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Existence and uniqueness of solution for a system of equations of microwave heating of the biologic tissues

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Abstract

The existence and uniqueness of solution for a system of equations of microwave heating of biologic issue is discussed. Using the Green function approach we establish the existence and uniqueness of solution.

Keywords*:* Existence, uniqueness solution, microwave heating, biologic issues

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1.0 Introduction

Since the time when Bush in 1866 gave his observation of the disappearance of cancerous cell on the skin surface after a patient had high fever; there has been a lot of work on the investigation of the effect of heat deposition and the consequent rise in temperature in an attempt of using heat to destroy or control the growth of cancerous cells.

Heat deposition is reported in the literature as non-uniform in tissue location [7-10]. Possibility of hot spots had also been reported in tissue and thus a mathematical model to promote insight to the heat deposition pattern and consequent use of temperature is advanced by different authors [1, 2, 4, 5, and 11].

A number of researches have been carried out; some of these are enumerated below. Saxena and Arya [9] investigated the steady state temperature distribution in human skin and sub dermal tissue exposed to a dry and cool environment with negligible perspiration; Pal and Pal [8] also studied the steady state temperature distribution in human skin and subcutaneous tissue (SST). Their model accounts for heat conduction, perfusion of the capitulary bed and metabolic heat production of the dermis and subcutaneous tissues. Kritikos et al. [6] used the Pennes model but introduced the evapotranspiration from the surface into their model and thus predicted the steady state temperature rise in a homogenous tissue sphere exposed to plane wave electromagnetic energy. Recently Jiang et al. [12] discussed the effects of thermal properties and geometrical dimension on the skin burns. Lui and Marchant [13] considered the microwave heating of three-dimensional blocks with a transverse magnetic wave-guide mode in a long rectangular wave-guide. El-dabe et al. [4] studied the effects of microwave

heating on the thermal states of biological tissues. The purpose of the paper is to establish qualitatively that solution exist and give condition for uniqueness of solution.

This work is presented in four sections. We gave the literature review in section 1, the mathematical formulation of our problem is presented in Section 2 while the existence, and the uniqueness theorems are given in Section 3. In section 4 we give brief discussion and conclusion.

2.0 Mathematical formulation

In the works of Kastella and Fox [5], Wulff [11] some equations of temperature variation were given in living tissue. Adebile [1] worked on the generalized equation, which is given as:

$$\frac{\partial}{\partial t'}\left(C'_{p_t}, p'_t, T\right) = Div\left(K'p \; grad \; T\right) - Div\left(p'_b \; Cp'_b \; \rho T \; \underset{\sim}{q}\right) - \left(p'_b Cp'_b \; \phi' T\right) + \varphi' + Hm' + Q\left(x', T, t'\right) \qquad \textbf{(2.1)}$$

and in conjunction with the conservation of mass and momentum equation which are respectively:

$$\frac{\partial}{\partial t'}\left(p'_b \; A'\right) + Div\left(p'_b \; A' \; q'\right) = 0 \qquad \textbf{(2.2)}$$

and

$$\frac{\partial}{\partial t'}\left(\rho_b A' \; q'\right) + \underline{q'} \cdot \text{grad}\left(p'_b \; A' \; q'\right) = -A' \text{grad } P' + \text{Div}\left(\mu' \; A' \; \text{grad } \underline{q'}\right) + \rho' A' g' \qquad \textbf{(2.3)}$$

with the initial and boundary conditions

$$T\left(\underline{x'}, 0\right) = T_0\left(\underline{x'}\right); \quad \underline{q}\left(x', 0\right) = \underline{q}(x), \quad T\left(a', t'\right) = F\left(t'\right); \quad T\left(b', t\right) = G'(t)$$

$$0 < a', \; b' < \infty, \quad t' > 0$$

$$q\left(C' \ ,t' \ \right)=m_1' \ (t'), \ \ \left(d' \ ,t'\right)=m_2\left(t' \ \right) \qquad \textbf{(2.4)}$$

