac{\Delta_{k-1,j}(z_j)}{\Delta_{k,j}(z_j)}\right)\frac{\bar{u}_j}{|u_j|^2-(\rho_j)^2};$$

$$\hat{r}_j=Rad\left(\hat{Z}_j\right)=\left|\frac{\Delta_{k-1,j}}{\Delta_{k,j}}\right|\frac{\rho_j}{|u_j|^2-(\rho_j)^2} \qquad (9)$$

Because

$$\frac{\Delta_{k-1,j}\left(z_j^{(s)}\right)}{\Delta_{k,j}\left(z_j^{(s)}\right)}=(z_j-\lambda_j)\left[1-\frac{1}{\Delta_{k,j}\left(z_j^{(s)}\right)}B_k\left(S_{1,j}^{(s)}(\lambda),S_{2,j}^{(s)}(\lambda),\Lambda\ ,S_{k,j}^{(s)}(\lambda)\right)\right] \qquad (10)$$

where we have set

$$S_{v,j}(\lambda)=\sum_{\substack{i=1\\i\neq j}}^{n}\left(\frac{1}{z_j-\lambda_i}\right)^v;v=1(1)k \qquad (11)$$

the method (3) reduces to the error relation

$$\hat{\varepsilon}_j=\frac{\varepsilon_j\left(T_{2,j}(\beta)t_j-t_j+\left(\frac{1}{\Delta_{k,j}(z)}B_k\left(S_{1,j}(\lambda),S_{1,j}(\lambda),\Lambda\ ,S_{k,j}(\lambda)\right)-T_{2,j}(\beta)\right)\right)}{\left(1-T_{2,j}(\beta)\right)\left(1-t_j\right)} \qquad (12)$$

On simplification, employing the identities

$$A^{\frac{1}{k}}-B^{\frac{1}{k}}=\frac{A-B}{\sum_{j=0}^{k-1}A^{\frac{k-j-1}{k}}B^{\frac{j}{k}}};A^k-B^k=(A-B)\sum_{j=0}^{k-1}B^jA^{k-j-1},\ k\geq 1 \qquad (13)$$

establish that $\frac{1}{\Delta_{k,j}\left(z_j^{(s)}\right)}=o\left(\varepsilon^k\right)$. Furthermore, it can be seen that

$$T_2 = \frac{1}{\Delta_{k,j}(z)} B_k\left(S_{1,j}(\lambda),\Lambda, S_{k,j}(\lambda)\right) - T_{2,j}(\beta)$$

$$= \begin{cases} o\left(\varepsilon^{p+k}\right);\ \beta = 0 \\ o\left(\varepsilon^{p+k}\right) + o\left(r^2\varepsilon^k\right);\ \beta = 1 \end{cases};\ p \ge 2 \tag{14}$$

in the notations of [9,16], in which then $\rho_j = o\left(\varepsilon^k r\right)$. Conclusively, is the error relation

$$\hat{r}_j = o\left(\varepsilon^{k+1} r\right);$$

$$\hat{\varepsilon}_j = o\left(r^2\varepsilon^{3k+1}\right) + o\left(r^2\varepsilon^{2k+1}\right) + \begin{cases} o\left(\varepsilon^{p+k+1}\right) + \beta\, o\left(r^2\varepsilon^{k+1}\right);\ \beta = 1 \\ o\left(\varepsilon^{p+k+1}\right);\ \beta = 0 \end{cases} \tag{15}$$

of the methods in (3) for a general k and p. Considering dominant terms, the fact that $|\varepsilon_i| < \varepsilon \le r$ and we can put $\varepsilon = o(r)$, thus the errors in the family of methods (3) propagates its effect as

$$r_j^{(s+1)} = o\left(r^{(s)}\left(\varepsilon^{(s)}\right)^{k+1}\right);$$

$$\varepsilon_j^{(s+1)} = \begin{cases} \left[o\left(r^{(s)2}\right) + o\left(\varepsilon^{(s)p}\right)\right] o\left(\left(\varepsilon^{(s)}\right)^{k+1}\right);\ \beta = 1 \\ \left[o\left(r^{(s)2}\right) + o\left(\left(\varepsilon^{(s)}\right)^{p-k}\right)\right] o\left(\left(\varepsilon^{(s)}\right)^{2k+1}\right);\ \beta = 0 \end{cases} \tag{16}$$

