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Modelling chaotic Hamiltonian systems as a Markov Chain

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Abstract

The behaviour of chaotic Hamiltonian system has been characterised qualitatively in recent times by its appearance on the Poincaré section and quantitatively by the Lyapunov exponent. Studying the dynamics of the two chaotic Hamiltonian systems: the Henon-Heiles system and non-linearly coupled oscillators as their trajectories intersect Poincaré section $q_1 = 0$, $p_1 > 0$, ***these intersections are random. To determine how random they are we shall model the intersections as a Markov chain and show that these intersections describe a closed ergodic Markov chain with a doubly stochastic matrix*** π_{ij}, $\sum_i \pi_{ij} = \sum_j \pi_{ij} = 1$. ***This is true for these systems with an error of*** $\pm 2\%$.

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1.0 Introduction

An important class of dynamical systems is the Hamiltonian dynamical systems. They can be described by a set of 2*n* first-order Hamilton-Jacobi equations. These equations are regarded as a vector field, which define a flow in phase space [3]. The vector field is given by the flow of Hamiltonian $H(q,p)$, which is the total energy of the system. Conservation of the energy in the Hamiltonian systems requires the value of $H(q,p)$ remain constant along the trajectory therefore $H(q,p) = E$. Under this condition one degree of freedom is lost and the trajectories are bound to a condition $H(q,p) = E$ or to a 2*n* - 1–dimensional surface on the phase space or to a surface, Σ of constant energy, known as the Poincaré section.

Akin-Ojo [1] considered a closed bounded system with Hamiltonian $H_n(q_1,...,q_n,p_1,...,p_n)$ of n degrees of freedom, $n \geq 2$, where $H_n = E$ is the only integral (or constant) of motion. The chaotic behaviour of this system can be modelled with a mapping of M of Σ into itself. The Σ can be celled into m cells with a probability distribution which predicts the randomness on Σ. Akin-Ojo established that the dynamics on Σ follows a Markov chain, with doubly stochastic matrix. Then as a totally chaotic system (no constant of motion at all) the chain is ergodic.

In this paper the two systems Hénon-Heiles system [4] and the nonlinearly coupled oscillators system [2], NLCO, are of two degrees of freedom i.e. n=2. These closed Hamiltonian $H(q_1,q_2,p_1,p_2)$ of two degrees of freedom and one constant of motion $H(q,p) = E$ on the Poincaré section $q_1 = a,\ p_1 > 0$ are non–integrable, so they exhibit chaos. These systems being nonlinear can only be analysed by numerical computations. Their behaviours are then exhibited on the Poincaré section P.S. Studying the dynamics of these systems as their trajectories intersect the P.S. these points of intersections are random. The complex appearance of the various intersections of the systems on the P.S. leads to the question of a relationship between statistics and chaos. We model the intersections in terms of a Markov chain (process) by celling Σ into two. Then we show that as a totally chaotic system they describe a closed ergodic chain with a doubly-stochastic matrix π_{ij}, $\sum_i \pi_{ij} = \sum_j \pi_{ij} = 1$.

2.0 Computational analysis

The chaotic Hamiltonian system of the form $H(q_1,q_3,p_1,p_2) = (p_1^2 + p_2^2)/2 + V(q_1,q_2)$ can be solved for using Hamilton's equations of motion,

$$\dot{q}_1 = \partial H/\partial p_1 \qquad \dot{p}_1 = -\partial H/\partial q_1$$
$$\dot{q}_2 = \partial H/\partial p_2 \qquad \dot{p}_2 = -\partial H/\partial q_2 \qquad (2.1)$$

Since H is nonlinear and non integrable with $H = E$ the only constant of motion, we cannot solve this problem analytically. Using Runge-Kutta's fourth order method to solve numerically, data points are generated as the trajectory intersects the Poincaré $q_1 = a,\ p_1 > 0$.

The result on the P.S. is subject to the initial condition. The degree of chaoticity of the system depends on the total energy.

