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Dynamic stability of a lightly damped column trapped by a harmonically slowly varying explicitly time dependent load

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Abstract

In this paper we initiate an analytical approach for determining the dynamic buckling load of a finite viscously damped column acted upon by a harmonically slowly varying explicitly time dependent load. The viscous damping is considered light and the column rests on an elastic foundation that produces a nonlinear restoring force per unit length. Unlike most similar analyses, the time variable appears explicitly making the problem non-autonomous the formulation contains two small but unrelated parameters upon which asymptotic expansions are initiated. The coefficients are sinusoidally slowly varying and problem is solved using a generalization of Lindsted-Poincare method in a mulit-timing regular perturbation technique. Simple asymptotic results implicit in the load parameter are obtained.

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1.0 Introduction

The dynamic buckling load of any material is an important design factor that determines the state of dynamic stability of most materials. In this paper we consider the case of a finite lightly damped column acted by a harmonically slowly varying explicitly time dependent dynamic load. The column rests on a cubic nonlinear elastic foundation. Earlier dynamic bucking analyses on a column tended to concentrate on loading histories that made the resultant equations autonomous of time. Such studies include Amazigo [1], Amazigo and Frank [2] and Amazigo et al [3] among others. Analysis presented here is similar in spirit to those in Schaller [4], Pinna and Ronalds [5], Popov [6], Zhu et al [7], and Aksogan and Sofiyer [8]. Mention must also be made of Heinen and Bullesdach [9], Michel et al [10], Ulo Lepik [11] and Hunt et al [12] among others.

2.0 Differential equation

The relevant differential equation [1-3] satisfied by the lateral displacement $W(X, T)$, where X and T are the spatial and time variables respectively, of an imperfect damped column trapped by an arbitrary time dependent load $P(T)$ is

$$m_0 W_{,TT} + QW_{,T} + EIW_{,XXXX} + P(T)W_{,XX} + K_1 W - \alpha k_3 W^3 = -P(T)\frac{d^2\overline{W}}{dX^2}, T > 0 \qquad (2.1)$$

where m_0 is the mass per unit length, Q is the damping constant, EI is the bending constant where E is the Young's modulus and I is the moment of inertia. The finite column rests on an elastic foundation that produces a restoring force per unit length of $K_1W - \alpha K_3W^3$ where $K_1 > 0, K_3 > 0$, and α is the imperfection- sensitivity parameter which is such that α = 1, if the foundation behaves as a "softening" spring where as α take value α = -1 if the foundation behaves as a "hardening" spring. In this investigation, the foundation shall be deemed to be a "softening" type and so we shall assume that α = 1 throughout this work. $\overline{W}$ is the initial imperfection and a subscript following a comma indicates partial differentiation. We shall neglect axial inertia as well as all nonlinear geometric effects and shall assume homogenous initial displacement and velocity. We now introduce the following non-dimensional quantities

$$X = \left(\frac{k_1}{EI}\right)^{\frac{1}{4}} X;\; w = \left(\frac{k_3}{k_1}\right)^{\frac{1}{2}} W;\; \lambda f(\bar{t}) = \frac{p}{2(EIk_1)^{\frac{1}{2}}};\; \delta = \frac{Q}{(m_0k_1)^{\frac{1}{2}}};\; \varepsilon\overline{W} = \left(\frac{k_3}{k_1}\right)^{\frac{1}{2}}\overline{W};\; \bar{t} = \left(\frac{k_1}{m_0}\right)^{\frac{1}{2}} T$$

Here ε is a small amplitude of the imperfection satisfying the $|\varepsilon| << 1$. Similarly λ is the amplitude of the load function $f(\bar{t})$ and is nondimensionalized in such a manner that the perfect undamped column on linear elastic

foundation has the classical buckling load λ_c given by $\lambda_c = 1$. The damping constant δ is assumed small relative to unity and so satisfies the inequality $0 < \delta << 1$. We assumed that the small parameter ε and δ are not related. Thus the non-dimensional form of the governing equation (2.1) becomes

$$w_{,\bar{t}\bar{t}} + \delta w_{,\bar{t}} + w_{,xxxx} + 2\lambda f(\bar{t}) w_{,xx} + w - w^3 = -2\lambda f(\bar{t})\varepsilon \frac{d^2\overline{w}}{dx^2}, \bar{t} > 0;\; 0 < x < \pi \quad (2.1a)$$

$$w = w_{,xx} = 0\; at\; x = ,\pi \quad (2.1b)$$

$$w(X,0) = w_{,\bar{t}}(X,0) = 0,\;\; 0 < x < \pi \quad (2.1c)$$

The aim is to determine a particular value of λ say λ_D called the dynamic buckling load satisfying the inequality $0 < \lambda_D < \lambda_c \leq 1$ for which the structure buckles dynamically under the load function $f(\bar{t})$ taken as

$$f(\bar{t}) = cos\,\delta\bar{t} \quad (2.2)$$

We note specifically that $|f(\bar{t})| \leq 1$ for. $\bar{t} \geq 0$. Following Budiansky [13], the dynamic buckling load λ_D is defined as the maximum load parameter for which the problem has a bounded solution for all time $\bar{t} > 0$. On substituting (2.2) into (2.1a) we observe that the problem has sinusoidally slowly varying coefficients. Such a problem is at time solved using Mathieu-type of instability [13, 14]. However as noted by Budiansky [13] Mathieu-type of instability is usually associated with many cycles of oscillation as opposed to just one shot of oscillation that triggers off dynamic buckling. We note from the earlier substitution that the problem has tow scales namely the fast time scale t and slow time scale $\tau = \delta\bar{t}$ we shall assume

$$\overline{w}(X) = \bar{a}_m \sin mx,\;\; |\bar{a}_m| << 1, m = 1,2,3,\Lambda \quad (2.3)$$

