Journal of the Nigerian Association of Mathematical Physics,
***Volume 8* (*November 2004*).**

Some example of modelling with super-diagonal bilinear moving average time series

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Abstract

In this paper the modeling of super diagonal bilinear moving average time series models are considered. Other determination of bilinear models based on the observed covariance structure of the data is pointed out. Linear and bilinear moving average models that have identical covariance structure are fitted to both simulated and real-time series data. Forecasts obtained for stationary and invertible linear and bilinear models are compared

Keywords and Phrases: Super-diagonal bilinear moving average time series; stationarity; ergodicity; invertibility; covariance structure.

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1.0 Introduction

Let $X_t, t \in Z$ and $e_{t,}, t \in Z$ be two stochastic processes defined on some probability space (Ω, F, P), where $Z = \{, \Lambda, -1, 0, 1, \Lambda\}$. We assume that $e_t, t \in Z$ is independent, identically distributed with $E(e_t) = 0$ and E $(e_t^2) = \sigma^2 < \infty$. A bilinear model is one which is linear in both $X_t, t \in Z$ and $e_t, t \in Z$ but not in those variables jointly. Let $a_1, a_2, \Lambda, a_r, b_1, b_2 \Lambda, b_h$ and $\theta_{ij}, 1 \le i \le m, 1 \le j \le k$ be real constants. The general model of order (*r, h, m k*) or SBARMA (*r, h, m,* and *k*) is

$$X_t = e_t + \sum_{j=1}^{r} a_j x_{t-j} + \sum_{j-1}^{h} b_j e_{t-j} + \sum_{i=1}^{m} \sum_{j=1}^{k} \theta_{ij} x_{t-i} e_{t-j} \qquad (1.1)$$

for every *t* in *Z*. The first part on the right side of (1.1) can be identified as the autoregressive part of the process $X_t, t \in Z;$ the second part as the moving average part and the third part as the 'pure' bilinear part. The super-diagonal model (1.1) is completely bilinear if $r = h = 0$. In [3] after obtaining the vectorial representation of (1.1) showed that all bilinear models having that vector form are strictly stationary, ergodic and unique under suitable conditions on a matrix built on $a_1, a_2, K, a_r, b_1, b_2, K, b_h, \theta_{ij}, 1 \le i \le m, 1 \le j \le k, i \ge j \text{ and } \sigma^2$. the two important features that emerge from their work are the following:

(1) The presence of the moving average part makes no impact on the existence problem of (1.1)

(2) The conditions for strict stationarity of autoregressive moving average or ARMA models are obtainable from the conditions for strict stationarity of bilinear models

We do not know of any nice condition under which the model (1.1) is invertible. The inevitability of special cases of (1.1) have been studied by [8], [11], [13], [14], [15]. [11] established that the presence of autoregressive part makes no impact on the invertibility problem of his special case of (1.1). A sufficient condition for invertibility of diagonal bilinear models have been derived [9].

In [2] it has been shown that if $X_t, t \in Z$ is a stationary bilinear process defined by (1.1), then its covariance function is the same as that of an ARMA process of order (*r*, max (*h*,*g*))g = *min*(*m*,*k*) with autoregressive coefficients being functions of $a_1, a_2 \Lambda, a_r, b_1, b_2 \Lambda, b_h$ and $\theta_{ij}, 1 \le i \le m, 1 \le j \le k, i \ge j$. [15] also arrived at the same conclusion after obtaining a Markovian representation of bilinear models. [15 considered the estimation of the parameter of the bilinear models.

$$X_t = a_0 + \sum_{j=1}^{p} a_j X_{t-j} + \sum_{i=1}^{p}\sum_{j=1}^{q} \theta_{ij} X_{t-i} e_{t-j} + e_t \tag{1.2}$$

[7] have considered the estimation and prediction of the subset bilinear model

$$X_t = a_0 + \sum_{i=1}^{u} a_{k_i} X_{t-k_i} + \sum_{j=1}^{m} br_j s_j X_{t-r\,j} e_{t-s\,j} + e_t \tag{1.3}$$

where $1 \le k_1 \le k_2 \Lambda \le k_n$ are subsets of the integers (1, 2…., *p*) and *p* is the order of the best linear autoregressive

model that fits the data The statistical properties, such as stationary , invertibility and covariance structure of (1.2) and (1.3) are not yet known.

