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On the convergence of the dynamic series solution of a constrained elastic column subjected to wind gust.

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Abstract

The one dimensional problem of analysing the dynamic behaviour of an elevated water tower with elastic deflection–control device and subjected to a dynamic load was examined in [2]. ***The constrained elastic system was modeled as a column carrying a concentrated mass at its top and elastically constrained at a point along its length. A new solution technique, which yielded a series solution to the problem, was then developed. This paper is basically concerned with establishing the convergence of the series solution obtained in*** [2] ***and hence, the acceptance of this solution as the actual solution of the constrained elevated water tower vibration problem. Damping is neglected.***

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1.0 Introduction

Several structures (especially tall ones) collapse as a result of poor design and/or analysis concepts, some causing damage to lives and properties. Hence, the design and analysis of such structures are of practical interest to Engineers, Applied Mathematicians and Physicists [1-4]. This paper is concerned with the dynamic analysis of such structures and this is normally carried out by, first, considering such a structure as an elastic cantilever column, which may or may not carry a concentrated mass at its free end and subjected to a time-dependent load [1-4]. Next a mathematical model comprising of the governing equation, the boundary conditions and the initial condition describing the behaviour of the column model is then developed. Thirdly, the initial–boundary value problem is then solved using an appropriate technique. Furthermore we remark that the column model may either be constrained or not.

Gbadeyan and Titiloye [2] studied the dynamic response of an elevated water tower, which is a typical example of the structures under consideration. The water tower is assumed to be elastically constrained using elastic tendons, and is subjected to a strong gust of wind. The water tower is modeled as a column carrying a concentrated mass at its top elastically constrained at a point along its length and under the influence of the strong gust of wind. Of keen interest however, is the development of the versatile technique used to solve the dynamical problem. This technique involves the use of generalized integral transform and modified strubles method [1, 2]. A key feature of the technique, which is peculiar to the problem under consideration, is that its solution can be easily adjusted to handle problems having more than one constraint located at any point along the length of the column. Impressive though the work in [2] is, the discussion there in fails to treat an important issue likely to be encountered. In particular, the issue of convergence in relation to the series solution obtained is not addressed.

The primary motivation for this presentation derives therefore from the issue which is very important yet un-addressed in [2]. In particular, in this presentation the convergence of the series solution obtained in [2] is examined. It is shown that the series solution, which describes, at least formally, the dynamic deflection of a constrained un-damped elevated water tower subjected to wind gust is uniformly convergent.

In the following sections, a brief discussion of the mathematical formulation of the problem and its solution, details of the proof of the convergence of the series solution and the conclusion are presented.

2.0 **The mathematical formulation and the series solution**.

The equations governing the response of the elastic column system already alluded to [2, 4] are:

$$W^{iv}(x,t)+\frac{\gamma}{\lambda}W''(x,t)+\frac{\mu g}{\lambda}[(l-x)W'(x,t)]+\left[\frac{\mu}{\lambda}+\frac{\gamma}{\lambda g}\delta(x-l)\right]W(x,t) = \frac{q(x,t)}{\lambda}-\frac{K}{\lambda}W(x,t)\delta(x-a) \tag{2.1}$$

$$\left.\begin{aligned} &W(0,t)=W'(0,t)=0\\ &W''(l,t)=W'''(l,t)+\frac{\gamma}{\lambda}W'(l,t)=0 \end{aligned}\right\} \tag{2.2}$$

$$W(x,0)=W(x,0)=0 \tag{2.3}$$

In (2.1) – (2.3) the following notation have been used

$(\cdot)'$ denotes differentiation with respect to *x*

x is the spatial co-ordinate

$(\dot{\cdot})$ denotes differentiation with respect to *t*

t is the time

W(*x*, *t*) is the deflection of the column

K is the springs constant

Q(*x*, *t*) is the transverse wind load

μ is the constant mass per unit length of the column

γ is the weight of the tank (and its content)

δ(.) is the Dirac–delta function

ρ is the constant bending stiffness of the column

l is the length of the system

g is the acceleration due to gravity.

