Journal of the Nigerian Association of Mathematical Physics,
Volume 8 (*November 2004*)

Flexural motions of uniform beam under the actions of concentrated mass traveling with variable velocity

S. T. Oni
Department of Industrial Mathematics and Computer Science,
Federal University of Technology, Akure, Nigeria.

Abstract

This paper presents the dynamic analysis of the vibrations of a uniform beam under the action of a concentrated mass travelling with variable velocity. The solution technique discussed involves the expansion of Dirac delta function in cosine series form, a modification of the Stumble's asymptotic method and the use of the generating function of the Bessel function type. Analytical solution are obtained and the numerical results in plotted curves show that for the moving force and moving mass problems, the response amplitudes of the bean traversed by a load moving with variable velocity decrease with an increase in the foundation constant K. Similarly, the critical speed for the system traversed by a moving force is found to be smaller than that under the influence of moving mass showing that resonance is reached earlier in moving mass problem. Also, the displacement amplitude of the moving mass is greater than of the moving force. This further confirms the non-reliability of the moving force solution as safe approximation to the moving mass problem.

pp 215 – 224

1.0 Introduction

The problem of assessing the dynamic behaviour of structures carrying moving loads has been almost exclusively reversed in literature for moving loads moving at constant speeds. Among these is the work of Stanisic et al [1], Milormir et al [2], Sadiku and Leipholz [3], Oni [4, 5], Gbadeyan and Oni [6] to mention a few.

The more practical cases when velocities at which these loads move are no longer constants but vary with the time have received little attention in literature 97, 10]. This may be as a result of the complex space-time dependencies inherent in such problem. Specifically, even when the inertia effects of the moving load are neglected analytical solutions involving integral transforms are both intractable and cumbersome. However, such practical problems as acceleration and braking of automobile on roadways and highway bridges, taking off and landing of air-crafts on runway and braking and acceleration forces in the calculation of rails and railway bridges in which the motion is not uniform but a function of time have intensified the need for the study of the behaviour of structures under the action of loads moving with variable velocity. The class of problems was first tackled by Lowan [8] who solved the problem of the transverse oscillations of beams under the action of moving variable loads. Much later, Kokhmanyuk and Filippov [9] treated the dynamic effects on the transverse motion of a uniform beam of a load moving at variable speed. The work of Gbadeyan and Aiyesimi [10] is a recent development in this area of study. In particular, they undertook the analysis of the dynamic response of a finite beam continuously supported by a viscoelastic foundation to a moving load moving at variable speed. It was found that the period of the resonating vibration decreases with increasing value of lateral frequency of the load. However, in this work, the inertia effects of the moving load are assumed negligible and only the force solution is not an upper bound for the actual deflection of an elastic system.

Thus, this work is concerned with the flexural motions of a uniform beam under the actions of a concentrated mass travelling with a variable velocity. The main objective of this paper is to provide a closed form solution to this problem and to classify the effect of various parameters of the dynamical system on the response of the beam.

2.0 Formulation of the initial boundary valve problem

The undamped motion of a Bernoulli-Euler beam resting on an elastic foundation and under the action of a load moving with variable velocity is governed by the partial differential equation

$$EJ\,\frac{\partial^4 U(x,t)}{\partial x^4} - N_\sigma\,\frac{\partial^2 U(x,t)}{\partial x^2} + \mu_b\,\frac{\partial^2 U(x,t)}{\partial t^2} + KU(x,t) = Q(x,t) \qquad (2.1)$$

where *EJ* is flexural rigidity of the beam, N_a is the axial force, μ_b is the mass per unit length of the beam, K is the elastic foundation, *U*(*x*,*t*) is the transverse displacement, *x* and *t* are the spatial and time coordinates respectively, and *Q*(*x*,*t*) is the concentrated load moving with variable velocity. The structure under consideration is simply supported and carrying a concentrated mss *M*, which is moving at variable velocity. Consequently, the boundary conditions are

$$U(0,t) = 0 = L(L,t) = \frac{\partial^2 U(0,t)}{\partial x^2} = 0 = \frac{\partial^2 U(L,t)}{\partial x^2} \qquad (2.2)$$