$$0<c' \ , \ d' \ <1, \ \ t'>0$$

using dimensionless variables as in (2.1) we have the hydrodynamic fluid (blood) equation as

$$\frac{\partial}{\partial \tau}(\rho_b A)+\frac{\partial}{\partial \eta}(\rho_b A q_\eta)=0 \qquad \textbf{(2.5)}$$

and

$$\frac{\partial}{\partial \tau}(\rho_b A q)+q_\eta \frac{\partial}{\partial \eta}\left(\rho_b A q_\eta\right)=-R_o A\frac{\partial \rho}{\partial z}+R_1\frac{\partial}{\partial \eta}\left(A\frac{\partial q_n}{\partial \eta}\right)+R_2 \rho A g \qquad (2.6)$$

with the non-dimensional energy equation as

$$\frac{\partial}{\partial \tau}\left(F_t G_t \theta\right)=\frac{\partial}{\partial \eta}\left(H\frac{\partial \theta}{\partial \eta}\right)-\rho_b \ c_{p_b}\rho_o Div\left(F_b G_b q_c \theta\right)$$

$$-\left(\rho_b \ c_{p_b} m\chi \rho_t\right)\theta+\varphi+Hm+Q \ (\eta,\theta,t) \qquad (2.7)$$

with the initial and boundary condition being

$$\theta\left(\eta,0\right)=\theta_o(\eta), \qquad q\left(\eta,0\right)=q_o(\ \eta\ ) \qquad (2.8)$$

$$\theta\left(a,\tau\right)=F(\tau), \quad \theta\left(b,\tau\right) \ = G(\tau), \quad 0<a, \quad b<\infty, \quad \tau>0 \qquad (2.9)$$

$$q\left(c,\tau\right)=m_1(\tau), \quad q\left(d,\tau\right)=m_2(\tau), \quad 0<c, \quad d<1, \quad \tau>0 \qquad (2.10)$$

we have some physically responsible assumptions.

3.0 Existence and uniqueness theorems

3.1 Preliminaries

The equation (2.5) – (2.10) is simply put in the system of equation below

$$\frac{\partial u}{\partial \tau}-\alpha_1\frac{\partial^2 u}{\partial \eta^2}=f(\eta,\tau,u,\theta), \ \ \eta\in\left(0,1\right), \ \ \tau>0 \qquad (3.1)$$

$$\frac{\partial \theta}{\partial Z}-\alpha_2\frac{\partial^2 \theta}{\partial \eta^2}=g(\eta,\tau,u,\theta), \ \ \eta\in(0,1), \ \ \tau>0 \qquad (3.1a)$$

$$u\left(\eta_1 0\right)=\hat{u}_o(\eta), \quad \theta\left(\eta_1 0\right)=\hat{\theta}_o(\eta) \qquad (3.2)$$

$$u\left(0,\tau\right)=u\left(1,\tau\right)=0, \quad \theta\left(0_1,\tau\right)=0\left(1,\tau\right) \qquad (3.3)$$

we state some conditions on the dependent variables before we proceed to the theorems and proofs.

(S.1) $\hat{u}_o(\eta)$ and $\hat{\theta}_o(\eta)$ are bounded for $\eta\in(0,1)$ and has at most a countable number of discontinuities

(S.2) *f* and *g* satisfy the uniform Lipshitz condition.

$$\left|\psi(\eta,\tau,u_1,\theta_1)-\psi\ (\eta,\tau,u_2,\theta_2)\right| \le k_1\left(\left|u_1-u_2\right|+\left|\theta_1-\theta_2\right|\right)$$

$(\eta,\tau)\in \overline{D}$, and $D=\left\{\left(\eta,\tau\right), \ \eta\in\left(0,1\right), \ 0<\tau\le A\right\}$ the solution of (3.1)-(3.3) is the dual (u,θ) which is defined and continuous in $\overline{D}$ and which at each point of *D* has continuous uniformly bounded partial derivatives satisfying the system of parabolic equations in (3.1) – (3.3).