α is the height of the constraint from the lower end of the system

To solve the initial boundary value problem ((2.1) – (2.3)), Gbadeyan and Titiloye in [2] used the generalized finite integral transform and modified asymptotic technique. This resulted in

$$\ddot{W}(q,t)+P_m^2W(q,t)=H_{OR}Q^*(q,t) \tag{2.4}$$

where P_m is the extended modified frequency defined as $P_m=\frac{\Gamma_q}{2\mu_q}\left(2\mu_q-v\Gamma_1(q,q)\right)$, $v=\frac{\varepsilon}{1+\varepsilon}\pi 1$, and

$Q*(q,t)$,

$$\Gamma_q^2=\left(\omega_q^2+\varepsilon a^4\mu R_a-HR_b\right),\ \varepsilon=\frac{\lambda}{\mu g l},\ a^4=\frac{\omega_q^2\mu}{\rho},\ H=\frac{K}{\alpha},\ R_A=\left[\Lambda_1(q,s)+2\sum_{m=1}^{\infty}\cos m\pi\Lambda_1(q,s)\right],$$

$$R_B = \left[\Lambda_1(q,s) + \frac{2l}{\alpha}\sum_{m=1}^{\infty} \cos m\pi \Lambda_2(q,s) \right],\ R_{AA} = R_A\big|_{s=q},\ \ R_{BB} = R_B\big|,\ \ R_a = \frac{R_{AA}}{\mu_q},\ R_a = \frac{R_{BB}}{\mu_q},\ \ \Lambda_1(q,q) = \int_0^l W_q^2(x)dx;$$

$$\Lambda_1(q,q) = \int_0^l W_s(x) W_q(x)\,dx;\ \ \Lambda_2(q,s) = \int_0^l \cos\frac{m\pi x}{l} W_s(x) W_q(x)\,dx;\ \mu_q = \int_0^l W_q^2(x)dx.$$

$W_q(x)$ is the kernel of the generalized integral transform, $W(q\ ,t)$ is the generalized integral transform of $W(x,t),\ H_{OR} = \dfrac{v\,g\,l}{\gamma},\ \upsilon = \dfrac{\varepsilon}{1+\varepsilon}\ \pi\,1,$ and $Q^*(q,t)$ is the generalized integral transform of the transverse wind load. Taking the Laplace transform of (2.4) and using equation (2.3) we obtained

$$W(q,s) = \frac{H_{OR} Q^*_{BL}(q,s)}{S^2 - P_m^2} \qquad (2.5)$$

where $Q^*_{BL}(q,s)$ is the Laplace transform of $Q^*(q,t)$

The Laplace inversion of the expression (2.5) using the convolution theorem yields

$$W(x,t) = \sum_{q=1}^{\infty}\left[\frac{H_{OR}}{p_M} \int_0^L\!\int_0^L \sin P_m(t-u) Q(v.u) W_m(x) W_m(v)\,dvdu \right] \qquad (2.6)$$

Equation (2.6) describes at least formally, the dynamic deflection of a constrained undamped elevated water tower subjected to wind gust Q(x, t).

3.0 The convergence issue

Equation (2.6) may not be the actual solution to the problem of obtaining the dynamic deflection of the constrained undamped water tower subjected to a wind gust. As a matter of fact, one can consider it to be the non-formal solution only if the series solution can be shown to be uniformly convergent. This we proceed to do in this section.

Theorem

Suppose the following improper integral

$$\int_o^{\infty} \frac{\partial Q(x,t)dx}{\partial X} < \infty \qquad (3.1)$$

is bounded, then the series solution (2.6) is uniformly convergent.

Proof

It is known that for a stable system [5, 6] such that the one under consideration, the eigen function $W_q(x)$ are uniformly bounded in the domain $0 \le x \le l$, that is $|W_q(x)| \le v_1 < \infty\ \forall q$. Consider

$$R_a = \left(\Lambda_1(q,q) + 2\sum_{m=1}^{\infty} \cos m\pi\, \Lambda_2(q,s) \right)$$

then
$$|R_a| = \left| \frac{1}{\mu_q}\left(\Lambda_1(q,q) + 2\sum_{m=1}^{\infty} \cos m\pi\, \Lambda_2(q,q) \right) \right| \le \left| \frac{1}{\mu_q} \Lambda_1(q,q) \right| + 2\left| \frac{1}{\mu_q} \sum_{m=1}^{\infty} \cos m\pi\, \Lambda_2(q,q) \right|$$