It is assumed that the initial conditions of the motion are $$U(x,t) = 0 = \frac{\partial U(x,0)}{\partial x^2} \qquad (2.3)$$

If we consider not only the force effects of the concentrated moving load but its inertial effects as well and the motion of the contact point of the moving load is given by $$X_p = f(t) \qquad (2.4)$$

then according to d'Alemberts principle [7], the load is of the form

$$Q(x,t) = Mg\partial(x - f(f))\left[1 - \frac{1}{g}\frac{d^2 U(x_p,t)}{\partial t^2}\right] \qquad (2.5)$$

where the acceleration, $\frac{d^2 U(x_p,t)}{\partial t^2}$ of the mass is computed from the total differential of the second order of function *U*(*x*,*t*) with respect to *t*.

$$\frac{d^2 U(x_p,t)}{\partial t^2} = \frac{d^2 U(x,t)}{\partial t^2} + 2\,\frac{d^2 U(x,t)}{\partial x \partial t}\frac{\partial f(t)}{dt} + \frac{d^2 U(x,t)}{\partial x^2}\left(\frac{\partial f(t)}{dt}\right)^2 + \frac{dU(x,t)}{\partial x}\frac{\partial^2 f(t)}{dt^2} \qquad (2.6)$$

If we take $f(t)$ to be of the form $$f(t) = x_0 = \Lambda \sin \beta t \qquad (2.7)$$

where x_0 is the equilibrium position of the longitudinally oscillating load, Λ is the longitudinal amplitude of oscillation of the load and β is the longitudinal frequency of the load equation (2.1) by virtue of (2.4) to (2.6) after some simplifications and rearrangements, becomes

$$\begin{aligned}&EJ\,\frac{\partial^4 U(x,t)}{\partial x^4} - N_\sigma\,\frac{\partial^2 U(x,t)}{\partial x^2} + \mu_b\,\frac{\partial^2 U(x,t)}{\partial t^2} + KU(x,t)\\ &+ M\partial[x_0 + \Lambda \sin \beta t]\left\{\frac{\partial^2 U(x,t)}{\partial t^2} + 2\beta\Lambda \cos \beta t\,\frac{\partial^2 U(x,t)}{\partial t \partial x} + \beta^2 \Lambda^2 \beta t\,\frac{\partial^2 U(x,t)}{\partial x^2}\right.\\ &\left. + \Lambda \sin \beta t \frac{\partial^2 U(x,t)}{\partial x}\right\} = Mg\partial[x - (x_0 + -)\Lambda \sin \beta\, t]\end{aligned} \qquad (2.8)$$

3.0 Transformation of equation

Equation (2.8) is a fourth order partial differential equation, which in addition to being singular has variable coefficients. Firstly, by virtue of the boundary conditions the fourth order equation will be reduced to second order equation by applying the finite Fourier sine integral transform with respect to *x*. The integral transform is defined as

$$U(j,k) = \int_0^L U(x,t) \sin \frac{j\pi\, x}{L}\,dx \qquad (3.1)$$

with the inverse $$U(x,t) = \frac{2}{L}\sum_{j=1}^{\infty} U(x,t) \sin \frac{j\pi\, x}{L}\,dx \qquad (3.2)$$

Thus applying (3.1) to equation (2.8), one obtains

$$U_{tt}(j,t) + \gamma_j^2 U(j,t) + \frac{M}{\mu_b}\left[P_a(j,k,t) + P_a(j,k,t) + P_c(j,k,t) + P_d(j,k,t)\right] = \frac{P}{\mu_b}\sin \frac{j\pi}{L}(x_0 + \Lambda \sin \beta t)$$ where

$$\gamma_j^2 = \left[\frac{EI}{\mu_b}\left(\frac{j\pi}{L}\right)^4 + \frac{N}{\mu_b}\left(\frac{j\pi}{L}\right)^2 + \frac{K}{\mu_b}\right] \qquad (3.3)$$

