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On the Cooley-Turkey Fast Fourier algorithm for arbitrary factors

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Abstract

Atonuje and Okonta in [1] ***developed the Cooley-Turkey Fast Fourier transform algorithm and its application to the Fourier transform of discretely sampled data points N, expressed in terms of a power y of 2. In this paper, we extend the formalism of*** [1] ***Cookey-Turkey Fast Fourier transform algorithm. The method is developed in this paper to guarantee the application of (C-TFFT) algorithm for arbitrary factors say*** $N = P_1 P_2$ **.**

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1.0 Introduction

We shall consider the discrete Fourier Transform of *N* points, usually written for any continuous function $f(x)$ as

$$F(u) = \sum_{x=0}^{N-1} f(x) e^{-2 \sin l N} \tag{1.1}$$

for $u = 0,1,,2,\Lambda\ ,N-1$. By defining *W* as the complex number

$$W = e^{-2 \sin l N} \tag{1.2}$$

The Fourier transform now becomes

$$F(u) = \sum_{x=0}^{N-1} f(x) W^{xx} \tag{1.3}$$

(1.3) simply means that the vector $f(x)$ is multiplied by a matrix whose $(u,x)^{th}$ element is a constant W to the power *ux*. Thus the matrix multiplication produces a vector result whose complements are $F(u)'s$. It is a documented fact that the general discrete Fourier transform (1.1) for *N* values of *u* corresponding to the discrete values $U = 0,1,2,\Lambda\ N-1$ is evaluated as follows

$$\begin{aligned}
F(0) &= f(0) exp(-2l\pi(0))x\frac{0}{N} + f(1) exp(-2l\pi(0))x\frac{1}{N} + \Lambda \\
F(1) &= f(0) exp(-2l\pi(1))x\frac{0}{N} + f(1) exp(-2l\pi(1))x\frac{1}{N} + \Lambda \\
&\text{M} \qquad\qquad\qquad \text{M} \qquad\qquad\qquad \text{M} \\
F(N-1) &= f(0) exp(-2l\pi(N-1))x\frac{0}{N} + f(1) exp(-2l\pi(N-1))x\frac{1}{N} + \Lambda
\end{aligned} \tag{1.4}$$

Evaluating of $F(0)$ requires a total of $(2N - 1)$ operations, since we have to perform N multiplications, (that is $f(x)exp(\cdot)$ and $N - 1$ additions.

Similarly, evaluation of $F(1)$ requires a total of (2N - 1) operations and so on unit the evaluation of $F(u)$ over all *N* values of *u*. This requires a total number of N^2 and $N(N-1)$ operations.

The Cooley-Turkey algorithm results in matrix factorization process, which introduces zeros into the factored matrices and as a result reduces the required number of multiplications and additions. The case where the number of sampled points is $N = 2^y$, a base 2 operation was presented by [1].

In this paper, we develop the fast Fourier transform algorithm which removes the assumption that $N = 2^y$, an integer. We show that significant time-savings can be obtained as long as *N* is highly composite, that is, $N = P_1 P_2 \Lambda\ P_{m,}$ where *P* is an integer.

2.0 **The new (C-T) Fast Fourier transform for arbitrary factors** $n = p_1 p_{2:}$

Assume that number of points N satisfies the relationship, $N = P_1 P_{2,}$ where P_1 and P_2 are integer-valued. To formulate the FFT algorithm for this case, we first express *n* and *k* indices in the discrete Fourier transform

$$f(n) = \sum_{2-0}^{N-1} f_0(k) w^{nk},\ n = 0,1,2\Lambda\ N-1 \tag{2.1}$$

as

$$\left.\begin{aligned} n &= n_1 p_1 + n_0 ; n_0 = 0,1,\mathrm{K}\ \ p_{1-}1; n_1 = 0,1,\mathrm{K}\ \ p_2 - 1 \\ k &= k_1 p_2 + k_0 ; k_0 = 01,\mathrm{K}\ \ p_2 - 1; k_1 = 0,1,\mathrm{K}\ \ p_1 - 1 \end{aligned}\right\} \tag{2.2}$$

We observe that the method (2.2) of writing the indices allows us to give a unique representation of each decimal integer. Using equation (2.2), w can re-write (2.1) as

$$f(n_1 n_0) = \sum_{k_0=0}^{p_2-1} \left[\sum_{k_1=0}^{p_2-1} f_0(k_1, k_0) w^{nk_1 p_2} \right] w^{nk_0} \tag{2.3}$$