3.2 Existence result

Theorem 3.1

Let $\hat{u}_o(\eta)$, $\hat{\theta}_o(\eta)$ *and f and g satisfy (S.1) and (S.2) respectively. Then there exists a solution of problem* (3.1) – (3.3).

Proof

We construct the sequence (u_k) and (θ_k) which satisfy

$\frac{\partial u_k}{\partial t}-\alpha_1\frac{\partial^2 u_k}{\partial\eta^2}=F(\eta,t,u_{k-1},\theta_{k-1}),k=1,2,\Lambda$ and $\frac{\partial\theta_k}{\partial t}-\alpha_2\frac{\partial^2\theta_k}{\partial\eta^2}=g(\eta,t,u_{k-1},\theta_{k-1}),k=1,2,\Lambda$,
$u_k(\eta,0)=\hat{u}_0(\eta)$,

$\theta_k(\eta,0)=\hat{\theta}_0(\eta)$, $u_k(0,t)=u_k(1,t)=0$, $\theta_k(0,t)=\theta_k(1,t)=0$, $\frac{\partial u_0}{\partial t}-\alpha_1\frac{\partial^2 u_0}{\partial\eta^2}=0$, $\frac{\partial u_0}{\partial t}-\alpha_2\frac{\partial^2 u_0}{\partial\eta^2}=0$,
$u_0(\eta,0)=\hat{u}_0(\eta)$, $\theta_0(\eta,0)=\hat{\theta}_0(\eta)$, $u_0(0,t)= \ =u_0(1,t)=0$, $\theta_0(0,t)=\theta_0(1,t)=0$ .Let $G(\eta,t;\xi)$ be the Green function of the heat equation, then $G(\eta,t;\xi)=2\sum_{n=1}^{\infty}e^{-\beta\lambda_n t}\sin n\pi\xi\sin n\pi\eta,\ t\ \phi\ 0,$
$\eta\in(0,1), G(\eta,t;\xi)=2\sum_{n=1}^{\infty}e^{-\beta\lambda_n(t-s)}\sin n\pi\xi\sin n\pi\eta, t>0$, $s\le t$, $\eta\in(0,1)$, where $\beta=\alpha_1,\alpha_2$ with
$\lambda_n=(n\pi)^2$,
$n\ge 1, u_k=u_0(\eta,t)+\int_0^t ds\int_0^1 G_{\beta=\alpha_1}(\eta,t;s;\xi)f(\eta,s;u_{k-1},\theta_{k-1})d\xi$
$\theta_k=\theta_0(\eta,t)+\int_0^t ds\int_0^1 G_{\beta=\alpha_2}(\eta,t;s;\xi)g(\eta,s'',u_{k-1}$,
$\theta_{k-1})d\xi$ where $u_0(\eta,t)=\int_0^1 G_{\beta=\alpha_1}(\eta,t;\xi)\hat{u}_0(\xi)d\xi$,
$\theta_0(\eta,t)=\int_0^1 G_{\beta=\alpha 2}(\eta,t;\xi)\hat{\theta}_0(\xi)d\xi\ \ G_{\beta=\alpha_1}(\eta,t;\xi)=2\sum_{n=1}^{\infty}e^{-\alpha_1\lambda_n t}$ -
$\sin n\pi\xi\sin n\pi\eta,\ t\ \phi\ 0,\ \eta\in(0,1),\ G_{\beta=\alpha_2}(\eta,t;\xi)=2\sum_{n=1}^{\infty}e^{-\alpha 2\lambda_n t}inn\pi\xi;\ \sin n\pi\eta,\ t\ \phi\ 0,\ \eta\in(0,1)$.