$$P_a(j,k,t) = \int_0^L \delta[x - (x_0 + \Lambda \sin \beta t)]\frac{\partial^2 U(x,t)}{\delta t^2}\sin \frac{j\pi\, x}{L}\,dx \qquad (3.4)$$

$$P_a(j,k,t)=\int_0^L 2\beta\Lambda\delta[x-(x_0+\Lambda\sin\beta t)]\frac{\partial^2 U(x,t)}{\partial t\partial x}\sin\frac{j\pi x}{L}dx \tag{3.5}$$

$$P_c(j,k,t)=\int_0^L \beta^2\Lambda^2\cos^2\beta t\delta[x-(x_0+\Lambda\sin\beta t)]\frac{\partial^2 U(x,t)}{\delta x^2}\sin\frac{j\pi x}{L}dx \tag{3.6}$$

$$P_d(j,k,t)=-\int_0^L \beta^2\Lambda\sin\beta t\delta[x-(x_0+\Lambda\sin\beta t)]\frac{\partial U(x,t)}{\delta x}\sin\frac{j\pi x}{L}dx \tag{3.7}$$

Next, we evaluate the integrals in equations (3.4) to (3.7) To this end; use is made of argument similar to these in [1,5]. Thus, in equation (3.5), the property of the Dirac-delta function as an even function is used to express it in Fourier Cosine series given by

$$\delta\left[x-(x_0+\wedge\sin\beta t)\right]=\frac{1}{L}\left[1+2\sum_{n=1}^{\infty}\cos\frac{n\pi}{L}(x_0+\Lambda\sin\beta t)\frac{\cos n\pi}{L}\right] \tag{3.8}$$

Also, in view of equation (3.2), we have $$U_{tt}(x,t)=\frac{2}{L}\sum_{k=1}^{\infty}U_{tt}(k,t)\sin\frac{k\pi x}{L} \tag{3.9}$$

Substituting (3.8) and (3.9) into (3.4), one obtains

$$p_a(j,k,t)-\frac{2}{L^2}\sum_{n=1}^{\infty}\sum_{k-1}^{\infty}U_{tt}(k,t)\left[I_{tt}+2\cos\frac{n\pi}{L}(x_0+\Lambda\sin\beta t)I_b\right] \tag{3.10}$$

where $$I_a=\int_0^l\sin\frac{k\pi x}{L}\sin\frac{m\pi x}{L}dx \tag{3.11}$$

$$I_b=\int_b^l\cos\frac{n\pi x}{L}\sin\frac{k\pi x}{L}\sin\frac{j\pi x}{L}dx \tag{3.12}$$

Carrying not the integration in (3.11) and (3.12) and simplifying, the desired transform is obtained as:

$$p_a(j,k,t)\frac{1}{l}\left[U_{tt}(j,k)+\sum_{k=1}^{\infty}U_{tt}(k,t)\sin\frac{j\pi}{L}\left(x_0+\wedge\sin\beta t.\sin\frac{k\pi}{L}(x_0+\Lambda\sin\beta t)\right)\right] \tag{3.13}$$

Following similar argument as in the previous analysis, equation (3.5) becomes

$$p_b(j,kt)=\frac{4\wedge\beta k\pi}{L^3}\cos\beta t\sum_{n=1}^{\infty}U_t(j,k,t)\left[I_c+2\sum_{n=1}^{\infty}\cos\frac{n\pi}{L}(x_0+\beta\sin\beta t)I_d\right] \tag{3.14}$$

where $$I_c=\int_0^l\cos\frac{k\pi x}{L}\sin\frac{m\pi x}{L}dx \tag{3.15}$$

$$I_d=\int_0^l\cos\frac{n\pi x}{L}\cos\frac{k\pi x}{L}\sin\frac{j\pi x}{L}dx \tag{3.16}$$