The main objective of this paper is to study the estimation of the model (1.1) with $r = 0$. From the works of [2] and [12] the model (1.1) with $r = 0$ has same covariance function as some moving average process of order $q = \max(h,g)$ or MA(q). One benefit of determining the covariance structure of the time series data is that it enables us to partially answer the question of order determination for bilinear models. In what follows, we refer to (1.1) with $r = 0$ as a super diagonal bilinear moving average model of order (*h*, *m*, *k*) or SBMA(*h*, *m*, *k*,)

2.0 Estimation and order determination

In this section, we consider the estimation of the parameters of the super-diagonal bilinear moving average model

$$X_t = \sum_{j=1}^{h} b_j e_{t-j} + \sum_{i=1, i\ge j}^{m} \sum_{j=1}^{k} \theta_{ij} X_{t-i} e_{t-j} + e_t \tag{2.1}$$

where the $e_t, t \in Z$ are independent and each is distributed $N(0,\sigma^2)$ Here we assume the model is stationary and invertible. We also assume we have a realization $\{X_1, X_2, \Lambda, X_n\}$ of the time series $X_t, t \in Z$. To obtain estimates of the parameters we proceed as in [7], [15] and apply the method of least squares to minimize

$$S(\underline{\theta}) = \sum_t e_t^2 \tag{2.2}$$

with respect to the parameters $\underline{\theta} = (b_1, b_2, \Lambda, b_h, \theta_{11}, \theta_{21}, \Lambda, \theta_{m1}, \theta_{22}, \Lambda, \theta_{m2}, \Lambda, \theta_{gg}, \theta_{g+1,g}, \mathrm{K}\, \theta_{mg})^T$. When minimizing $S(\underline{\theta})$ with respect to $\underline{\theta}$, the normal equations are nonlinear in $\underline{\theta}$. The solution of these equations require the use of nonlinear algorithm such as Newton-Raphson. The variance of $\hat{\underline{\theta}}$ is estimated from

$$V(\hat{\underline{\theta}}) \cong 2\hat{\sigma}^2 H^{-1}(\hat{\underline{\theta}}) \tag{2.3}$$

where

$$H(\hat{\underline{\theta}}) = (\partial^2 S(\hat{\underline{\theta}}) / \partial\hat{\underline{\theta}}_i\, \partial\hat{\underline{\theta}}_j) \tag{2.4}$$

is the matrix of second-order derivatives. Consistency and asymptotic normality for conditional least squares are complicated to verify because of the invariability problem of bilinear models. For diagonal bilinear models consistency has been proved by [10]. If the

option linear model is an $MA(q)$, the and analyst is considering super diagonal bilinear time series models, then (2.1) is worthy of further investigation. The maximum lag of the input process $e_t, t \in Z$ involved in (2.1) is *q* =max(*h,g*) where *g* =min(*m,k*). In considering (2.1), one can always take *k* $\leq$ *m*, so that *q* = *max*(*h,g*), with *g* = (*m,k*) = *k*. This implies that one can always take *k* $\leq$ *q* and *k* $\leq$ *q* in (2.1), leading to $X_t = e_t + \sum_{j=1}^{q} b_j e_{t-j} + \sum_{1=1, i \geq j}^{m} \sum_{j=1}^{q} \theta_{ij} X_{t-i} e_{t-j}$

(2.5)

The choice of the value of *m* is made on the basis of the information criteria of [1], which is given by

$$AIC = N \log \hat{\sigma}^2 + 2 \text{ (independent number of parameters)} \qquad (2.6)$$

where

$$\hat{\sigma}^2 = \frac{S(\hat{\underline{\theta}})}{N} \qquad (2.7)$$

and *N* is the number of observations used for calculating $S(\hat{\underline{\theta}})$.