In anticipation of the problem to be solved, we make the following transformation

$$\omega(\tau) = (m^4 - 2m^2\lambda\cos\tau + 1) = (m^4 - 2m^2\lambda\cos\delta\bar{t} + 1) \quad (2.4)$$

For m as in (2.3) we now let

$$\frac{d\hat{t}}{d\bar{t}} = \omega^{\frac{1}{2}},\; t = \hat{t} + \frac{1}{\delta}\left(\varepsilon^2\mu_2 + \varepsilon^3\mu_3 + \Lambda\right), \quad (2.5a,b)$$

$$\mu_i = \mu_i(\tau),\;\; \mu_i(0) = 0,\; i = 2,3,4,\Lambda \quad (2.5c)$$

Now we let $w(x,\bar{t}) = U(x,t,\tau,\varepsilon,\delta)$ and obtain

$$w_{,\bar{t}} = \omega^{\frac{1}{2}} U_{,t} + (\varepsilon^2\mu_2' + \varepsilon^3\mu_3' + K)U_{,t} + \delta U_{,\tau} \quad (2.6a)$$

$$w_{,\overline{tt}} = \omega U_{,tt} + (\varepsilon^2\mu_2' + \varepsilon^3\mu_3' + \mathrm{K})^3 U_{,tt} + \delta^2 U_{,\tau\tau} + 2\omega^{\frac{1}{2}}(\varepsilon^2\mu_2' + \varepsilon^3\mu_3' + \mathrm{K})U_{,tt}$$

$$+ 2\delta(\mu_2'\varepsilon^2 + \mu_3'\varepsilon^3 + \mathrm{K})U_{,tt} + 2\omega^{\frac{1}{2}}\delta U_{,tt} + \frac{\delta\omega^{-\frac{1}{2}}\omega' U_{,t}}{2} + \delta(\mu_2''\varepsilon^2 + \mu_3''\varepsilon^3 + \Lambda) \quad (2.6b)$$

where $\frac{d(\ )}{d\tau} = (\ )'$. We shall let $\quad U(x,t,\tau,\varepsilon,\delta) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} U^{1j}\varepsilon^i\delta^j \quad (2.7)$

where the ij in U^{ij} are superscript and not powers. On substituting (2.5a)-(2.7) into (2.1a-c), using (2.2) and (2.3) and simplifying we get

$$MU^{10} = U_{,tt}^{10} + \frac{1}{\omega}(U_{,xxxx}^{10} + 2(\lambda\cos\tau)w_{,xx} + w) = \frac{2\overline{a}_m m^2(\cos\tau)\sin mx}{\omega} \quad (2.8)$$

$$MU^{11} = -2\omega^{\frac{1}{2}}U_{,tt}^{10} - 2\left(\frac{\omega'}{\omega^{\frac{3}{2}}}\right)U_{,t}^{10} - \omega^{\frac{1}{2}}U_{,t}^{10} \quad (2.9)$$

$$MU^{20} = 0 \quad (2.10)$$

$$MU^{21} = -2\omega^{-\frac{1}{2}}U_{,t\tau}^{20} - 2\left(\frac{\omega'}{\omega^{\frac{3}{2}}}\right)U_{,t}^{20} - \omega^{-\frac{1}{2}}U_{,t}^{20} \quad (2.11)$$

$$MU^{30} = -2\omega^{-\frac{1}{2}}\mu_2' U_{tt}^{10} + \frac{(U^{10})^3}{\omega} \quad (2.12)$$

$$MU^{31} = -2\omega^{-\frac{1}{2}}\mu_2' U_{,tt}^{11} - 2\left(\frac{\omega'}{\omega^{\frac{3}{2}}}\right)U_{,t}^{30} - \frac{\mu_2'' U_{,t}^{10}}{\omega} - 2\omega^{-\frac{1}{2}}U_{,t\tau}^{30} + \frac{3U^{11}(U^{10})^2}{\omega} - \frac{\mu_2' u_{,t}^{10}}{\omega} - (-2)\omega^{\frac{1}{2}}U_{,t\tau}^{30} \quad (2.13)$$

The boundary conditions are $\quad U^{ij} = U_{,xx}^{ij} = 0$ at $X = 0, \pi \quad (2.14)$

The initial conditions evaluated at $t = \tau = 0$ are $\quad U^{ij} = 0,\ \forall i$ and $j \quad (2.15)$

$$U_{,t}^{10} = 0,\ U_{,t}^{1r} + \omega^{-\frac{1}{2}}(0)U_{,\tau}^{1s} = 0;\ \ s = r-1,\ r = 1,2,3,\Lambda \quad (2.16a)$$

$$U_{,t}^{20} = 0,\ U_{,t}^{2r} + \omega^{-\frac{1}{2}}(0)U_{,\tau}^{2p} = 0,\ \ p = r-1,\ r = 1,2,3,\Lambda \quad (2.16b)$$

$$U_{,t}^{30} + \omega^{-\frac{1}{2}}(0)\mu_2'(0)U_{,t}^{10} = 0 \quad (2.17a)$$

$$U_{,t}^{3r} + \omega^{\frac{1}{2}}(0)\left\{\mu_2'(0)U_{,t}^{1r} + U_{,\tau}^{3s}\right\} = 0\ \ s = r-1,\ r = 1,2,3,\Lambda \quad (2.17b)$$