Rewrite $w^{(n_1 p_1 + n_0) k_1 p_1}$ and we obtain

$$\begin{aligned} w^{nk_1 p_2} &= w^{(n_1 p_1 + n_0) k_1 p_2} \\ &= w^{p_1 p_2 n_2 k_1} w^{n_0 k_1 k_2} = \left[w^{p_1 p_2} \right]^{n_1 k_2} w^{n_0 k_1 p_2} = w^{n_0 k_1 p_2} \end{aligned} \tag{2.4}$$

where we use the fact that $w^{p_1 p_2} = w^N = 1$

From (2.4), we rewrite the inner sum of equation (2.3) as a new array

$$f_1(n_0, k_0) = \sum_{k_1=0}^{p_1-1} f_0(k_1 k_0) w^{n_0 k_1 k_2} \tag{2.5}$$

If we expand the terms w^{nk_e} the outer loop can be written as

$$f_2(n_0, n_1) = \sum_{k_1=0}^{p_1-1} f_1(n_0, k_0) w^{(n_1 p_1 + n_0) k_0} \tag{2.6}$$

The final result can be written as $f(n_1, n_0) = f_2(n_0, n_1)$ (2.7)

Thus as in base 2 algorithm presented by [1], equations (2.5), (2.6) and (2.7) are defining the FFT algorithm for the case $N = P_1 P_{2.}$

3.0 **Application**

3.1 **The formulation of the base 4 C-TFFT algorithm**

Consider the case *N* = 16. By the C-T algorithm, we write $N = P_1 P_2 = 4 \times 4$. That is, we will develop the base 4 fast Fourier transform for the case N = 16. Using arbitrary form, we can represent the variables *n* and *k* in the discrete Fourier transform $f(n) = \sum_{k-0}^{N-1} f_0(k) w^{nk}, n = 0,1,2,\mathrm{K}\ N-1$

(3.1)

In a base4 or a quaternary number system

$$\left.\begin{aligned} n &= 4n_1 + n_0 ;\ n_1, n_0 = 0,1,2,3 \\ k &= 4k_1 + k_0 ;\ k_1, k_0 = 0,1,2,3 \end{aligned}\right\} \tag{3.2}$$

Equation (2.3) then becomes $f(n_1, n_0) = \sum_{k_0=0}^{3} \left[\sum_{k_1=0}^{3} f_0(k_1, k_0) w^{nk_1} \right] w^{nk_0}$ (3.3)

Rewriting w^{4nk_1} we have

$$w^{4nk_1} = w^{4(kn_1 + n_0)k_1} = w^{16 n_1 k_1} w^{4 n_0 k_1} = \left[w^{16} \right]^{n_1 k_2} w^{4 n_0 k_1},\ w^{4nk_1} = w^{4 n_0 k_1} \tag{3.4}$$

The term in the brackets is equal to unit, since $w^{16} = 1$. we can now rewrite (3.1) as

$$f(n_1,n_0)=\sum_{k_0=0}^{3}\sum_{k_1=0}^{3}f_0(k_1,k_0)w^{(4n_1+n_0)(4k_1+k_0)}=\sum_{k_0=0}^{3}\sum_{k_1=0}^{3}f_0(k_1,k_0).w^{(4n_1+n_0)4k_1}w^{(4n_2+n_0)k_0}$$

$$=\sum_{k_0=0}^{3}\sum_{k_1=0}^{3}f_0(k_1,k_0)\left[w^{16n_1k_1}\right]w^{4n_0k_1^{(4n_1+n_0)}k_0}$$

The term in the brackets becomes 1 and we have $\sum_{k_0=0}^{3}\left[\sum_{k_1=0}^{3}f_0(k_1,k_0)\,w^{4n_0k_1}\right]w^{(4n_2+n_0)k_0}$ ***(3.5)***