$$\le \left| \frac{1}{\mu_q} \Lambda_1(q,q) \right| + 2\sum_{m=1}^{\infty} \left| \frac{1}{\mu_q} \cos m\pi\, \Lambda_2(q,q) \right| = \left| \frac{1}{\mu_q} \right| \left\{ |\Lambda_1(q,q)| + 2\sum_{m=1}^{\infty} |\cos m\pi| |\Lambda_2(q,q)| \right\} = \left| \frac{1}{\mu_q} \right| \left\{ |\Lambda_1(q,q)| + 2\sum_{m=1}^{\infty} |\Lambda_1(q,q)| \right\}$$

$$= \left| \frac{1}{\mu_q} \right| \left\{ \left| \int_0^l W_q(x) W_q(x) dx \right| + 2\sum_{m=1}^{\infty} \left| \int_0^l \cos m\pi x W_q(x) W_q(x) dx \right| \right\} \le \left| \frac{1}{\mu_q} \right| \left\{ \left| \int_0^l W_q(x) W_q(x) dx \right| + 2\sum_{m=1}^{\infty} \left| \int_0^l \cos m\pi x \right| \left| W_q(x) \right| \left| W_q(x) dx \right| \right\}$$

$$= \left| \frac{1}{\mu_q} \right| \left\{ \left| \int_0^l W_q(x) W_q(x) dx \right| + 2\left| \int_0^l W_q(x) W_q(x) dx \right| \right\} \le \mu\, v_1^2 l (v_1^2 l + 2 v_1^2 l).$$

Similarly $\left|R_b\right| = \left|\frac{1}{\mu_q}\left(\Lambda_1(q,q)\right) + \frac{2l}{\alpha}\sum_{m=1}^{\infty} \cos m\pi\, \Lambda_2(q,q)\right| \le v_3 \ \pi \ \infty.$ Also consider $P_m = \frac{\Gamma_q}{2\mu_q}\left(2\mu_q - \Lambda_1(q,q)\right)$, and let d^*_j denote $\left(1 - \frac{\Lambda_1(q,q)}{2\mu_q}\right)$, so that $P_m = \Gamma_q d^*_j$

(3.2)

and $\left|d_j *\right| = \left|\left(1 - \frac{\Lambda_1(q,q)}{2\mu_q}\right)\right| \le \left(1 + \frac{\frac{1}{2} v_1^2 l}{\mu v_1^2 l} = 1 + \frac{1}{2\mu}\right) = v_4 \ \pi \ \infty$. At this juncture we make an assumption, which is consistent with practical conditions viz. The load $Q(x,t)$ is uniformly bounded that is there exist Q**:|Q(x,t)| ≤ Q** < ∞, ∀ $t \ge 0$. Furthermore, the eigenvalues ω_q^2 are known [1, 5] to be real and hence form a countable set, which does not possess a finite accumulation point, except possibly for a finite number of $\left|\omega_q\right|$. Generally, if they are ordered such that $\omega_1^2 \ \pi \ \omega_2^2 \ \pi \ \omega_3^2 \ \pi \ \Lambda \ \omega\Lambda \ \pi \ \omega_{q-1}^2 \ \pi \ w_q^2$, then $\lim_{q\to\infty}\left|\frac{\frac{1}{\omega_{q+1}}}{\frac{1}{\omega_q}}\right| = \lim_{q\to\infty}\left|\frac{\omega_q}{\omega_{q+1}}\right| = R \ \pi \ 1$. Thus the ratio test for convergence is satisfied by the series $\sum_{q=1}^{\infty}\frac{1}{\omega_q}$. Hence $\sum_{q=1}^{\infty}\left|\frac{1}{\omega_q}\right|$ is convergent. In view of this $\sum_{q=1}^{\infty}\left(\frac{1}{\omega_q}\right)^n$, $n \ge 1$ is also convergent. For convenience, we set $\sum_{q=1}^{\infty}\frac{1}{\omega_q} = v_5 \ \pi \ \infty$. Now carrying out an integration by parts of the right hand side of equation (2.6) we obtain

$$W(x,t) = \sum_{Q=1}^{\infty} H_{OR}\int_0^L \left[\frac{Q(v,t)}{P_q^2} - \frac{\cos p_m t}{P_q^2} Q(v,u)\right] W_q(x) W_q(r)\, dv - \sum_{q=1}^{\infty} H_{OR}\int_0^i\int_0^t \frac{w_q u \cos p_m (t-u)\,\partial Q(v,u)\, dv du}{P_q^2\,\partial u} \tag{3.3}$$