The sequence of equations (2.8)-(2.19b) is now solved by letting

$$U_{ij} = \sum_{n=1}^{\infty} U_u^{ij}(t,\tau)\sin nx \quad (2.18)$$

We now substitute (2.18) into (2.8) for i = 1, j = 0 and after, multiply through by *sinmx* and observe that when n = m we have

$$U_{m,tt}^{10} + U_m^{10} = \frac{2m^2\overline{a}_m\cos\tau}{\omega(\tau)} \equiv B(\tau) \quad (2.19a)$$

$$U_m^{10}(0,0) = U_{m,t}^{10}(0,0) = 0 \quad (2.19b)$$

$$B(0) \equiv B_o = \frac{2m^2\overline{a}\lambda}{\omega_o};\ \omega_o = \omega(0) \quad (2.19c)$$

The solution of (2.19a-c) is $\quad U_m^{10}(t,\tau) = \alpha_1(\tau)\cos t + \beta_1(\tau)\sin t + B(\tau) \quad (2.20a)$

$$\alpha_1(0) = B_o,\ \beta_1(0) = 0 \quad (2.20b)$$

We shall now substitute (2.18) into (2.9) for i = 1, j = 1 and to ensure a uniformly valid solution, set to zero the coefficients of *cost* and *sint* and simplify to get

$$\beta_1' + \left\{\frac{1}{4}\left(\frac{\omega'}{\omega}\right) + \frac{1}{2}\right\}\beta_1 = 0;\ \ \alpha_1' + \left\{\frac{1}{4}\left(\frac{\omega'}{\omega}\right) + \frac{1}{2}\right\}\alpha_1 = 0 \quad (2.21a,b)$$

The solution of (2.21a, b) are $\beta_1(\tau)\equiv 0;\ \ \alpha_1(\tau)=B_o\left(\frac{\omega_o}{\omega}\right)^{\frac{1}{4}}e^{\frac{\tau}{2}}$ (2.21c)

The remaining equation in (2.9) is solved to get $U_m^{10}(t,\tau)=\alpha_2(\tau)\cos t+\beta_2(\tau)\sin t$ (2.22a)

$$\alpha_2(0)=0,\beta_2(0)=-\frac{\omega_o^{\frac{1}{2}}B_o}{2} \tag{2.22b}$$

So far write $U_m^{11}(t,\tau)=\alpha_1(\tau)\cos t+B;\quad U^{10}=U_m^{10}=(t,\tau)\sin mx$ (2.22c)

We expect that on full solution we shall have $\alpha_2(\tau)\equiv 0$ so that we get

$$U_m^{11}(t,\tau)=\beta_2(\tau)\sin t;\ \ U^{11}=U_m^{11}(t,\tau)\sin mx \tag{2.22d}$$

The substitutions into (2.10) and (2.11) easily yield $U^{20}=U^{21}=0$ (2.23)

We now substitute (2.18) into (2.12) for i = 3, j = 0 and set n = m to get

$$U_{m,tt}^{30}+U_m^{30}=-2\mu'_2\omega^{-\frac{1}{2}}U_{m,tt}^{10}+\frac{3}{4\omega}\left[r_o+r_1\cos t+r_2\cos 2t+r_3\cos 3t\right] \tag{2.24a}$$

$$U_m^{30}(0,0)=0,U_{m,t}^{30}(0,0)+\mu'_2(0)\omega_o^{\frac{1}{2}}U_{m,t}^{10}(0,0)=0 \tag{2.24b}$$

where $r_o(\tau)=B^3+\frac{3B\alpha_1^2}{2},\ \ r_o(0)=\frac{5B_0^3}{2};\ \ r'_o(0)=-\frac{3B_0^3}{4}$ (2.25a)

$$r_1(\tau)=3B^2\alpha_1+\frac{3\alpha_1^3}{2},\ r_1(0)=-\frac{15B_0^3}{4},\ \ r'_1(0)=\frac{21B_0^3}{8} \tag{2.25b}$$

$$r_2(\tau)=\frac{3B\alpha_1^2}{2},\ \ r_2(0)=\frac{3B_0^3}{2},\ \ r'_1(0)=-\frac{3B_0^3}{2} \tag{2.25c}$$

$$r_3(\tau)=\alpha_1^3,\quad r_3(0)=-B_0^3,\ \ r_3(0)=\frac{3B_0^3}{2} \tag{2.25d}$$

Similarly when n = 3m we get a second equation in the substitutions in (2.12) as

$$U_{3m,tt}^{30}+\phi_{3m}U_{3m}^{30}=-\frac{1}{4\omega}\left[r_0+r_1\cos t+r_2\cos 2t+r_3\cos 3t\right] \tag{2.26a}$$

$$U_{3m}^{30}(0,0)=U_{3m,t}^{30}(0,0)=0 \tag{2.26b}$$

$$\phi_{3m}(\tau)=\left(\frac{81m^4-18m^2\lambda\cos\tau+1}{m^4-2m^2\lambda\cos\tau+1}\right)>0,\ \forall\tau \tag{2.26c}$$

