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Multi-valued solution of the Burgers' equation and shock Determination I.

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Abstract

We present the Burgers' equation as a balance between time evolution, non-linearity and dissipation and use these properties to examine the vanishing behaviour of the dissipation coefficient. Furthermore, we undertake a rigorous mathematical analysis which gives rise to multi-valued solutions after sufficient time and discontinuities. Though the complete solution is single-valued for all time, t, revelations from the equation of shock determination is interesting in the determination of the random properties of the wave.

Keywords: Burgers' equation, time evolution, non-linearity, dispersion, dissipation, discontinuity, shock, hump.

pp 145 - 148

1.0 Introduction

The initial formulation of the Burgers' equation was as a turbulence model combining both non-linear propagation effects and diffusive effects. The literature is very vast, and when considered as a partial differential equation, it can be used to describe and analyse pulses or shock waves. It can also be considered as a special form of the momentum equation for irrotational, incompressible flows in which pressure gradients are neglected. In its simplest form, it is presented as

$$c_t + cc_x = \nu c_{xx} \qquad (1.1)$$

where c_{xx} is the non-linear dispersive term. The two terms on the LHS of (1.1), $c_t + cc_x$, governs the wave evolution with speed, c. Following from the shallow water non-linear wave equations with the effect of dispersion written as $c_t + cc_x + g\eta_x = 0$ (c is particle velocity and η is free surface elevation), we can consider the Burgers' equation as a balance between time evolution, non-linearity and dispersion. In another consideration, the Burgers' equation can be considered as an interplay between the non linear steeping and the diffusion of a wave, [1]. This poses a great challenge for both analytical and numerical modelling. For instance, for a finite body like the Earth with a free surface and several internal zones of differing physical properties, observed modes of propagation presents the wave velocity as a function of the frequency. Such modes themselves exhibit dispersive effects. This study suggests that for the limiting case of a vanishing dissipation coefficient, ν, i.e. $\nu \to 0$, the solution of (1.1) reduces to the solution of $c_t + cc_x = 0$ which is a simple non linear partial differential equation often used as a model problem for fluid dynamical systems. This is also known as the in viscid Burgers' equation, [2]. We thus produce an exact solution of the Burgers' equation as an initial value problem. The multi-valued solution to the Burgers' equation is obtained after sufficient time and discontinuities as $\nu \to 0$. Furthermore, the equation for shock determination is obtained and analysed for a shock of single hump.

2.0 Governing equations and their specifications

The equation (1.1) can be considered as an exact solution for waves described by

$$\rho_t + q_x = 0$$
$$q = \varphi(\rho) - \nu\rho_x \tag{2.1}$$

with $\varphi(\rho)$ considered as a quadratic function of ρ and $c(\rho) = \varphi'(\rho)$, [3]. For a general form of $\varphi(\rho)$, (1.1) takes the form

$$c_t + c\,c_x = \nu\, c_{xx} - \nu\, c''(\rho)\rho_x^2 \tag{2.2}$$

The ratio of the two terms in the RHS of (2.2) is of the order of the amplitude of disturbance, which makes (1.1) a good approximation for small amplitude oscillation with $\nu\, c''(\rho)\rho_x^2$ been smaller than $\nu\, c_{xx}$ in the strength of the shock.

3.0 Mathematical formulation

This formulation will show that as $\nu \to 0$, the solution of (1.1) reduces to the solution of

$$c_t + cc_x = 0 \tag{3.1}$$

with discontinuous shocks which satisfy

$$U = \frac{1}{2}(c_1 + c_2), \qquad c_2 > U > c_1 \tag{3.1a}$$

with (3.1) re-written as

$$x = \xi + t\,F(\xi) \tag{3.2}$$

Following [4] and Hopf [5] we have

$$c = -\frac{2\nu\varphi_x}{\varphi} = -2\nu\frac{d}{dx}\ln\varphi \tag{3.3}$$

as a non linear transformation that could reduce (1.1) to the linear heat equation. To do this, we introduce

$$c = \psi_x \tag{3.4a}$$

Thus,

$$\frac{d}{dx}\psi_t + \tfrac{1}{2}\frac{d}{dx}\psi_x^2 = \nu\frac{d}{dx}\psi_{xx} \tag{3.4b}$$