Model (2.1) may have $MA(\lambda)$ autocorrelation function, and yet $\lambda \neq q =$ max (*h*, *g*). Consider for example, the strictly stationary process $X_t, t \in Z$ satisfying

$$X_t = be_{t-1} + \theta X_{t-3} e_{t-2} + e_t \qquad (2.8)$$

where $e_t, t \in Z$ are independent and each e_t is distributed *N*(0, 1). For model (2. 8), *h* = 1, *m* = 3, *k* = 2, $\theta_{11} = \theta_{21} = \theta_{31} = \theta_{22} = 0, \theta_{32} = \theta$ and *g* = *q* =2. It can easily be checked that $E(X_t) = 0$

$$E(X_t X_{t-k}) = \begin{cases} \sigma^2(1+b^2)/(1-\sigma^2\theta^2), k = 0 \\ \sigma^2 b, \quad k = \underline{+}1 \\ 0, \ k \neq 0, \underline{+}1 \end{cases} \qquad (2.9)$$

Model (2.8) has MA (1) autocorrelation function. Our simulation results haves shown that model (2.1) has the same covariance function as some *MA(q)* provided $h \geq g$ = min (*m, k*). When $g > h$ and θ_{gg} = *0* model (2.1) has the same covariance function as some $MA(\lambda), h \leq \lambda \leq g$.

In considering model (2.1) for a series with MA(λ) autocorrelation function, we consider model (2.5) for $q \leq \lambda$ and use *AIC* criteria to determine m that produces the minimum AIC value. After the minimum AIC value has been obtained, some moving average and bilinear coefficients may not be significantly different from zero. A parsimonious (subset super-diagonal moving average) model can then be achieved by eliminating the coefficients that will not lead to further reduction in the *AIC* value.

3.0 Simulation results

In this section we give a brief summary of our simulation results to demonstrate the use of the order of the optimal linear models for order determination of bilinear models. In what flows, $e_t, t \in Z$ is a sequence of independent identically distributed random variables with e_t having normal distribution with mean 0 and variance $\sigma^2 < \infty$. Methods of moment calculation are those used in [8] and only the results are given.

3.1 **Example 1**

Let $X_t, t \in Z$ be the strictly stationary ergodic process satisfying

$$X_t = (b + \theta_{11} X_{t-1} + \theta_{21} X_{t-2})e_{t-1} X\, e_t \qquad (3.1)$$

for every *t* in *Z*. The strict stationary condition of [3] implies that the roots (in modulus) of the equation

$$y^2 - \sigma^2 \theta_{11} y - \sigma^2 \theta_{21}{}^2 - 0 \qquad (3.2)$$

lie inside the unit circle. It can easily be checked that

$$\mu = E(X_t) = \sigma^2 \theta_{11,} \qquad (3.3)$$

$$(1 - \sigma^2 \theta_{11}{}^2 - \sigma^2 \theta_{21}{}^2) E(X_t^2) = \sigma^2 (1 + b^2) + 2\sigma^4 \theta_{11} [\theta_{11} + b(\theta_{11} + \theta_{21})]$$

$$+ 2\sigma^4 \theta_{11} \theta_{21} [b + \sigma^2 \theta_{11} (2\theta_{11} + \theta_{21})], \qquad (3.4)$$

$$E(X_t X_{t-1}) = \sigma^2 [b + \sigma^2 \theta_{11} (\theta_{11} + \theta_{21}) + \sigma^4 \theta_{11}{}^2, \qquad (3.5)$$

$$E(X_t X_{t-k}) = \sigma^4 \theta_{11}{}^2, k > 1 \qquad (3.6)$$

$$R(K) = \begin{cases} E(X_t{}^2) - \sigma^4 \theta_{11}^2, \ k = 0 \\ \sigma^2 [b + \sigma^2 \theta_{11} (\theta_{11} + \theta_{21})], \ k = 1 \\ 0, \ k \neq 0,1 \end{cases} \qquad (3.7)$$

where

$$R(K) = E[(X_t - \mu)(X_{t-k} - \mu)] \qquad (3.8)$$

A sufficient condition for the invertibility of (3.1) is (see [11])

$$1 > b^2 + 2\sigma^2 b \theta_{11} (\theta_{11} + \theta_{21}) + (\theta_{11}{}^2 + \theta_{21}{}^2) E(X_t{}^2) + 2\theta_{11} \theta_{21} E(X_t X_{t-1}) \qquad (3.9)$$