Evaluating I_c and $I_{d,}$ equation (3.14) after some simplifications and rearrangements yield

$$P_b(j,k,t)=-\frac{8\Lambda\beta}{L^2}\cos\beta t\sum_{k=1}^{\infty}U(k,t)\left[S_a(j,k)+2\sum_{n=1}^{\infty}S_b(j,k,n)\cos\frac{n\pi}{L}(x_0\Lambda\sin\beta t)\right] \tag{3.17}$$

where $S_a(j,k)-\frac{jk}{k^2-j^2}$ and $S_b(j,k,n)=\frac{jk\left[k^2+n^2-j^2\right]}{\left[(n+k)^2-j^2\;\middle|(n-k)^2-j^2\right]}$ (3.18)

Equation (3.16), in the same vein leads to

$$p_c(j,k,t)=\frac{-2\beta^2\Lambda^2k^2\pi^2}{L^4}\cos^2\beta t\sum_{k=1}^{\infty}U(k,t)\left\{I_a+\sum_{n=1}^{\infty}\cos\frac{n\pi}{L}(x_0+\Lambda\sin\beta t)I_b\right\} \tag{3.19}$$

where I_a and I_b, are as defined in equation (3.11) and (3.12). Thus, it follows that

$$p_c(j,k,t)=\frac{(\beta\Lambda\pi)^2}{L}\cos^2\beta t\left[j^2U(j,t)+\sum_{k=1}^{\infty}k^2.\sin\frac{j\pi}{L}(x_0+\sin\beta t)\sin\frac{k\pi}{L}\right.$$
$$\left.(x_0+\Lambda\sin\beta t)U(k,t)\right] \tag{3.20}$$

The same argument leads to

$$p_d(j,k,t)=\frac{-2\Lambda\beta^2\pi k}{L^3}\sin\beta t\sum_{k=1}^{\infty}U(k,t)\left[I_c+\sum_{n=1}^{\infty}sos\frac{n\pi}{L}(x_0+\Lambda\sin\beta t)I_d\right] \tag{3.21}$$

where I_c and I_d are defined in (3.15) and (3.16). Consequently equation (3.21) leads to

$$p_d(j,k,t)=\frac{-4\Lambda\beta^2}{L^2}\sin\beta t\sum_{k=1}^{\infty}U(k,t)\left[S_a(j,k)+2\sum_{n=1}^{\infty}S_b(j,k,n)con\frac{n\pi}{L}(x_0+\Lambda\sin\beta t)\right] \quad (3.22)$$

where $S_a(j,k)$ and $S_b(j,k,n)$ are as defined in (3.18). Using equation (3.13), (3.17), (3.20) and (3.22), equation (3.3) can be simplified and rearranged in the form

$$U_{tt}(j,t)+\gamma_j^{\prime}U(j,t)+\varepsilon_0\Big\{U_{tt}(j,t)+\left(\frac{\beta\Lambda j\pi}{L^2}\right)^2\cos^2\beta tU(j,t)+\sum_{k=1}^{\infty}U_{tt}(k,t)\sin$$

$$\frac{n\pi}{L}(x_0+\Lambda\sin\beta t)\sin\frac{k\pi}{L}(x_0+\Lambda\sin)-U(k,t)\frac{8\Lambda\beta}{L}\big[\cos\beta tS_a(j,k)+$$

$$2\sum_{n=1}^{\infty}S_b(j,k,n)\cos\frac{j\pi}{L}(x_0+\Lambda\sin\beta t\ )\big]-U(k,t)\frac{\Lambda\beta}{L}\left[\frac{\beta\Lambda k^2\pi^2}{L}\cos^2\beta t\sin\frac{j\pi}{L}(x_0+\Lambda\sin\beta t)\right. \quad (3.23)$$

$$\left.\sin\frac{k\pi}{L}(x_0+\Lambda\sin\beta t)+4\beta S_a(j,k)\sin\beta t+\sum_{n=1}^{\infty}8\beta S_b(j,k,n)\sin\beta t\cos\frac{n\pi}{L}(x_0+\Lambda\delta m\beta t)\right]\Big\}$$

$$=\frac{p\sin j\pi}{\mu_b L}(x_0+\Lambda\delta m\beta t)$$

where $\varepsilon_0=\dfrac{M}{\mu_b L}$ (3.24)

Equation (3.23) represents the transformed equation of uniform elastic beam under a load moving with a variable velocity. Evidently, an exact closed form solution to this equation is impossible. Consequently, in what follows two cases of the coupled equation are considered.