Integrating,

$$\psi_t + \tfrac{1}{2}\psi_x^2 = \nu\psi_{xx} \tag{3.5a}$$

$$\psi = -2\nu\int\frac{\varphi_x}{\varphi} = -2\nu\ln\varphi \tag{3.5b}$$

$$\psi_t = \frac{-2\nu\left[\varphi\varphi_x - \varphi_x^2\right]}{\varphi^2} \tag{3.5c}$$

Substituting the values of (3.5b, and c) into (3.5a), we have

$$-\frac{2\nu\,\varphi_t}{\varphi} + \frac{2\nu^2\varphi_x^2}{\varphi^2} = -2\nu^2\left[\frac{\rho_{xx}}{\varphi} - \frac{\varphi_x^2}{\varphi^2}\right] \tag{3.5d}$$

and so

$$\varphi_t = \nu\,\varphi_{xx} \tag{3.6}$$

which is now a linear diffusion equation. We shall now consider the exact solution of the Burgers' equation as an initial value problem. In that consideration, for the initial value problem, we have

$$c = F(x) \text{ at } t = 0, \;\; x \in R \tag{3.7}$$

which by appropriate transformations using (3.3) becomes another initial value problem for the heat equation

$$\varphi(x) = \Phi(x) = exp\left\{-\frac{1}{2\nu}\int_0^\eta F(\eta)d\eta\right\} \qquad \text{at} \quad t = 0 \tag{3.7a}$$

The solution for (3.6) is

$$\varphi(x,t)=\frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty}\Phi(\eta)\exp\left\{-\frac{(x-\eta)^2}{4\nu t}\right\}d\eta \qquad (3.7b)$$

By the definition of (3.7a), we can then rewrite (3.7b) as

$$\varphi(x,t)=\frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty}exp\left\{-\frac{1}{2\nu}\int_0^{\eta}F(\eta)\,d\eta\right\}\exp\left\{-\frac{(x-\eta)^2}{4\nu t}\right\}d\eta \qquad (3.7c)$$

A solution for (3.7) can now be easily obtained from (3.3) by writing

$$\varphi(x,t)=\frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty}e^{-\frac{1}{2\nu}G(\eta:x,t)}d\eta \qquad (3.8)$$

where $$G(\eta;x,t)=\int_0^{\eta}F(\dot{\eta})\,d\dot{\eta}+\frac{(x-\eta)^2}{2t} \qquad (3.9)$$

so that

$$c(x,t)=\frac{\int_{-\infty}^{\infty}\left(\frac{x-\eta}{t}\right)e^{-\frac{1}{2\nu}G(\eta;x,t)}d\eta}{\int_{-\infty}^{\infty}e^{-\frac{1}{2\nu}G(\eta;x,t)}d\eta} \qquad (3.10)$$

which is now an exact solution of the Burgers' equation as an initial value problem. We will now allow x,t and $F(x)$ to remain fixed whilst contributions to (3.10) comes from the neighbourhood of the stationary point of G, i.e. where

$$\frac{\partial G}{\partial\eta}=F(\eta)-\frac{x-\eta}{t}=0 \qquad (3.11)$$

Let $\eta=\xi(x,t)$ be the point i.e. the solution of (3.11), then

$$F(\xi)=\frac{x-\xi}{t} \qquad (3.12$$

Using the method of stationary phase, and following the definition above, we know that the contribution from the neighbourhood of the stationary point $\eta=\xi$ in an integral $\int_{-\infty}^{\infty}g(\eta)e^{-\frac{G(\eta)}{2\nu}}d\eta$ is

$$\int_{-\infty}^{\infty}g(\eta)e^{-\frac{G(\eta)}{2\nu}}d\eta=g(\xi)\sqrt{\frac{4\pi\nu}{G''(\xi)}}e^{-\frac{G(\xi)}{2\nu}} \qquad (3.13)$$

Thus, by assuming that there is only one stationary point $\xi(x,t)$ satisfying (3.12), then

$$\int_{-\infty}^{\infty}\frac{x-\eta}{t}e^{-\frac{G}{2\nu}}d\eta\sim\frac{x-\xi}{t}\sqrt{\frac{2\pi\nu}{G''(\xi)}}e^{-\frac{G(\xi)}{2\nu}} \qquad (3.14)$$

and $$\int_{-\infty}^{\infty}e^{-\frac{G}{2\nu}}d\eta\approx\sqrt{\frac{2\pi\nu}{G''(\xi)}}e^{-\frac{G(\xi)}{2\nu}} \qquad (3.15)$$

Therefore, from (3.10),

$$c(x,t)\cong\frac{x-\xi}{t} \qquad (3.16)$$

with $\xi(x,t)$ as defined in (3.12).