To ensure a uniformly valid solution in (2.24a) we set to zero the coefficient of *cost* on the right side and get

$$\mu'_2(\tau)=-\frac{3r_1\omega^{\frac{1}{2}}}{8\alpha_1};\ \mu'_2(0)=-\frac{45\omega_0^{-\frac{1}{2}}B_0^2}{32};\ \ \mu''_2(0)=\frac{9B_0(7B-5)}{64\omega_{0\frac{1}{2}}} \tag{2.27}$$

The solution of the remaining equation in (2.24a) is

$$U_m^{30}(t,\tau)=\alpha_3(\tau)\cos t+\beta_3(\tau)\sin t+\frac{3}{4\omega}\left[r_0-\frac{r_2\cos 2t}{3}+\frac{r_3\cos 3t}{8}\right] \tag{2.28a}$$

$$\alpha_3(0)=\frac{195B_0^3}{128};\ \ \beta_3(0)=0 \tag{2.28b}$$

To solve (2.26a-c) we note that $\phi_{3m}(\tau)=\phi_{3m}(0)+\tau\phi'_{3m}(0)+\frac{\tau^2}{2}\phi''_{3m}(0)+\Lambda$ (2.29a)

It is to be noted that the accuracy to be maintained in this investigation is up to the term in δ (and not δ^2). We equally note that $\phi'_{3m}(0)=0$ in (2.29a) and the second team there will eventually contribute to team in δ^2. We can justifiable write $\phi_{3m}(\tau)\cong\phi_{3m}(0)=\phi^2=\left(\frac{81m^4-18m^2\lambda+1}{m^4-2m^2\lambda+1}\right)>0$ (2.29b)

The introduction of (2.29b) into (2.26a) and eventual solution yield

$$U^{30}_{3m}(t,\tau)=\alpha^{4}(\tau)cos\phi t+\beta_4(\tau)sin\phi t-\frac{1}{4\omega}\left[\frac{r_0}{\phi_2}+\frac{r_1 cos t}{\phi^2-1}+\frac{r_2 cos 2t}{\phi^2-4}+\frac{r_3 cos 3t}{\phi^2-9}\right] \tag{2.30a}$$

$$\alpha_4(0)=\frac{B_0^3 L_0}{8\omega_0},\quad L_0=\left[\frac{5}{\phi^2}-\frac{15}{\phi^2-4}-\frac{3}{\phi^2-4}-\frac{2}{\phi^2-9}\right] \tag{2.30b}$$

Now substituting (2.18) for i = 3, j = 1 into (2.13), we see that when n = m we get

$$U^{31}_{m,tt}+U^{31}_{m}=2\omega^{\frac{1}{2}}\mu_2'\beta_2\, sin\, t-\left\{\frac{1}{2}\left(\frac{\omega'}{\omega^{\frac{3}{2}}}\right)+\omega^{\frac{1}{2}}\right\}U^{30}_{m,t}-2\omega^{-\frac{1}{2}}U^{30}_{m,t\tau}$$
$$+\frac{1}{\omega}(\mu_2''+\mu_2')\alpha_1\, sin\, t-\frac{9\left(U^{10}_{m}\right)^2 U^{11}_{m}}{4} \tag{2.31a}$$

$$U^{31}_{m}(0,0)=0,\quad U^{31}_{m,t}(0,0)+\omega_0^{\frac{1}{2}}U^{30}_{m,\tau}(0,0)=0 \tag{2.31b}$$

Similarly when n = 3m in the substitution in (2.13) we get, using (2.29b)

$$U^{31}_{3m,tt}+\phi^2 U^{31}_{3m}=-\left\{\frac{1}{2}\left(\frac{\omega'}{\omega^{\frac{3}{2}}}\right)+\omega^{\frac{1}{2}}\right\}U^{30}_{3,mt}-2\omega^{-\frac{1}{2}}U^{30}_{3m,t\tau}-\frac{3(U^{10}_{m})^2U^{11}_{m}}{4} \tag{2.32a}$$

$$U^{31}_{3m}(0,0)=0,\ U^{31}_{3m,t}(0,0)+\varpi_0^{-\frac{1}{2}}U^{31}_{3m}(0,0)=0 \tag{2.32b}$$

To ensure a uniformly valid solution in (2.31a), we equate to zero the coefficients of *cost* and *sint* and get

$$\beta_3'+\left\{\frac{1}{4}\left(\frac{\omega'}{\omega}\right)+\frac{1}{2}\right\}\beta_3=0,\quad \alpha_3'+\left\{\frac{1}{4}\left(\frac{\omega'}{\omega}\right)+\frac{1}{2}\right\}\alpha_3=\frac{H(\tau)\omega^{\frac{1}{2}}}{2} \tag{2.33a}$$

$$H(\tau)=\left[\frac{9\beta_2}{4}\left(\frac{\alpha_1^2}{4}+B^2\right)-\frac{1}{\omega}(\mu_2''+\mu_2')\alpha_1-2\mu_2'\beta_2\omega^{-\frac{1}{2}}\right] \tag{2.33b}$$

$$H(0)=\frac{1}{\omega_0^{\frac{1}{2}}}\left[\frac{45B_0^3}{32}\left(1+\omega_0^{-1}-\omega_0^{\frac{-3}{2}}\right)+\frac{9B_0(7B_0-5)}{64_0}\right] \tag{2.33c}$$