Equation (3.5) represents the formulation of the FFT algorithm. Considering the inner brackets of (3.5), we have

$$f_1(n_0,k_0)=\sum_{k_1=0}^{3}f_0(k_1,k_0)w^{4n_0k_1}$$

$$\begin{aligned}
f_1(0,0)&= f_0(0,0)W^0+0+0+0+f_0(1,0)W^0+0+0+0+f_0(2,0)W^0+0+0+0+f_0(3,0)W^0+0+0+0\\
f_1(0,1)&= 0+f_0(0,1)W^0+0+0+0+f_0(1,1)W^0+0+0+0+f_0(2,1)W^0+0+0+0+f_0(3,1)W^0+0+0\\
f_1(0,2)&= 0+0+f_0(0,2)W^0+0+0+0+f_0(1,2)W^0+0+0+0+f_0(2,2)W^0+0+0+0+f_0(3,2)W^0+0\\
f_1(0,3)&= 0+0+0+f_0(0,3)W^0+0+0+0+f_0(1,3)W^0+0+0+0+f_0(2,3)W^0+0+0+0+f_0(3,3)W^0\\
f_1(1,0)&= f_0(0,0)W^0+0+0+0+f_0(1,0)W^4+0+0+0+f_0(2,0)W^8+0+0+0+f_0(3,0)W^{12}+0+0+0\\
f_1(1,1)&= 0+f_0(0,1)W^0+0+0+0+f_0(1,1)W^4+0+0+0+f_0(2,1)W^8+0+0+0+f_0(3,1)W^{12}+0+0\\
f_1(1,2)&= 0+0+f_0(0,2)W^0+0+0+0+f_0(1,2)W^4+0+0+0+f_0(2,2)W^8+0+0+0+f_0(3,2)W^{12}+0\\
f_1(1,3)&= 0+0+0+f_0(0,3)W^0+0+0+0+f_0(1,3)W^4+0+0+0+f_0(2,3)W^8+0+0+0+f_0(3,3)W^{12}\\
f_1(2,0)&= f_0(0,0)W^0+0+0+0+f_0(1,0)W^8+0+0+0+f_0(2,0)W^{16}+0+0+0+f_0(3,0)W^{24}+0+0+0\\
f_1(2,1)&= 0+f_0(0,1)W^0+0+0+0+f_0(1,1)W^8+0+0+0+f_0(2,1)W^{16}+0+0+0+f_0(3,1)W^{24}+0+0\\
f_1(2,2)&= 0+0+f_0(0,2)W^0+0+0+0+f_0(1,2)W^8+0+0+0+f_0(2,2)W^{16}+0+0+0+f_0(3,2)W^{24}+0\\
f_1(2,3)&= 0+0+0+f_0(0,3)W^0+0+0+0+f_0(1,3)W^8+0+0+0+f_0(2,3)W^{16}+0+0+0+f_0(3,3)W^{24}\\
f_1(3,0)&= f_0(0,0)W^0+0+0+0+f_0(1,0)W^{12}+0+0+0+f_0(2,0)W^{24}+0+0+0+f_0(3,0)W^{36}+0+0+0\\
f_1(3,1)&= 0+f_0(0,1)W^0+0+0+0+f_0(1,1)W^{12}+0+0+0+f_0(2,1)W^{24}+0+0+0+f_0(3,1)W^{36}+0+0\\
f_1(3,2)&= 0+0+f_0(0,2)W^0+0+0+0+f_0(1,2)W^{12}+0+0+0+f_0(2,2)W^{24}+0+0+0+f_0(3,2)W^{36}+0\\
f_1(3,3)&= 0+0+0+f_0(0,3)W^0+0+0+0+f_0(1,3)W^{12}+0+0+0+f_0(2,3)W^{24}+0+0+0+f_0(3,3)W^{36}
\end{aligned}\qquad (3.6)$$