4.0 Discussion of Equation (3.16)

(3.16) is the asymptotic solution of the Burgers' equation and can be re-written as

$$\begin{aligned} c &= F(\xi) \\ x &= \xi+tF(\xi) \end{aligned} \qquad (4.1)$$

which is the exact solution of (3.1) with the stationary point $\xi(x,t)$ as the characteristic variable. (4.1) gives rise to multi-valued solutions after sufficient time and therefore discontinuities. But the complete solution (3.10) is single-valued for all t since (3.12) has two solutions for sufficiently large value of t. We shall therefore introduce some modifications as follows:

Let the solutions of (3.12) be ξ_1 and ξ_2 with $\xi_1 > \xi_2$. So, ξ_1 and ξ_2 will contribute to (3.14) and (3.15). Thus,

$$c(x,t) \sim \frac{\frac{x-\xi_1}{t}\sqrt{G''(\xi_1)}\,e^{-G(\xi_1)/2\nu} + \frac{x-\xi_2}{t}\sqrt{G''(\xi_2)}\,e^{-G(\xi_2)/2\nu}}{\sqrt{G''(\xi_1)}\,e^{-G(\xi_1)/2\nu} + \sqrt{G''(\xi_2)}\,e^{-G(\xi_2)/2\nu}} \qquad (4.2)$$

If $G(\xi_1) \neq G(\xi_2)$, then as $\nu \to 0$ the exponent makes one of the terms very large compared with the other and vice versa. Thus if $G(\xi_1) << G(\xi_2)$, we have

$$c(x,\xi,t) \cong \frac{x-\xi_1}{t} \qquad (4.3)$$

But if $G(\xi_1) >> G(\xi_2)$, we have

$$c(x,\xi,t) \cong \frac{x-\xi_2}{t} \qquad (4.4)$$

This makes ξ_1 and ξ_2 both functions of (x,t) and the criterion $G(\xi_1) <> G(\xi_2)$ will determine the appropriate choice of ξ_1 or ξ_2 for given (x,t). Inflections from ξ_1 to ξ_2 will therefore occur at those values of (x,t) for which $G(\xi_1) = G(\xi_2)$, i.e. when

$$\int_0^{\xi_2} F(\eta')\,d\eta' + \frac{(x-\xi_2)^2}{2t} = \int_0^{\xi_1} F(\eta')\,d\eta' + \frac{(x-\xi_1)^2}{2t} \qquad (4.5)$$

But both ξ_1 and ξ_2 satisfy (3.12), i.e.

$$F(\xi) = \frac{x-\xi}{t} \quad \text{or} \quad \frac{x-\xi}{2}F(\xi) = \frac{(x-\xi)^2}{2t} \qquad (4.6)$$

Thus,

$$\int_0^{\xi_1} - \int_0^{\xi_2} F(\eta')\,d\eta' = \frac{x-\xi_2}{2}F(\xi_2) - \frac{x-\xi_1}{2}F(\xi_1) \qquad (4.7a)$$

which can be written analytically as

$$\int_{\xi_1}^{\xi_2} F(\eta')\,d\eta' = \frac{1}{2}\{F(\xi_1) + F(\xi_2)\}(\xi_1 - \xi_2) \qquad (4.7b)$$

(4.7a) is called shock determination. The discussion of this equation is in another paper. Great insight can be obtained from [3]. The changeover in the choice of terms in (4.2) leads to discontinuity in $c(x,t)$ as $\nu \to 0$. Following this analysis, we conclude that solutions of Burgers' equation approach those described by (3.1) and (3.1a) as $\nu \to 0$.

5.0 Further discussions and conclusions

In [4] and Hopf [5], we see the proposition of the Burgers' equation as a turbulence model lacking certain properties for proper modelling. Even if there is no physical behaviour to be modelled by this equation, it is still an interesting study because of its light approach to non linear equations. With appropriate choice of initial conditions (a decreasing speed with *x*), the equation leads to the formation of shocks. In case of an inviscid fluid, multi-valued solutions appear. After the shock formation the solution decays while the maximum moves away from the shock position as a result of the

viscosity. The dissipation coefficient, ν, is fixed in reality and also relatively small and so $\nu \to 0$ is a good approximation. Even so, there are still distinctions between the limit solution $\nu \to 0$ and the solution for fixed small ν, [3], and [6].

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