The solutions of (2.33a-c) are

$$\beta_3(\tau)=0;\quad \alpha_3(\tau)=\omega^{-\frac{1}{4}}e^{-\frac{\tau}{2}}\left[\int_0^\tau\frac{\omega^{\frac{1}{4}}e^{-\frac{s}{2}}H(s)ds}{2}+\omega_0^{\frac{1}{4}}\alpha_3(0)\right] \tag{2.34a,b}$$

$$\alpha_3'(0)=\frac{B_0^3R_5}{\omega_0^{\frac{3}{2}}},\quad R_5=-\frac{195}{256}+\frac{\omega_0^{\frac{7}{2}}H(0)}{2} \tag{2.34b}$$

The remaining equation in (2.31a) is $U^{31}_{m,tt}+U^{31}_{m}=r_4\ sin\,2t+r_5\ sin\,3t$ (2.35a)

$$r_4(\tau)=-\left\{\frac{1}{2}\left(\frac{\omega'}{2\omega^{\frac{5}{2}}}+\frac{1}{\omega^{\frac{3}{2}}}\right)+\omega^{-\frac{1}{2}}\left(\frac{r_2}{\omega}\right)+\frac{9B\alpha_1\beta_2}{4}\right\} \tag{2.35b}$$

$$r_5(\tau)=-\left\{\left\{\frac{9r_3}{32}\left\{\frac{1}{2}\left(\frac{\omega'}{2\omega^{\frac{5}{2}}}+\frac{1}{\omega^{\frac{3}{2}}}\right)\right\}+\frac{9\omega^{-\frac{1}{2}}\left(\frac{r_2}{\omega}\right)'}{16}+\frac{9B\alpha_1^2\beta_2}{16}\right\}\right\} \tag{2.35c}$$

The solution of (2.35a-c), using (2.31b), is

$$U_{m}^{31}(t,\tau)=\alpha_5(\tau)\cos t+\beta_5(\tau)\sin t-\frac{r_4\sin 2t}{3}-\frac{r_5\sin 3t}{8} \qquad (2.36a)$$

$$\alpha_5(0)=0,\ \beta_5(0)-\left(\frac{2r_4}{3}+\frac{3r_5}{8}\right)\Bigg|_{\tau=0}+\omega_0^{-\frac{1}{2}}\left[\alpha_3'+\frac{3}{4}\left(\frac{r_0}{\omega}\right)'-\frac{1}{3}\left(\frac{r_2}{\omega}\right)'-\frac{1}{8}\left(\frac{r_3}{\omega}\right)'\right]\Bigg|_{\tau=0}=0 \qquad (2.36b)$$

We now simplify (2.32c), set to zero the coefficients of *cost* and *sint* and get respectively

$$\beta_4'+\frac{1}{2}\left\{\left(\frac{\omega'}{2\omega}\right)+1\right\}\beta_4=0 \text{ and } \alpha_4'+\frac{1}{2}\left\{\left(\frac{\omega'}{2\omega}\right)+1\right\}\alpha_4=0 \qquad (2.37a)$$

The solutions of (2.37a) are
$$\beta_4(\tau)\equiv 0;\ \alpha_4(\tau)=\alpha_4(0)\left(\frac{\omega_0}{\omega}\right)^{\frac{1}{4}}e^{\frac{\tau}{2}} \qquad (2.37b)$$

$$\alpha_4'(0)=-\frac{\alpha_4(0)}{2}=\frac{B_0^3L_0}{16\omega_0} \qquad (2.37c)$$

The remaining equation in the substitution in (2.32a)

$$U_{3m,tt}^{31}+\phi^2U_{3m}^{31}=-r_6\sin t+r_7\sin 2t+r_8\sin 3t \qquad (2.38a)$$

$$r_6(\tau)=-\frac{1}{4}\left(\frac{\omega'}{2\omega^{\frac{5}{2}}}+\frac{1}{\omega^{\frac{3}{2}}}\right)\left(\frac{r_1}{\phi^2-1}\right)-\frac{\omega^{-\frac{1}{2}}\left(\frac{r_2}{\omega}\right)'}{2(\phi^2-1)}+\frac{3}{4}\left\{\beta_2\left(\frac{\alpha_1^2}{4}+B^2\right)-\frac{\beta_2\alpha_1^2}{4}\right\} \qquad (2.38b)$$

$$r_6(0)=\frac{\omega_0^{-\frac{1}{2}}B_0^3R_0}{16},\ R_0=\frac{81}{\omega_0(\phi^2-1)}+6 \qquad (2.38c)$$

$$r_7(\tau)=-\frac{1}{2}\left(\frac{\omega'}{2\omega^{\frac{5}{2}}}+\frac{1}{\omega^{\frac{3}{2}}}\right)\left(\frac{r_2}{\phi^2-1}\right)-\frac{\omega^{-\frac{1}{2}}\left(\frac{r_2}{\omega}\right)'}{2(\phi^2-4)}+\frac{3B\alpha_1\beta_2}{4} \qquad (2.38d)$$

$$r_7(0)=\frac{3\omega_0^{-\frac{1}{2}}B_0^3R_1}{8},\ R_1=1+\frac{3}{\omega_0(\phi^2-4)} \qquad (2.38e)$$

$$r_8(\tau)=-\frac{3}{4}\left(\frac{\omega'}{2\omega^{\frac{5}{2}}}+\frac{1}{\omega^{\frac{3}{2}}}\right)\left(\frac{r_1}{\phi^2-9}\right)-\frac{3\omega^{-\frac{1}{2}}\left(\frac{r_3}{\omega}\right)'}{2(\phi^2-1)}+\frac{3\alpha_1^2\beta_2}{4} \qquad (2.38f)$$

$$r_8(0)=\frac{3\omega_0^{-\frac{1}{2}}B_0^3R_2}{32},\ R_2=1+\frac{16}{\omega_0(\phi^2-1)} \qquad (2.38g)$$