We have generated 100 observations from the model (3.1) with $b = 0.5, \theta_{11} = 0.3, \theta_{21} = 0.2$ *and* $\sigma^2 = 1.0$. The theoretical and estimated autocorrelations; sample mean and variance of the generated model (3.1) are given in Table 1

The estimated autocorrelations suggest an MA (1) process. The fitted MA (1) model is

$$X_t = \underset{(\pm 0{\cdot}1963)}{0.3473} + \underset{(\pm 0{\cdot}0857)}{0.3717}\, a_{t-1} + a_t \qquad (3.9)$$

with Var(a_t) =2.0622, leading to 106 percent increase in the error variance. Since the optimal linear model is MA(1), we consider (2.5) with q ≥ 1. The AIC value is found to be minimum when *q* = 1, *m* = 2 and estimates obtained are: $\hat{b} = 0.5153 \pm 0.0465$, $\hat{\theta}_{11} = 0.2585 \pm 0.0453$, $\hat{\theta}_{21} = 0.2327 \pm 0.0341, \hat{\sigma}^2 = 0.9850$ and AIC = 4.49.

3.2 **Example 2**

Let X_t, $t \in Z$ be the strictly stationary ergodic process satisfying

$$X_t = \theta_{11} X_{t-1} e_{t-1} + \theta_{22} X_{t-2}\, e_{t-2} + \theta_{33} X_{t-3} e_{t-3} + e_t \qquad (3.10)$$

for every *t* in *Z.*

Table 1: Theoretical and estimated autocorrelations, mean and variance of simulated series

LAG	Model (3.1)		Model (3.10)		Model (3.19)	
	Autocorrelations					
k	THE	EST	THE	EST	THE	EST
1	0.36	0.38	0.50	0.57	0.59	0.60
2	0.00	0.12	0.27	0.34	0.18	0.20
3	0.00	0.11	0.08	0.09	0.00	0.08

4	0.00	0.06	0.00	0.04	0.00	0.05
5	0.00	-0.02	0.00	-0.03	0.00	-0.02
6	0.00	0.00	0.00	0.02	0.00	-0.03
7	0.00	-0.10	0.00	-0.03	0.00	-0.04
8	0.00	-0.03	0.00	-0.12	0.00	-0.08
9	0.00	-0.13	0.00	-0.16	0.00	-0.15
10	0.00	-0.14	0.00	-0.13	0.00	-0.17
11	0.00	-0.13	0.00	-0.14	0.00	-0.13
12	0.00	-0.08	0.00	-0.08	0.00	-0.08
13	0.00	-0.05	0.00	-0.06	0.00	-0.05
14	0.00	0.02	0.00	-0.05	0.00	-0.06
15	0.00	-0.09	0.00	-0.08	0.00	-0.09
16	0.00	-0.11	0.00	0.02	0.00	-0.05
17	0.00	-0.01	0.00	0.06	0.00	0.04
18	0.00	0.03	0.00	0.01	0.00	0.01
19	0.00	-0.03	0.00	-0.03	0.00	-0.08
20	0.00	-0.06	0.00	-0.07	0.00	-0.09
	MEAN					
	0.30	0.35	0.90	0.95	0.70	0.79
	VARIANCE					
	1.83	2.59	2.15	3.03	3.19	5.19