(1) The Hénon-Heiles potential [4] with potential function $V(q_1,q_2) = (q_1^2 + q_2^2)/2 + q_1^2 q_2 - q_2^3/3$ and Hamiltonian $H = (p_1^2 + p_2^2)/2 + (q_1^2 + q_2^2)/2 + q_1^2 q_2 - q_2^3/3$ the dynamics of the system are given by

$$\dot{q}_1 = \partial H/\partial p_1 = p_1 \qquad \dot{p}_1 = -\partial H/\partial q_1 = -(q_1 + 2q_1 q_2)$$
$$\dot{q}_2 = \partial H/\partial p_2 = p_2 \qquad \dot{p}_2 = -\partial H/\partial q_2 = -(q_2 + q_1^2 - q_2^2)\ . \qquad (2.2)$$

This system is totally chaotic for $E = 1/6$ (Figure 1a). For this particular energy, the intersections on the P.S.are thereby modelled as Markov chain.

(2) The nonlinearly coupled oscillators [2] with potential function

$$V(q_1,q_2) = q_1^2/2 + 3q_2^2/2 + \alpha q_1^4/4 + \alpha q_2^4/4 + 3\alpha q_1^2 q_2^2$$

where $\alpha = 1$ and Hamiltonian $H = (p_1^2 + p_2^2)/2 + q_1^2/2 + 3q_2^2/2 + \alpha q_1^4/4 + \alpha q_2^4/4 + 3\alpha q_1^2 q_2^2$ where $\alpha = 1$. The dynamics of the system are given by

$$\dot{q}_1 = \partial H/\partial p_1 = p_1 \qquad \dot{p}_1 = -\partial H/\partial q_1 = -(q_1 + q_1^3 + 6q_1 q_2^2)$$
$$\dot{q}_2 = \partial H/\partial p_2 = p_2 \qquad \dot{p}_2 = -\partial H/\partial q_2 = -(3q_2 + q_2^3 + 6q_1^2 q_2) \qquad (2.3)$$

This system is totally chaotic for $E = 100$ (Figure 1b). For this particular energy, the intersections on the P.S. are thereby modelled as Markov chain.

3.0 Markov Chain

There are systems of finite number of states (classical or quantal), which can be modelled as a Markov chain or (process) [1, 6]. For these systems their trajectories are chaotic and the points of intersections are random. Let us take the point of intersection say, X, as a random variable. If the trajectory intersects the P.S. at point j at step r, then it will do so again at some region. Hence, $\sum_j P(r) = 1$. But one does not know where it will intersect the P.S at step $(r+1)$. We assume there is a probability that it does at region k. Then the probability that the system is in state k at step $(r+1)$ is

$$P_k(r+1) = \sum_{j=1}^{m} P_j(r)\pi_{jk} \qquad (3.1)$$

The Markovian property is that at any given time the probability of transition (or movement) from one state j to state k does not depend on how one arrived in one's present state. The Markovian chain is a process without memory of the past [5].

$$P_k(r+1) = \sum P_j(r)\pi_{jk} = \sum\sum P_i(r-1)\pi_{ij}\pi_{jk} \qquad (3.2)$$

$$P_k(r) = \sum_{i=1}^{m} P_i(0)\pi_{ik}^r\ , \qquad r = 0,1,2,\Lambda \qquad (3.3)$$

where $\{P_i(0)\}_{i=1}^m$ is the initial probability distribution $(r = 0)$, with $\pi_{ik} \geq 0$.

$$P_i(r) \geq o \text{ such that } \sum_i^m P_i(r) = 1,\ \sum_{k=1}^m \pi_{ik} = 1 \qquad (3.4)$$

If condition $\sum_{k=1}^m \pi_{ik} = 1$ holds, then π is a double stochastic matrix and is independent of time r.