The solution of (2.38a-g) is

$$U_{3m}^{31}(t,\tau)=\alpha_6(\tau)\cos\phi t+\beta_6(\tau)\sin\phi t+\frac{r_6\sin t}{\phi^2-1}+\frac{r_7\sin 2t}{\phi^2-4}+\frac{r_8\sin 3t}{\phi^2-9} \qquad (2.39a)$$

$$\alpha_6(0)=0,\ \beta_4(0)-\left[\frac{r_6}{\phi^2-1}+\frac{2r_7}{\phi^2-4}+\frac{3r_8}{\phi^2-9}\right]\Bigg|_{\tau=0}+\omega_0^{-\frac{1}{2}}\left[\alpha_4'-\frac{1}{4}\left\{\frac{1}{\phi^2}\left(\frac{r_0}{\omega}\right)'+\frac{1}{\phi^2-1}\left(\frac{r_2}{\omega}\right)'\right.\right.$$

$$\left.\left.+\frac{1}{\phi^2-4}\left(\frac{r_2}{\omega}\right)'+\frac{1}{\phi^2-9}\left(\frac{r_3}{\omega}\right)'\right\}\right]\Bigg|_{\tau=0}=0 \tag{2.39b,c}$$

A detailed simplification of (2.39c) gives

$$\beta_6(0)=\omega_0^{-\frac{1}{2}}B_0^3R_3,\ R_3=\left[\left\{\frac{R_0}{16(\phi^2-1)}-\frac{3R_1}{4(\phi^2-4)}+\frac{9R_2}{32(\phi^2-9)}\right\}+\frac{1}{\omega}\left\{\left\{\frac{L_0}{16}+\frac{3}{8}\left\{\frac{7}{4(\phi^2-1)}\right.\right.\right.\right.$$

$$\left.\left.\left.\left.+\frac{1}{(\phi^2-4)}-\frac{1}{2\phi^2}+\frac{1}{(\phi^2-9)}\right\}\right\}\right\}\right] \tag{2.39d}$$

Thus, we have

$$U(x,t,\varepsilon,\delta)=\varepsilon(U_m^{10}+\delta U_m^{11})\sin mx+\varepsilon^3[(U_m^{30}+\delta U_m^{31})\sin mx+(U_{3m}^{31}+\delta U_{3m}^{31})\sin 3mx]$$
$$+0(\varepsilon\delta^2)+0(\varepsilon^3\delta^2) \tag{2.40}$$

We shall now determine the maximum displacement $U(x_a,t_a,\varepsilon,\delta)\equiv U_a$ and the conditions for this are

$$U_{,x}=0;,\ U_{,t}+\omega^{-\frac{1}{2}}(\varepsilon^2\mu_2'+\Lambda+\delta U_{,\tau})=0 \tag{2.41a,b}$$

which are determined at the values $x_a, t_a, and\ \tau_a$ associated respectively with $x,t,$ and τ. Upon substituting (2.40) in (2.41a), we get $x_a=\left(\frac{1+2r}{2m}\right)\pi,\ r=0,1,2,3,\Lambda$. We shall however set r = 0 and thus get

$$x_a=\frac{\pi}{2m} \tag{2.42a}$$

We shall next let

$$t_a=t_0+\delta t_{01}+\varepsilon^2(t_{20}+\delta t_{21}+\Lambda)+\Lambda \tag{2.42b}$$

$$\bar{t}_a=\bar{t}_0+\delta\bar{t}_{01}+\varepsilon^2(\bar{t}_{20}+\delta\bar{t}_{21}+\Lambda)+\Lambda \tag{2.42c}$$

$$\hat{t}_a=\hat{t}_0+\delta\hat{t}_{01}+\varepsilon^2(\hat{t}_{20}+\delta\hat{t}_{21}+\Lambda)+\Lambda \tag{2.42d}$$

$$\tau_a=\delta\bar{t}_a+\delta\{\bar{t}_0+\delta\bar{t}_{01}+\varepsilon^2(\bar{t}_{20}+\delta\bar{t}_{21}+\Lambda)+\Lambda\} \tag{2.42e}$$

where $\hat{t}_a$ and $\bar{t}_a$ are the critical values of $\hat{t}_a$ and $\bar{t}_a$ respectively. By evaluating the coefficients of $\varepsilon,\ \varepsilon\delta$ and ε^3 in (2.41b), we get respectively

$$U_{m,t}^{10}=0;\ t_{01}U_{m,tt}^{10}+U_{m,t}^{11}=0;\ t_{20}U_{m,tt}^{10}+U_{m,t}^{30}-U_{3m,t}^{30}=0 \tag{2.43a,b,c}$$

where (2.43a-c) are evaluated at the critical values. From (2.43a), we get $t_0=\pi r,\ r=0,1,2,3,\Lambda$. Since we need the least nontrivial value of t_0, we set r = 1 and get $t_0=\pi$ (2.44a)