Let attention be focused on the first series of (3.3) so that we have

$$\left|H_{OR}\sum_{Q=1}^{\infty}\left\{\int_0^L\left[\frac{Q(v,t)}{P_q^2} - \frac{Q(v,0)}{P^2}\cos p_m t\right] W_q^X W_q^v\right\}\right| \le v_1^2 l H_{OR}\sum_{Q=1}^{\infty}\left\{\left|\int_0^L \frac{Q(v,t)}{P_q^2} dr\right| + \left|\int_0^L \frac{Q(v,0)\cos p_m t}{P_q^2} dv\right|\right\}$$

$$\le v_1^2 l H_{OR}\sum_{q=1}^{\infty}\frac{Q**}{\Gamma_q^2 v_4^2} + \frac{Q**}{\Gamma_q^2 v_4^2} \Rightarrow \frac{2v_1 l}{v_4^2} H_{OR} Q**\sum_{Q=1}^{\infty}\frac{1}{\Gamma_q^2} = \frac{2v_1 l}{v_4^2} H_{OR} Q**\sum_{Q=1}^{\infty}\frac{1}{\omega_q^2\left(1 + \varepsilon\,\mu^2 v_2\right) + HR_b}$$

$$\le \frac{2v_1 l}{v_4^2} H_{OR} Q**\sum_{Q=1}^{\infty}\frac{1}{\omega_q^2\left(1 + \varepsilon\,\mu^2 v_2\right) + Hv_3} = \frac{2v_1 l}{v_4^2 A_5} H_{OR} Q**\sum_{Q=1}^{\infty}\frac{1}{\omega_q^2}\frac{1}{\left(1 - \frac{A_6}{\omega_q^2}\right)} \tag{3.4}$$

where $A_5 = \left(1 + \frac{\varepsilon}{\lambda}\mu^2 v_2\right)$, $A_6 = \frac{Hv_3}{A_5}$. Hence equation (3.4) $\le \frac{2v_1 l}{O_4^2 A_5} H_{OR} Q** F * \sum_{q=1}^{\infty}\frac{1}{w_q^2}$ (3.5) where $F* = \dfrac{1}{\left(1-\dfrac{A_6}{\omega_1^2}\right)}$, $\dfrac{A_6}{\omega_1^2} \pi 1$, and equation (3.5) $\leq \dfrac{2v_1 l H_{OR} Q**F*}{v_4^2 A_5} v_5^2 = S_1 \pi \infty$ (3.6)

Now, let the second series of (3.3) be considered, setting $\int_0^\infty \left|\dfrac{\partial Q}{\partial u}(u,v)\right| du = v_6 \pi \infty$, we have

$$\left|\sum_{q=1}^{\infty} H_{OR} \int_0^l \int_0^l \frac{W_q^{(x)} W_q^{(v)} \cos P_m (t-u)}{P_q^2} \frac{\partial Q}{\partial u}(v,u) dvdu\right| \leq H_{OR} \sum_{Q=1}^{\infty} \int_0^L \int_0^L \left|\frac{W_q^{(x)} W_q^{(v)}}{P_q^2}\right| \left|\frac{\partial Q}{\partial u}\right| dvdu$$

$$\leq H_{OR} v_1^2 v_6 \sum_{q=1}^{\infty} \frac{1}{P_q^2} \leq H_{OR} v_1^2 v_6 v_2^4 \sum_{q=1}^{\infty} \frac{1}{\Gamma_q^2} \leq H_{OR} \frac{v_1^2 v_6 v_4^2}{A_5} \sum_{q=1}^{\infty} \frac{1}{\omega_q^2} \frac{1}{\left(1-\dfrac{A_6}{\omega_q^2}\right)} \qquad (3.7)$$

$$\leq H_{OR} v_1^2 v_6 v_4^2 F* \sum_{q=1}^{\infty} \frac{1}{\omega_q^2} \leq H_{OR} v_1^2 v_6 v_4^2 F* v_5^2 = S_2 \pi \infty$$

The inequalities (3.6) and (3.7) show that the series occurring in (2.6) or (3.3) are uniformly convergent thus establishing that $W(x, t)$ as given in (2.6) is the actual solution to the constrained column–mass problem.

4.0 Conclusion

In this paper we have discussed the dynamic problem of analysing the response of an elastic column carrying a concentrated mass at its top having an elastic constraint at a point along its length under the influence of dynamic load. It was also remarked that such a system is a modeled of an elevated water tower with elastic deflection control device, which is subjected to a dynamic load (such as a strong gust of wind). Finally we have shown that the corresponding series solution converges uniformly.

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