(a) ***Moving Force***

If we neglect the inertia term, we the classical case of a moving force problem. Under this assumption $\varepsilon_0=0$ and equation (3.24) after some simplifications and rearrangement becomes.

$$U_{tt}(j,t)+\gamma_j^2U(j,k)=\frac{p}{\mu_b}\left[\sin F\cos(G\sin\beta t)+\cos F\sin(G\sin\beta t)\right] \quad (3.25)$$

where $F=\dfrac{j\pi x_0}{L}$ and $G=\dfrac{j\pi\wedge}{L}$ (3.26)

In order to solve this equation, the generating function of the Bessel function of order K given by $e^{\frac{G}{2}(t=1/t)}=\sum_{k=-00}^{\infty}t^kJ_k(G)$ where $J_k(z)=\sum_{m=0}^{\infty}(-1)^m\left(\dfrac{z}{2}\right)^{k+2m}\dfrac{1}{(k+m)!\,m!}$ is used to derive the following Bessel function relations

$$(i)\ \cos(G\sin\beta t)=J_0(G)+2\sum_{m=1}^{\infty}J_{2m}(G)\cos 2m\beta t \quad (3.27)$$

$$(ii)\ \sin(G\sin\beta t)=2\sum_{m=0}^{\infty}J_{2m+1}(G)\sin(2m+1)\beta t \quad (3.28)$$

Firstly, it is straight forward to show that the general solution of the homogenous part of (3.25) is given by;

$$U_c(j,t)=C_1\cos\gamma_jt+C_2\sin\gamma_jt \quad (3.30)$$

where C_1 and C_2 are constants. Thus a particular solution to equation (3.25) takes the form

$$U_p(j,t)=p_1(t)\cos\gamma_jt+p_2(t)\sin\lambda_jt \quad (3.31)$$

where $p_1(t)r$ and $p_2(t)$ are function to be determined. From equation (3.31), it is straight forward to show that $p_1(t)=\frac{-p}{\mu_b\gamma_j}\int \sin\gamma_j t[\sin F\cos(G\sin\beta t)+\cos F\sin(G\sin\beta t)]dt$ (3.32)

and $p_2(t)=\frac{p}{\mu_b\gamma_j}\int \cos\gamma_j t[\sin F\cos(G\sin\beta t)+\cos F\sin(G\sin\beta t]dt$ (3.33)

Using the Bessel relations in equation (3.27) and some trigonometric identities yields

$$p_1(t)=\frac{p}{\mu_b\gamma_j}\left\{\sin FJ_0(G)\frac{\cos b_0t}{b_0}+2\sin F\sum_{k=1}^{\infty}J_{2k}(G)\left[\frac{\cos b_1t}{2b_1}+\frac{\cos b_2t}{2b_2}\right]+2\cos F\sum_{k=0}^{\infty}J_{2kH}(G)\right\}$$
$$\left[\frac{\sin b_4t}{b_4}-\frac{\sin b_3t}{2b_3}\right] \quad (3.34)$$

and
$$P_2(t)=\frac{p}{\mu_b\gamma_j}\left\{(G)\frac{Sinb_0t}{b_0}+2\sin f\sum_{k=1}^{\infty}J(G)\left[\frac{sinb_1t}{2b_1}+\frac{sinb_2t}{b_2}\right]+2cosF\sum_{k=0}^{\infty}J_{2kH})G)\right.$$
$$\left.\left[\frac{cosb_4t}{2b_4}-\frac{cosb_3t}{2b_3}\right]\right\} \quad (3.35)$$

where $b_0=\gamma_1 b_1=\gamma_i+2k\beta b_1=\gamma_i-2k\beta, b_1=\gamma_i+(2k+1)\beta b_4=\gamma_i-(2k+1)\beta$ (3.36)