If we rewrite (3.6) in matrix notion, we get

$$\begin{bmatrix}f_1(0,0)\\f_1(0,1)\\f_1(0,2)\\f_1(0,3)\\f_1(1,0)\\f_1(1,1)\\f_1(1,2)\\f_1(1,3)\\f_1(2,0)\\f_1(2,1)\\f_1(2,2)\\f_1(2,3)\\f_1(3,0)\\f_1(3,1)\\f_1(3,2)\\f_1(3,3)\end{bmatrix}=\begin{bmatrix}1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0\\0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0\\0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0\\0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1\\1&0&0&0&w^4&0&0&0&w^8&0&0&0&w^{12}&0&0&0\\0&1&0&0&0&w^4&0&0&0&w^8&0&0&0&w^{12}&0&0\\0&0&1&0&0&0&w^4&0&0&0&w^8&0&0&0&w^{12}&0\\0&0&0&1&0&0&0&w^4&0&0&0&w^8&0&0&0&w^{12}\\1&0&0&0&w^8&0&0&0&w^0&0&0&0&w^8&0&0&0\\0&1&0&0&0&w^8&0&0&0&w^0&0&0&0&w^8&0&0\\0&0&1&0&0&0&w^8&0&0&0&w^0&0&0&0&w^8&0\\0&0&0&1&0&0&0&w^8&0&0&0&w^0&0&0&0&w^8\\1&0&0&0&w^{12}&0&0&0&w^8&0&0&0&w^4&0&0&0\\0&1&0&0&0&w^{12}&0&0&0&w^8&0&0&0&w^4&0&0\\0&0&1&0&0&0&w^{12}&0&0&0&w^8&0&0&0&w^4&0\\0&0&0&1&0&0&0&w^{12}&0&0&0&w^8&0&0&0&w^4\end{bmatrix}\begin{bmatrix}f_0(0,0)\\f_0(0,1)\\f_0(0,2)\\f_0(0,3)\\f_0(1,0)\\f_0(1,1)\\f_0(1,2)\\f_0(1,3)\\f_0(2,0)\\f_0(2,1)\\f_0(2,2)\\f_0(2,3)\\f_0(3,0)\\f_0(3,1)\\f_0(3,2)\\f_0(3,3)\end{bmatrix}\qquad (3.7)$$

The matrix (3.7) is obtained from equation (3.6) by using the relationship $w^{nk}=w^{nk_{mod}(N)}$. Recall that [nkmod(*N*)] is the remainder upon division of *nk* by N. Similarly, we write the outer summation of (3.5) as

$$f_2(n_0,n_1)=\sum_{k_0=0}^{3}f_1(n_0,k_0)w^{(4n_1+n_0)k_0}\qquad (3.8)$$

Equation (3.8) can be written as

$$
\begin{aligned}
f_2(0,0) &= f_1(0,0)W^0 + f_1(0,1)W^0 + f_1(0,2)W^0 + f_1(0,3)W^0 + 0+0+0+0+0+0+0+0+0+0+0+0\\
f_2(0,1) &= f_1(0,0)W^0 + f_1(0,1)W^4 + f_1(0,2)W^8 + f_1(0,3)W^{12} + 0+0+0+0+0+0+0+0+0+0+0+0\\
f_2(0,2) &= f_1(0,0)W^0 + f_1(0,1)W^8 + f_1(0,2)W^{16} + f_1(0,3)W^{24} + 0+0+0+0+0+0+0+0+0+0+0+0\\
f_2(0,3) &= f_1(0,0)W^0 + f_1(0,1)W^{12} + f_1(0,2)W^{24} + f_1(0,3)W^{36} + 0+0+0+0+0+0+0+0+0+0+0+0\\
f_2(1,0) &= 0+0+0+0+ f_1(1,0)W^0 + f_1(1,1)W^1 + f_1(1,2)W^2 + f_1(1,3)W^3 + 0+0+0+0+0+0+0+0\\
f_2(1,1) &= 0+0+0+0+ f_1(1,0)W^0 + f_1(1,1)W^5 + f_1(1,2)W^{10} + f_1(1,3)W^{15} + 0+0+0+0+0+0+0+0\\
f_2(1,2) &= 0+0+0+0+ f_1(1,0)W^0 + f_1(1,1)W^9 + f_1(1,2)W^{18} + f_1(1,3)W^{27} + 0+0+0+0+0+0+0+0\\
f_2(1,3) &= 0+0+0+0+ f_1(1,0)W^0 + f_1(1,1)W^{13} + f_1(1,2)W^{26} + f_1(1,3)W^{39} + 0+0+0+0+0+0+0+0\\
f_2(2,0) &= 0+0+0+0+0+0+0+0+ f_1(2,0)W^0 + f_1(2,1)W^2 + f_1(2,2)W^4 + f_1(2,3)W^6 + 0+0+0+0\\
f_2(2,1) &= 0+0+0+0+0+0+0+0+ f_1(2,0)W^0 + f_1(2,1)W^6 + f_1(2,2)W^{12} + f_1(2,3)W^{18} + 0+0+0+0\\
f_2(2,2) &= 0+0+0+0+0+0+0+0+ f_1(2,0)W^0 + f_1(2,1)W^{10} + f_1(2,2)W^{20} + f_1(2,3)W^{30} + 0+0+0+0\\
f_2(2,3) &= 0+0+0+0+0+0+0+0+ f_1(2,0)W^0 + f_1(2,1)W^{14} + f_1(2,2)W^{28} + f_1(2,3)W^{42} + 0+0+0+0\\
f_2(3,0) &= 0+0+0+0+0+0+0+0+0+0+0+0+ f_1(3,0)W^0 + f_1(3,1)W^3 + f_1(3,2)W^6 + f_1(3,3)W^9\\
f_2(3,1) &= 0+0+0+0+0+0+0+0+0+0+0+0+ f_1(3,0)W^0 + f_1(3,1)W^7 + f_1(3,2)W^{14} + f_1(3,3)W^{21}\\
f_2(3,2) &= 0+0+0+0+0+0+0+0+0+0+0+0+ f_1(3,0)W^0 + f_1(3,1)W^{11} + f_1(3,2)W^{22} + f_1(3,3)W^{33}\\
f_2(3,3) &= 0+0+0+0+0+0+0+0+0+0+0+0+ f_1(3,0)W^0 + f_1(3,1)W^{12} + f_1(3,2)W^{24} + f_1(3,3)W^{36}
\end{aligned}
\qquad (3.9)
$$