Note: THE means Theoretical; EST means Estimated

The strict stationary condition implies that the roots (in modulus) of the equation

$$y^2 - \phi_1^{\,2} y^2 - \phi_2^{\,2} y - \phi_3^{\,2} = 0 \qquad (3.11)$$

lie inside the unit circle where $\phi_1 = \sigma\theta_i, i = 1,2,3.$ it can be checked that

$$u = E(X_t) = \sigma(\phi_1 + \phi_2 + \phi_3) \qquad (3.12$$

$$M_2 = E(X_t^{\,2}) = k_2(1 + 2\phi_1^{\,2} + 2\phi_2^{\,2} + 2\phi_3^{\,2} + 2\phi_1\phi_2 + 2\phi_1\phi_3 + 2\phi_2\phi_3) \qquad (3.13)$$

where $k_2 - \sigma^2 / \left(1 - \phi_1^{\,2} - \phi_2^{\,2} - \phi_3^{\,2}\right).$.

$$(X_t X_{t-1}) = (\phi_1\phi_2 + \phi_2\phi_3)M_2 + \sigma^2\phi_1(\phi_1 + \phi_2 + \phi_3) + \sigma^2(\phi_1\phi_2 + \phi_2\phi_3) + \mu^2 \qquad (3.14)$$

$$E(X_t X_{t-2}) = \phi_1\phi_3 M_2 + \sigma^2\phi_2(\phi_1 + \phi_2 + \phi_3) + \sigma^2\phi_1\phi_3 + \mu^2 \qquad (3.15)$$

$$E(X_t X_{t-3}) = \sigma^2\phi_3(\phi_1 + \phi_2 + \phi_3) + \mu^2 \qquad (3.16)$$

$$E(X_t X_{t-k}) = \mu^2,\ k > 3. \qquad (3.17)$$

Thus, $R(k) = 0, k > 3.$

We have generated 100 observations from the model (3.10) with $\theta_{11} = 0.4, \theta_{33} = 0.2$ and the theoretical and estimated autocorrelations, sample mean and variance of the generated model (3.10) are given in Table 1. The estimated autocorrelation suggest an MA(2) process. The optimal linear model obtained is

$$X_t = \underset{(\pm 0.2794)}{0.9450} + \underset{(\pm 0.0949)}{0.5695} a_{t-1} + \underset{(\pm 0.0947)}{0.3491} a_{t-2} + a_t \qquad (3.18)$$

with Var(a_t) = 2.0448, leading to 104 percent increase in the error variance. Since the option linear model's MA(2), we consider (2.5) with $q \geq 2$. The AIC value is found to be minimum when $q = 3, m = 3$ and without the moving average part. The estimates are:

$$\hat{\theta}_{11} = 0.4172 \pm 0.0630,\ \hat{\theta}_{21} = 0.1611 \pm 0.0645,\ \hat{\theta}_{31} = -0.0838 \pm 0.0598,\ \hat{\theta}_{22} = 0.2713 \pm 0.0598,$$

$$\hat{\theta}_{32} = 0.0884 \pm 0.0598,\ \hat{\theta}_{33} = 0.2184 \pm 0.0238,\ \hat{\sigma}^2 = 0.9442,\ AIC = 6.43.$$

The very small values of some of these coefficients suggest the parsimonious bilinear model (3.10) with the following estimates:

$$\hat{\theta}_{11} = 0.4351 \pm 0.0486,\ \hat{\theta}_{22} = 0.3107 \pm 0.0486,\ \hat{\theta}_{33} = 0.2128 \pm 0.0199,\ \hat{\sigma}^2 = 0.9856,\ AIC = 4.59.$$

3.3 **Example 3.**

Let X_t, $t \in Z$ be the strictly stationary ergodic process satisfying

$$X_t = \sum_{j=1}^{2} b_j e_{t-j} + \sum_{i=1}^{2} \sum_{i \geq j}^{} \sum_{j=1}^{2} \theta_{ij} X_{t-i} e_{t-j} + e_t \quad (3.19)$$

for every *t* in *Z*. The stationary condition implies that the eigen values of the matrix

$$L = 8 \times 8 \begin{bmatrix} \frac{L_1}{4x4}, & \frac{L_2}{4x4} \\ I_4 & {}_4\underline{0} \end{bmatrix} \quad (3.20)$$