Using (3.34) and (3.35). The particular solution of the non-homogenous second order differential equation takes the form.

$$U_p(j,t)=\frac{p}{\mu_b\gamma_j}\sin\left\{FJ_0(G)\left[\cos\frac{(\gamma_j-b_0)}{b_0}+\sin F\sum_{k=1}^{\infty}J_{2k}(G)\cos\frac{(\gamma_j-b_4)}{b_1}\right.\right.$$
$$\left.\left.+\cos\frac{(\gamma_j-b_2)}{b_2}+\cos F\sum_{k=0}^{\infty}J_{2k+1}(G)\left(\cos\frac{(\gamma_j-b)_4}{b_4}-\sin\frac{\gamma_j-b_3}{b_3}\right)\right]\right\} \quad (3.37)$$

Consequently, $U(j,t)=U_c(j,t)+U_n(j,t)$ (3.38)

Applying the initial conditions (2.3) to (3.28), the constants are found to be

$$C_1=\frac{-p}{\mu_b\gamma_j}\left\{\sin\frac{FJ_0(G)}{b_0}+\sin F\sum_{k=1}^{\infty}J_{2k}(G)\left[\frac{1}{b_1}+\frac{1}{b_0}\right]\right. \quad (3.39)$$

and $$C_2=\frac{-p}{\mu_b\gamma_j^2}\left\{\cos F\sum_{k=1}^{\infty}J_{2k+1}(G)\left[\frac{\gamma_j-b_4}{b_4}-\frac{\gamma_j-b_3}{b_3}\right]\right\} \quad (3.40)$$

Substituting (3.39) and (3.40) into (3.38) and inverting after some simplifications and rearrangements yield

$$U(x,t)=\frac{2}{L}\sum_{j=1}^{\infty}\frac{p}{\mu_b\gamma_j^2}\left\{\frac{sinFJ_0(G)\gamma_j}{b_0}[cos(\gamma_j-b_0)-cos\gamma_jt]\right.$$
$$+\gamma_j\, sinF\sum_{k=1}^{\infty}J_{2k}(G)\left[\frac{cos(\gamma_j-b_1)-cos\gamma_jt}{b_1}+\frac{cos(\gamma_j-b_2)t\cos\gamma_jt}{b_0}\right] \quad (3.41)$$
$$+cosF\sum_{k=0}^{\infty}J_{2k+1}(G)\left[\frac{\gamma_j\sin(\gamma_j-b_4)t-(\gamma_j-b_4)sin\gamma_jt}{b_4}-\frac{[\gamma_j\sin(\gamma_j-b_3)t-(\gamma_j-b_3)sin\gamma_jt]}{b_3}\right]\sin j\pi x$$

Equation (3.41) is the displacement response of the beam due to the moving force.

(b) **Moving mass–entire equation**

If the moving load has mass commensurable with that of the elastic bam, the inertia effect of the moving mass is not negligible. Thus, $\varepsilon_0 = 0$ and we are required to solve the entire equation (3.23). This is termed the moving mass problem. Evidently, an exact closed form solution of this equation is not possible. Thus, we resort to the approximate analytical solution techniques which is a modification of the asymptotic method of Struble discussed extensively in [5]

First, equation (3.32) is rearranged to take the form

$$U_{tt}(t,j) - \frac{G_1(j,n,t)}{G_0(j,\varepsilon,t)}U_t(j,t) - \frac{G_2(j,n,t)}{G_0(j,\varepsilon,t)}U(j,t) + \sum_{\substack{k=1\\k\neq j}}^{\infty}\left\{\frac{Q_a(j,k,t)}{G_0(j,\varepsilon,t)}U_{tt}(t,j)\right.$$

$$\left.+\frac{Q_b(j,k,n,t)}{G_0(j,\varepsilon,t)}U_t(k,t) + \frac{Q_c(j,k,n,t)}{G_0(j,\varepsilon,t)}U(k,t\right\}) = \frac{p}{\mu_b G_0(j,\varepsilon,t)}\sin\frac{j\pi}{L}(x_0 + \Lambda\sin\beta t) \quad (3.42)$$