Enumerating the outcome in (3.9), in matrix form we have

$$
\begin{bmatrix} f_2(0,0)\\ f_2(0,1)\\ f_2(0,2)\\ f_2(0,3)\\ f_2(1,0)\\ f_2(1,1)\\ f_2(1,2)\\ f_2(1,3)\\ f_2(2,0)\\ f_2(2,1)\\ f_2(2,2)\\ f_2(2,3)\\ f_2(3,0)\\ f_2(3,1)\\ f_2(3,2)\\ f_2(3,3) \end{bmatrix}
=
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & w^4 & w^8 & w^{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & w^8 & w^0 & w^8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & w^{12} & w^8 & w^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & w^0 & w^1 & w^2 & w^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & w^0 & w^5 & w^{10} & w^{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & w^0 & w^9 & w^2 & w^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & w^0 & w^{13} & w^{10} & w^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w^0 & w^2 & w^4 & w^6 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w^0 & w^6 & w^{12} & w^2 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w^0 & w^{10} & w^4 & w^{14} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w^0 & w^{14} & w^{12} & w^4 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w^0 & w^3 & w^6 & w^9\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w^0 & w^7 & w^{14} & w^5\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w^0 & w^{11} & w^6 & w^1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w^0 & w^{12} & w^8 & w^4
\end{bmatrix}
\begin{bmatrix} f_1(0,0)\\ f_1(0,1)\\ f_1(0,2)\\ f_1(0,3)\\ f_1(1,0)\\ f_1(1,1)\\ f_1(1,2)\\ f_1(1,3)\\ f_1(2,0)\\ f_1(2,1)\\ f_1(2,2)\\ f_1(2,3)\\ f_1(3,0)\\ f_1(3,1)\\ f_1(3,2)\\ f_1(3,3) \end{bmatrix}
\qquad (3.10)
$$

Again the matrix form (3.10) is obtained from (3.9) by using the relationship $W^{nk} = W^{nk \bmod(N)}$. From equation (3.5) and (3.8), we have by equating these two $F(n_1,n_0) = f_2(n_0,n_1)$ (3.11)

We note from (3.11) that the final result $f_2(n_0,n_1)$ as obtained from the outer sum is bit-reserved order with respect to the desired values $f(n_1,n_0)$. this is the scrambling resulting from the FFT algorithm.

We now combing the inner sum, outer sum and (3.11) as

$$
\begin{aligned}
f_1(n_1,k_0) &= \sum_{k_1=0}^{3} f_0(k_1,k_0)W^{4n_0k_1}\\
f_2(n_0,n_1) &= \sum_{k_1=0}^{3} f_1(n_0,k_0)W^{[4n_1+n_0]k_0}\\
f(n_1,n_0) &= f_2(n_0,n_1)
\end{aligned}
\qquad (3.12)
$$

4.0 Conclusion

Equation (3.12) represents the original C-T formulation of the discrete FFT algorithm for very composite sample point $N = P_1P_2 = 16 = 4\times 4$.

We term these equations as recursive in that the second equation is computed in terms of the first. The FFT algorithm formulated here, results in matrix factorization process, which algebraically decomposed the discrete Fourier transform (3.1) into factored matrices with zeros introduced. This results

in the reduction of the required number of complex multiplication from N^2 to $\frac{P_1 P_2}{2}$ and complex additions from N(N - 1) to $P_1 P_2$.

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