4.0 Modelling the Hénon-Heiles system and NLCO system as a Markov Chain

The Poincaré sections of the Hénon-Heiles system and the NLCO system can be partitioned into two cells, because of the symmetry of the curves. These are shown in Figures 2(a) and 2(b) for 100, 200, 300, 400 points of intersections. Let us take 100 points of intersections as our initial distribution. Starting in state i at time step $r = 100$, then our initial distribution

$$\{P_i(100)\}_{i=1}^{2} = [P_1(100) \quad P_2(100)] \tag{4.1}$$

The probability that the system is in state k at time $(r+1) = 200$ is

$$P_k(200) = \sum_{i=1}^{2} P_i(100)\pi_{ik} \tag{4.2}$$

where $\pi_{ik} = \begin{bmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{bmatrix}$ is the transition matrix. In order to calculate π_{ik} this process is taken up to

$$P_k(300) = \sum_{i=1}^{2} P_i(100)\pi_{ik}^2 \tag{4.3}$$

$$P_k(400) = \sum_{i=1}^{2} P_i(100)\pi_{ik}^2 \tag{4.4}$$

Note that $\sum_{i=1}^{2} P_i(r) = 1$. Solving equations ((8)-(11), we calculate the transition matrix of the systems. The transition matrix for Hénon-Heiles system $E = 1/6$ is given by

$$\pi_{ik} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \tag{4.5}$$

This transition matrix describes closed ergodic Markov chain with a doubly stochastic matrix within an error of 2%. The transition matrix for the nonlinearly coupled oscillator system for $E = 100$ is

$$\pi_{ik} = \begin{bmatrix} 0.54 & 0.46 \\ 0.47 & 0.53 \end{bmatrix} \tag{4.6}$$

This transition matrix describes a closed ergodic Markov Chain with a double stochastic matrix within an error of 1.2%.

5.0 Discussion and conclusion

For the totally chaotic systems (ergodic states) we have determined the transition matrices for the Hénon-Heiles system and the nonlinearly coupled oscillator. These are doubly stochastic within an error of 2%. the systems as a Markov chain we were able to determine the transition matrices of these systems, which give a quantitative result of chaos instead of the usual Lyapunov exponent, while the portraits of the systems on the P.S. give a qualitative result of chaos. A more

accurate celling of the Poincaré section into four or more will be carried out for further work.

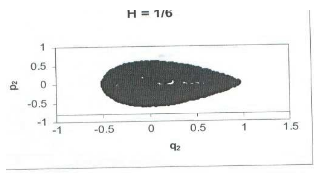

Figure 1(a): Hénon-Heiles system for E=1/6

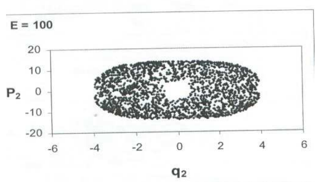

Figure 1(b): NLCO system for E=100

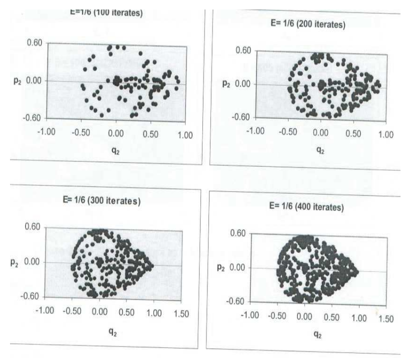

Figure 2(a): Hénon-Heiles system as a Markov chain

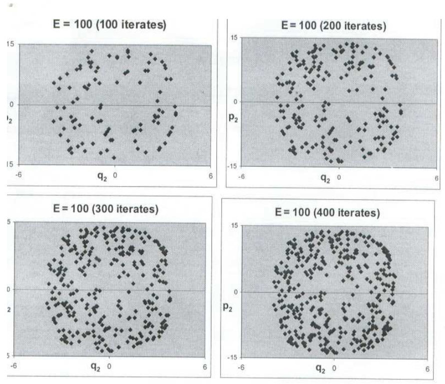

Figure 2(b): NLCO system as a Markov chain

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