Let $Q_k(t)=\sup_{\substack{\eta\\ t\ge s}}|u_k(\eta,t)-u_{k-1}(\eta,t)|,\ k=1,2,\Lambda$ $p_k(t)=\sup_{\substack{\eta\\ t\ge s}}|\theta_k(\eta,t)-\theta_{k-1}(\eta,t)|,\ k=1,2,\Lambda$, and let
$\sup_{\substack{\eta\\ t\ge s}}(\hat{u}_0(\eta),\ f(\eta,t,0,0,))$, be $\frac{Q_0}{2}$ $\sup_{\substack{\eta\\ t\ge s}}(\hat{\theta}_0(\eta),\ f(\eta,t,0,0,))$, be $\frac{P_0}{2}$,
$|f(\eta,t,u_0,\theta_0)|<|f(\eta,t,0,0)|\ k_1(|u_0|+|\theta_0|)$,
by hypothesis (S.2) $|g(\eta,t,u_0,\theta_0)|<|g(\eta,t,0,0)|+k_1(|u_0|+|\theta_0|)$ .By hypothesis
(S.2) $|u_0(\eta,t)|\le\frac{Q_0}{2}, |Q_0(\eta,$
$t)|\le\frac{P_0}{2}, Q_1(t)=\sup_{\substack{\eta\\ t\ge s}}|u_1(\eta,t),-u_0(\eta,t)|=\sup_{\substack{\eta\\ t\ge s}}\left|\int_0^t ds\int_0^1 G_{\beta=\alpha_1}(\eta,t,s;\eta)f(\eta,s,u_0,\theta_0)d\xi\right|$
$\le\int_0^t ds\int_0^1|G_{\beta=\alpha_1}(\eta,t,s,\eta)|$
$|F(\eta,s,u_0,\theta_0)|d\xi\le\int_0^t ds\int_0^1|G_{\beta=\alpha_1}(\eta,t,s;\eta)|\{|F(\eta,s,u_0,\theta_0|+(K_1(|u_0||\theta_0|))\}d\xi$.

Similarly $(p_1)\le\int_0^t ds\int_0^1|G_{\beta=\alpha_2}(\eta,t,s;\xi)|\{|g(\eta,t,0,0)|+(K_1|\theta_0|)+(|\mu_0|)\}d\xi$, but
$G(\eta,t,s;\xi)=2\sum_{n=1}^{\infty}e^{-\beta\lambda_n(t-s)}$
$\sin n\pi\xi\sin n\pi\eta\le 2\sum_{n=1}^{\infty}e^{-\beta\lambda_n(t-s)}=2\left\{e^{-\beta\pi^2[t-s]}+e^{-4\beta\pi^2(t-s)}+e^{-9\beta\pi^2(t-s)}+\Lambda\right\}$
$\le 2\left\{e^{-\beta\pi^2[t-s]}+e^{-2\beta\pi^2(t-s)}+e^{-3\beta\pi^2(t-s)}+\Lambda\right\}$

$\leq 2\left[\frac{e^{-\beta\pi^2(t-s)}}{1-e^{-\beta\pi^2(t-s)}}\right]\leq 2$, then

$Q_1(t)\leq\int_0^t ds\int_0^1 \left|G_{\beta=\alpha_1}(\eta,t,s;\eta)\right|\left\{\left|f(\eta,t,0,0)\right|+K_1\left|\mu_0\right|+\left|\theta_0\right|\right\}d\xi\leq\int_0^t 2(Q_0+K_1(Q_0)+$
$P_0)ds=2\{Q_0t+K_1Q_ot+P_ot\}$ then $Q_1(t)\leq 2\{(1+K_1)Q_0+P_0\}t$. Similarly $P_1(t)\leq 2\{Q_0+(1+K_1)P_0\}t$.
Again

$Q_2(t)=\sup_{\substack{\eta\\ t\geq s}}\left|u_2(\eta,t)-u_1(\eta,t)\right|=\sup_{\substack{\eta\\ t\geq s}}\left|\int_0^t ds\int_0^1 G_{\beta=\alpha_1}(\eta,t,s;\xi)f(\eta,s,u_0,\theta_0)d\xi\right.$

$-\int_0^t ds\int_0^1 G_{\beta=\alpha_1}(\eta,t,s;\xi)f(\eta,s,u_0,\theta$

$\left._0)'d\xi\right|\leq\sup_{\substack{n\\ t\geq s}}\int_0^t ds\int_0^1\left|G_{\beta=\alpha_1}(\eta,t,s;\xi)\right|\left|f(\eta,s,u_1,\theta_1)-f(\eta,s,u_0,\theta_0)\right|d\xi$