The R-order of convergence of (3) is deduced by application of the Schmidt theorem accordingly on this error-relations in (16) which result is stated in (2.28) and (2.29). The Gauss-Jacobi style of updating in (3) q-times is

$$Z_j^{(s+\frac{\delta+1}{q})} = z_j^{(s)} - \frac{\Delta_{k-1,j}\left(z_j^{(s)}\right)}{\Delta_{k,j}\left(z_j^{(s)}\right) - B_k\left(SI_{1,j}^{(s+\frac{\delta}{q})}(Z), SI_{2,j}^{(s+\frac{\delta}{q})}(Z), \Lambda, SI_{k,j}^{(s+\frac{\delta}{q})}(Z)\right)} \tag{17}$$

with

$$SI_{v,j}^{(s+\frac{\delta+1}{q})}(Z) = \sum_{\substack{i=1 \\ i \ne j}}^{n}\left(INV\left(z_j^{(s)} - Z_i^{(s+\frac{\delta}{q})} + C_i^{(s+\frac{\delta}{q})}\right)\right)^v;\ v = 1(1)k \tag{18}$$

$$\delta = 0,1,2,\Lambda, q-1;\ q = 1,2,3,\Lambda$$

the error propagates in the manner given by $r_j^{(s+\frac{\delta+1}{q})} = o\left(r^{(s+\frac{\delta}{q})}(\varepsilon^{(s)})^{k+1}\right);$

$$\varepsilon_j^{(s+\frac{\delta+1}{q})} = \begin{cases} \left[o\left(\left(r^{(s+\frac{\delta}{q})}\right)^2\right) + o\left(\left(\varepsilon^{(s+\frac{\delta}{q})}\right)^p\right)\right] o\left((\varepsilon^{(s)})^{k+1}\right);\ \beta = 1 \\ \left[o\left((\varepsilon^{(s)})^k\right) o\left(\left(r^{(s+\frac{\delta}{q})}\right)^2\right) + o\left(\left(\varepsilon^{(s+\frac{\delta}{q})}\right)^p\right)\right] o\left((\varepsilon^{(s)})^{k+1}\right);\ \beta = 0 \end{cases} \tag{19}$$

The Gauss-Seidel updating of the iterates on (3) gives

$$Z_j^{(s+\frac{\delta+1}{q})} = z_j^{(s)} - \frac{\Delta_{k-1,j}\left(z_j^{(s)}\right)}{\Delta_{k,j}\left(z_j^{(s)}\right) - B_k\left(SI_{1,j}^{(s+\frac{\delta+1}{q})}(Z), SI_{2,j}^{(s+\frac{\delta+1}{q})}(Z), \Lambda, SI_{k,j}^{(s+\frac{\delta+1}{q})}(Z)\right)} \tag{20}$$

where now

$$SI_{v,j}^{(s+\frac{\delta+1}{q})}(Z) = \sum_{i=1}^{j-1}\left(INV\left(z_j^{(s)} - Z_i^{(s+\frac{\delta+1}{q})}\right)\right)^v + \sum_{i=j+1}^{n}\left(INV\left(z_j^{(s)} - Z_i^{(s+\frac{\delta}{q})} + C_i^{(s+\frac{\delta}{q})}\right)\right)^v; \qquad (21)$$

$v = 1(1)k$

The error effect is

$$\varepsilon_j^{(s+\frac{\delta+1}{q})} = o\left(\left(\varepsilon_j^{(s)}\right)^{k+1}\right)\begin{cases}\left(\sum_{i=1}^{j-1} o\left(\varepsilon_i^{(s+\frac{\delta+1}{q})}\right) + \sum_{i=j+1}^{n} o\left(\left(\varepsilon_i^{(s+\frac{\delta}{q})}\right)^p\right)\right); \ \beta = 0\\ \left(\sum_{i=1}^{j-1} o\left(\varepsilon_i^{(s+\frac{\delta+1}{q})}\right) + \sum_{i=j+1}^{n} o\left(\left(r_i^{(s+\frac{\delta}{q})}\right)^2\right)\right); \ \beta = 1\end{cases} \qquad (22)$$

The lower bound of the R-order of convergence of (17) and (20) is similarly found by application of the Schmidt theorem on the error-relations respective, which is a subject of a future discussion, but how this can be done is highlighted in [9]. Conclusively, in the above, setting the correction term to zero in (4, 18, 20) imply that $p = 1$, in each case the resultant analysis gives the error propagation of the basic Wang and Zheng's Method accordingly.

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