From (2.43b), we get

$$t_{01}=-\frac{U_{m,t}^{11}}{U_{m,tt}^{10}}=-\frac{\omega_0^{-\frac{1}{2}}}{2} \tag{2.44b}$$

From (2.43c) we get

$$t_{20}=\left(\frac{U_{m,t}^{30}-U_{3m,t}^{30}}{U_{m,tt}^{10}}\right)=\frac{B_0^2L_0\phi\sin\phi t_0}{8\omega_0} \tag{2.44c}$$

So far we have used the fact that an arbitrary function $F(t_a,\tau_a)$ has the expansion about the point $(t_a,\tau_a)=(t_0,0)$.

$$F(t_a,\tau_a)=F(t_0,0)+\delta(t_{01}F_{,t}+\bar{t}_0F_{,\tau})+\varepsilon^2[t_{20}F_{,t}+\delta\{\bar{t}_{20}F_{,\tau}+t_{21}F_{,t}+t_{01}t_{21}F_{,tt}+\bar{t}_0t_{20}F_{,t\tau}\}]+\Lambda \tag{2.45}$$

We shall now determine the maximum lateral displacement U_a by evaluating (2.40) at the critical point using (2.44a-c). This gives

$$U_a = \varepsilon\left[(B-\alpha_1)+\delta\bar{t}_0(B'-\alpha_1')\right]\Big|_{\substack{t_0=\pi \\ \tau=0}} + \varepsilon^3\left[\left(U_m^{30}-U_{3m}^{30}\right)+\delta\{\bar{t}_{20}(B'-\alpha')\}+t_{01}t_{20}U_{m,tt}^{10}+t_{20}U_{m,t}^{11}\right.$$
$$\left.+\left(U_{m,t}^{30}-U_{3m,t}^{30}\right)t_{01}+\left(U_{m,t}^{30}-U_{3m,t}^{30}\right)\bar{t}_0+\left(U_m^{31}-U_{3,m}^{31}\right)\right]\Big|_{\substack{t_0=\pi \\ \tau=0}} + 0\left(\varepsilon\ \delta^2\right)+0\left(\varepsilon^3\delta^2\right) \qquad (2.46)$$

From (2.5a), evaluated at the critical point we get

$$\hat{t}_a = \int_0^{\hat{t}_a}\left(m^4-2m^2\lambda\cos\delta t_1+1\right)^{\frac{1}{2}} = \omega_0^{\frac{1}{2}}\left(\bar{t}_a+\frac{m^2\lambda\ \delta\bar{t}_a^{\,2}}{6\omega_0}\right)+\Lambda \qquad (2.47a)$$

Thus, we have $\hat{t}_{20} = \omega_0^{\frac{1}{2}}\bar{t}_{20};\quad \hat{t}_0 = \omega_0^{\frac{1}{2}}\bar{t}_0$ (2.47b)

From (2.5b) evaluated at the critical point we have

$$\hat{t}_{20} = t_{20}-\mu_2'(0)\bar{t}_0;\quad \hat{t}_0 = t_0 \qquad (2.47c)$$

Therefore we get $\bar{t}_{20} = \omega_0^{\frac{1}{2}}\left(t_{20}-\mu_2'(0)\bar{t}_0\right),\qquad \bar{t}_0 = \omega_0^{\frac{1}{2}}t_0$ (2.47d)

The terms in (2.46), evaluated at the indicated values, are now simplified as follows:

$$(B-\alpha_1) = 2B_0;\quad \delta\bar{t}_0(B'-\alpha_1') = \frac{B_0\delta\bar{t}_0}{2};\quad (B'-\alpha_1')\bar{t}_{20} = \frac{B_0\delta\bar{t}_{20}}{2} \qquad (2.48a)$$

$$t_{01}t_{20}U_{m,tt}^{10} = -B_0t_{01}t_{20};\quad \left(U_m^{30}-U_{3m}^{30}\right) = \frac{3B_0^3\left(1-\frac{R_4}{24}\right)}{\omega_0} \qquad (2.48b)$$

$$R_4 = \left[\frac{5(\cos\phi t_0-1)}{\phi^2}+\frac{15(1-\cos\phi t_0)}{2(\phi^2-1)}+\frac{3(\cos\phi t_0-1)}{\phi^2-4}+\frac{(1-\cos\phi t_0)}{(\phi^2-9)}\right] \qquad (2.48c)$$

$$t_{20}U_{m,t}^{11} = \frac{\omega_0^{-\frac{1}{2}}B_0t_{20}}{2};\quad \left(U_{m,t}^{30}-U_{3m,t}^{30}\right)t_{01} = \frac{B_0^3\phi L_0t_{01}\left(\sin\phi t_p\right)}{8\omega_0} \qquad (2.48d)$$

$$\left(U_{m,\tau}^{30}-U_{3m,\tau}^{30}\right)\bar{t}_0 = \frac{B_0^3R_7\bar{t}_0}{\omega_0};\quad R_7 = -\frac{R_5}{\omega_0^{-\frac{1}{2}}}+\frac{15}{64}-R_6;\quad R_5 = \frac{195}{256}+\frac{\omega_0^{-\frac{7}{2}}H(0)}{2} \qquad (2.48e)$$

$$R_6 = \frac{L_0\cos\phi t_0}{16}+\frac{3}{8}\left[\frac{1}{2\phi^2}+\frac{1}{\phi^2-4}+\frac{1}{\phi^2-9}\right] \qquad (2.48f)$$

$$\left(U_m^{31}-U_{3m}^{31}\right) = -\omega_0^{-\frac{1}{2}}B_0^3R_3\sin\phi t_0 \qquad (2.48g)$$