Have moduli less than unity, where

$$L_1 = \begin{bmatrix} \sigma^2\theta_{11}^{\ 2} & \sigma^2\theta_{11}\theta_{21} & \sigma^2\theta_{11}\theta_{21} & \sigma^2\theta_{21} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} \sigma^2\theta_{22}^{\ 2} & 0 & 0 & o \\ \sigma^2\theta_{11}\theta_{22} & \sigma^2\theta_{21}\theta_{22} & 0 & 0 \\ \sigma^2\theta_{11}\theta_{22} & 0 & \sigma^2\theta_{21}\theta_{22} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It can be checked that

$$\mu = E(X_t) = \sigma^2 (\theta_{11} + \theta_{22}). \quad (3.21)$$

$$(1 - \sigma^2(\theta_{11}^{\ 2} + \theta_{21}^2 + \theta_{22}^{\ 2})E(X_t^{\ 2}) = 2\sigma^2\theta_{11}\theta_{21}E(X_t X_{t-1}) + d_1. \quad (3.22)$$

$$(1 - \sigma^2\theta_{21}\theta_{22})E(X_t X_{t-1}) = \sigma^2\theta_{11}\theta_{22}X(X_t^{\ 2}) + d_2 + d_3. \quad (3.23)$$

$$E(X_t X_{t-2}) = \sigma^2 [b_2 + \mu(\theta_{11} + 2\theta_{22}) + 2\sigma^2\theta_{21}\theta_{22}]. \quad (3.24)$$

$$E(X_t X_{t-k}) = \mu^2, \ k > 2. \quad (3.25)$$

where $d_1 = \sigma^2 [1 + b_1^{\ 2} + b_2^{\ 2} + 2\mu(b_1(\theta_{11} + \theta_{21}) + b_2\theta_{22}) + 2\sigma^2(\theta_{11}^{\ 2} + \theta_{11}\theta_{22} + \theta_{21}^{\ 2})]$ (3.26)

$$d_2 = \sigma^2 [b_1 + b_2(b_1 + \mu(\theta_{11} + \theta_{22}))] \quad (3.27)$$

$$d_3 = \sigma^2 [\mu(2\theta_{11} + \theta_{21} + \theta_{22}) + \theta_{22}(\mu b_1 + \sigma^2\theta_{11})] \quad (3.28)$$

From (3.25), we obtain R (k) = 0, k > 2 (3.29)

We have generated 100 observations from the model (3.19) with $b_1 = 0.55, b_2 = 0.35, \theta_{11} = 0.40, \theta_{21}. =$ $0.05, \theta_{22} = 0.30$ *and* $\sigma^2 = 1.0$. The theoretical and estimated autocorrelations, sample mean and variance of the generated model (3.19) are given in Table 1. The estimated autocorrelations suggest an MA(2) process. The fitted MA(2) model is $X_t = \underset{(\pm 0.3630)}{0.7731} + \underset{(\pm 0.0983)}{0.7915} a_{t-1} + \underset{(\pm 0.0994)}{0.2566} a_{t-2} + a_t$

(3.30)

with Var$(a_t) = 3.1298$, leading to 213 percent increase in the error variances. Since the optimal linear model is MA(2) we consider (2.5) with $q \geq 2$. The AIC value is found to be minimum when q = 2, m = 2 and the estimates obtained are

$$\hat{b}_1 = 0.5453 \pm 0.0637, \ \hat{b}_2 = 0.4656 \pm 0.0537, \ \hat{\theta}_{11} = 0.4327 \pm 0.0470, \ \hat{\theta}_{21} = 0.0625 \pm 0.0275.$$

$$\hat{\theta}_{22}\ 0.2902 \pm 0{,}0414, \ \hat{\sigma}^2 = 0.9791, \ AIC = 7.95.$$

The estimates of the bilinear coefficient θ_{21} is very small when compared with the estimates of the other bilinear coefficients. The elimination of θ_{21} gave an AIC value of 11.16, leading to the retention of θ_{21} in the bilinear model.

3.3 **Remark.**

When we fit linear model to series X_t whose sample mean is not zero, the zero mean series $X_t - \overline{X}$ is considered or a constant term is added to the linear model. The mean of the bilinear model (2.1) is zero if and only if $\theta_{11} = \theta_{22} = \Lambda = \theta_{gg} = 0, \ g = min(m,k)$. The three-simulation results show that we do not always need to add a constant term or model the zero mean series when considering bilinear models.