$\leq\int_0^t ds\int_0^1\left|G_{\beta=\alpha_1}(\eta,t,s;\xi)\right|k_1\left(\left|u_1-u_0\right|+\left|\theta_1-\theta_0\right|\right)d\xi\leq 2\int_0^t ds\int_0^1 K_1\left(\left|u_1-u_0\right|+\left|\theta_1-\theta_0\right|\right)d\xi$. Let
$(1+K_1)Q_0+P_0=Q$, $Q_0+(1+K_1)P_0=P$, then

$Q_2(t)\leq 4\int_0^t K_1Q\,s\,ds=\frac{4k_1Qt^2}{2}$. Similarly, $Q_k(t)\leq\frac{2^kQK_1{}^{k-1}t^k}{k!}$, $k=1,2,\Lambda$ and similarly we obtain
$P_k(t)\leq$
$\frac{2^kPK_1{}^{k-1}t^k}{k!}$, $k=1,2,\Lambda$, so that $\lim_{k\to\infty}u_k(\eta,t)=u(\eta,t)$, $\lim_{k\to\infty}\theta_k(\eta,t)=\theta(\eta,t)$ and $u(\eta,t)=u_0(\eta,t)+\int_0^t ds\int_0^1 G_{\beta=\alpha_1}$
$(\eta,t,s;\xi)f(\eta,s,u,\theta)d\xi$, $\theta(\eta,t)=\theta_0(\eta,t)+\int_0^t ds\int_0^1 G_{\beta=\alpha_2}(\eta,t,s;\xi)g(\eta,s,u,\theta)d\xi$.

3.3 The uniqueness problem

We shall show that our problem has no more than one solution. We state a theorem and prove it.

Theorem 3.2

There exists at most one bounded solution of problem (3.1) – (3.3) which satisfies (S.1) and (S.2).

Proof

Let $(u,\theta),(u_1,\theta_1)$ be two bounded solutions then $u_1(\eta,t)-u_0(\eta,t)=\int_0^t ds\int_0^1 G_{B=\alpha_1}(\eta,t,s;\xi)$

$\{f(\eta,s,u_1;\theta)-f(\eta,s,u;\theta)\}d\xi$,
$\theta_1(\eta,t)-\theta(\eta,t)=\int_0^t ds\int_0^1 G_{B=\alpha_2}(\eta,t,s;\xi)\{g(\xi,s,u_1\theta_1)-g(\xi,s,u\ \theta)\}d\xi$.
Let $Q(t)=\sup_{\substack{\eta\\ t\geq s}}\left|u_1(\eta,s)-u(\eta,s)\right|$, $P(t)=\sup_{\substack{\eta\\ t\geq s}}\left|\theta_1(\eta,s)-\theta(\eta,s)\right|$. By hypothesis (S.2) we have
$Q(t)\leq k_1$
$\int_o^t\{Q(s)+P(s)\}ds$, $P(t)\leq k_1\int_o^t\{Q(s)+P(s)\}ds$. Thus
$Q(t)+P(t)\leq 2k_1\int_o^t\{Q(s)+P(s)\}ds\leq 2k_1\{Q$
$(s)+P(t)\}t$. Taking $t=\frac{1}{3k_1}$. Then $Q(t)+P(t)\leq\frac{2}{3}\{\{Q(t)+P(t)\}$. This is only possible when
$Q(t)=P(t)=0$ since the solutions are bounded. Then solution is unique, that is $\theta_1(\eta,t)=\theta(\eta,t)$ and
$u_1(\eta,t)=u(\eta,t)$

4.0 Result and discussion

Our result clearly gives us condition for the existence and uniqueness of a solution and if a solution does not exist the required rise in temperature that will promote a therapeutic effect on the tumour and cancer will not be achieved. This result also revealed that rise in temperature is a certainty consequent to microwave radiation. The uniqueness result gives a criterion for a non-multiple solutions.

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