We note specifically that B_0, among other terms, contains the load parameter λ. On simplifying (2.46), we get

$$U_a = 2B_0\left(1+\frac{\delta\bar{t}_0}{4}\right)+\frac{3B_0^3}{\omega_0}\left[\left(1-\frac{R_4}{24}\right)+\frac{\delta\,\omega_0R_8}{3B_0^3}\right]+0\left(\varepsilon\ \delta^2\right)+0\left(\varepsilon^3\ \delta^2\right) \qquad (2.49a)$$

$$R_8 = \left[\frac{B_0\bar{t}_{20}}{2}-B_0t_{01}t_{20}+\frac{\omega_0^{-\frac{1}{2}}B_0t_{20}}{2}+\frac{B_0^3\phi L_0t_{01}(\sin t_0)}{8\omega_0}+\frac{B_0^3\bar{t}_0R_7}{\omega_0}-\omega_0^{-\frac{1}{2}}B_0^3R_3(\sin t_0)\right] \qquad (2.49b)$$

We shall however write (2.49a) simply as $U_a = \varepsilon C_1+\varepsilon^3C_3+\Lambda$ (2.50a)

$$C_1 = 2B_0\left(1+\frac{\delta\bar{t}_0}{4}\right);\quad C_3 = \frac{3B_0^3}{\omega_0}\left[\left(1-\frac{R_4}{24}\right)+\frac{\delta\,\omega_0R_8}{3B_0^3}\right] \qquad (2.50b)$$

According to Budiansky [13], the dynamic buckling λ_D is obtained from the maximization

$$\frac{d\lambda}{dU_a} = 0 \qquad (2.51)$$

Before invoking (2.51) it is essential [1,3,15,16] to reserve the series (2.50a) and so obtain

$$\varepsilon = d_1 U_a + d_3 U_a^3 + \Lambda \tag{2.52a}$$

On substituting in (2.52a) for U_a and equating the coefficients of powers of ε, we have

$$d_1 = \frac{1}{C_1}; \ d_3 = -\frac{C_3}{C_4} \tag{2.52b}$$

The maximization in (2.51) is now easily accomplished through (2.52a) to get

$$\varepsilon = \frac{2}{3}\sqrt{\frac{C_1}{3C_3}} \tag{2.53}$$

which is evaluated at $\lambda = \lambda_D$. On simplification, equation (2.53) yields

$$\left(m^4 - 2m^2\lambda_D + 1\right)^{\frac{3}{2}} = \frac{9m^2\bar{a}_m \lambda_D \left|\varepsilon \bar{a}_m\right.}{\sqrt{2}}\sqrt{\left(1 - \frac{R_4}{24}\right) + \frac{\delta\,\omega_0 R_8}{3B_0^3}} \tag{2.54}$$

which is evaluated at buckling.

3.0 Analysis of result

The result (2.54) is valid if δ is small so that $\frac{\left|\delta \bar{t}_0\right|}{4} \pi 1$ and $\left|\frac{\omega_0 \delta R_8}{3B_0^3}\right| \pi 1$. In particular when $m = 1$, we get

$$\left(1 - \lambda_D\right)^{\frac{3}{2}} = \frac{9\left|\varepsilon \bar{a}_1\right|\lambda_D}{4}\sqrt{\frac{\left(1 - \frac{R_4}{24}\right) + \frac{\delta\,\omega_0 R_8}{3B_0^3}}{1 + \frac{\delta \bar{t}_0}{4}}} \tag{3.1}$$

where all the terms in (3.1) are evaluated at $m = 1$. When $\delta = 0$, we get the corresponding step loading (without damping) results corresponding to (2.54) and (3.1) which are

$$\left(m^4 - 2m^2\lambda_D + 1\right)^{\frac{3}{2}} = \frac{9m^2\left|\varepsilon \bar{a}_m\right|\lambda_D}{\sqrt{2}}\sqrt{\left(1 - \frac{R_4}{24}\right)} \ and \ \left(1 - \lambda_D\right)^{\frac{3}{2}} = \frac{9\left|\varepsilon \bar{a}\right|\lambda_D\sqrt{\left(1 - \frac{R_4}{24}\right)}}{4} \tag{3.2a,b}$$

where the latter is evaluated at $m = 1$. A result corresponding to (3.2b) was obtained in [2] as

$$\left(1 - \lambda_D\right)^{\frac{3}{2}} = \frac{9\lambda_D\left|\varepsilon \bar{a}_1\right|}{4} \tag{3.2c}$$

The disparity between the two results is accounted for by the fact that (3.2c) demanded that the buckling mode be in the shape of the imperfection while we have relaxed this assumption in the analysis that led (3.2b). All the results are implicit in the load parameter. We would normally expect $\left|\frac{\omega_0 \delta R_8}{3B_0^3}\right| \phi \frac{\left|\delta \bar{t}_0\right|}{4}$ and so the dynamic buckling load form (2.54) (or (3.1) is higher than the corresponding step loading result and so the step loading result provides a lower bound.

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