4.0 **Application to real time series data.**

We gave applications of our modeling procedure to two real time series data.

4.1 **IBM common stock closing prices**.

The original data, which consists of 360 observations, is series B in [5]. [5] fitted MA(1) model separately to the first and second halves of the differenced series as well as to the complete series. Using the results obtained, they produce evidence that in later periods the MA(1) model suffers a significant change in parameter value. We confine ourselves to the first half of the series. The MA (1) model fitted to the first 169

observations is

$$X_t = \underset{(\pm 0.0724)}{0.2634} a_{t-1} + a_t \quad (4.1)$$

with $\text{Var}(a_t) = 24.8043$, where X_t is change in price. Some of the autocorrelations of the squares of estimated residuals of (1.1) appear significant: suggesting non-linearity bilinear series. Based on our modeling procedure, we consider (2.5) with $q \geq 1$. The estimated bilinear model is

$$X_t = (\underset{(\pm 0.0704)}{0.2391} + \underset{(\pm 0.0085)}{0.0272} X_{t-1}) e_{t-1} + e_t \quad (4.2)$$

with $\hat{\sigma}^2 = 23.5327$. [8] considered the first 169 trading days. They fitted the bilinear model

$$Z_t = 0.02 Z_{t-1} e_{t-1} + e_t \quad (4.3$$

to the residuals Z_t obtained from the MA(1) model

$$Z_t = 0.26 Z_{t-1} + Z_t \quad (4.4)$$

On eliminating Z_t between (4.3) and (4.4) we obtain

$$X_t = 0.26 e_{t-1} + 0.02 X_{t-1} e_{t-1} + e_t \quad (4.5)$$

which is similar to the bilinear model (4.2) obtained using our modeling procedure. The forecasting performance of the linear and bilinear models is also given by [8].

4.2 **Ben Nevis temperatures.**

Next, consider the 200 daily dry bulb temperatures at noon on Ben Nevis referred to in [4] as series A*. [4] identified, estimated and diagnostically checked the MA(2) model

$$X_t = -0.238 a_{t-1} - 0.305 a_{t-2} + a_t \quad (4.6)$$

for the 200 observations with $\text{Var}(a_t) = 17 \cdot 91$ where X_t is change in temperature. However, some of the autocorrelations of the squares of estimated residuals of (4.6) appear significant.

Considering the 200-point data, and employing our modeling procedure, the resulting parsimonious bilinear model obtained is

$$X_t = -\underset{(\pm 0\cdot 0683)}{0\cdot 2267} e_{t-1} - \underset{(\pm 0\cdot 0688)}{0\cdot 2887} e_{t-2} + \underset{(\pm 0\cdot 0137)}{0\cdot 0270} X_{t-3} e_{t-1} + \underset{(\pm 0\cdot 0079)}{0\cdot 0124} X_{t-2} e_{t-2} - \underset{(\pm 0\cdot 0137)}{0\cdot 0213} X_{t-3} e_{t-2} + e_t \quad (4.7)$$

with $\hat{\sigma}^2 = 15 \cdot 6643$ leading to 12.6 percent decrease in the error variance.

The invertibility of the fitted bilinear model (4.7) has not been checked given the present state of knowledge. A comparison among the linear and bilinear models dealing with forecasting performance cannot, under this circumstance, be given.

5.0 **Conclusion**

We have developed a modelling procedure for superdiagonal bilinear moving average time series models that identify as moving average models under covariance analysis. We started by obtaining the optimal linear moving average model that fits the nonlinear time series data. The order of the optimal moving average model in conjunction with the information criterion of [1] constitute natural techniques for order determination of superdiagonal bilinear models. Our method of estimation has been applied to IBM Common Closing Stock Prices and Ben Nevis Temperature data.

Acknowledgement

The author is grateful to Professor G. C. Reinsel for a number of suggestions to improve the presentation of the material.

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