$$G_0(j,\varepsilon,t) = 1 + \varepsilon_0\left(2 - \cos\frac{2j\pi}{L}(x_0 + \Lambda\sin\beta t)\right)$$

where $$G_1(j,\varepsilon,t) = \frac{8\Lambda\beta}{L}\left[S_a(j,j)\cos\beta t + 2\sum_{n=1}^{\infty}S_a(j,j,n)\cos\frac{n\pi}{L}(x_0 + \Lambda\sin\beta t)\right] \quad (3.43)$$

$$G_2(j,\varepsilon,t) = \gamma_j^2 + L_1^2\cos^2\beta t - L_3\cos\beta t\sin\frac{j\pi}{L}(x_0 + \Lambda\sin\beta t)$$

$$+4\beta S_a(j,j)\sin\beta t + \sum_{n=1}^{\infty}8\beta S_b(j,j,n)\sin\beta t\cos\frac{n\pi}{L}(x_0 + \Lambda\sin\beta t \quad (3.44)$$

$$G_a(j,k,t) = \sin\frac{j\pi}{L}(x_0 + \Lambda\sin\beta t)\sin\frac{k\pi}{L}(x_0 + \Lambda\sin\beta t) \quad (3.45)$$

$$G_b(j,k) = \frac{8\Lambda\beta}{L}\left[S_a(j,k)\cos\beta t + 2\sum_{n=1}^{\infty}S_a(j,k,n)\cos\frac{n\pi}{L}(x_0 + \Lambda\sin\beta t)\right] \quad (3.46)$$

$$Q_c(j,k,\Lambda,t) = \frac{\beta\Lambda}{L}\left[L_2\cos^2\beta t\frac{\sin j\pi}{L}(x_0 + \Lambda\sin\beta t)\frac{\sin k\pi}{L}(x_0 + \Lambda\sin\beta t)\right)$$

$$+4\beta S_a(j,k)\sin\beta t + \sum_{n=1}^{\infty}8\beta s_a(j,k,n)\sin\beta t\frac{\cos n\pi}{L}(x_0 + \Lambda\sin\beta t)] \quad (3.47)$$

where $$L_1 = \frac{B\Lambda\, j\pi}{L} \quad (3.48)$$

$$L_2 = B\Lambda\frac{k^2\pi^2}{L} \quad (3.49)$$

$$L_3 = B\Lambda\frac{j^2\pi^2}{L} \quad (3.50)$$

By means of these techniques, one seeks the modified frequency corresponding to the frequency of the free system due to the presence of the moving mass. An equivalent free system operator defined by the modified frequency then replaces equation (3.4). Thus, we set the right hand side of (3.4) to zero and consider a parameter $\varepsilon_1 < 1$ for any arbitrary mass ratio ε_0 defined as: $$\varepsilon_1 = \frac{\varepsilon_0}{1+\varepsilon_0} \quad (3.51)$$

It follows that $$\varepsilon_0 = \varepsilon_1 + o(\varepsilon^2) \quad (3.52)$$

and

$$\frac{1}{1+\varepsilon_0\left[2-cos\frac{2j\pi}{L}\left(x_0+\Lambda sin\beta t\right)\right]}=1-\varepsilon_1\left(2-cos\frac{2j\pi}{L}(x_0+\Lambda sin\beta t)+0(\varepsilon^2)\right) \qquad (3.53)$$

$$\left|\varepsilon_1\left[2-cos\frac{2j\pi}{L}(x_0+\Lambda sin\beta t)\right]\right| \pi 1 \qquad (3.54)$$

When ε_1 is set to zero in equation (3.42), a situation corresponding to the case in which the inertial effect of the mass of the system is regarded as negligible is obtained. In such a lase the solution is of the form

$$U_{sm}(j,t) = C_0 cos(\gamma_j t - \phi_j) \qquad (3.55)$$

where $C_0\gamma_l$ and ϕ_j are constants. Furthermore, as $\varepsilon_1 < 1$ Strubles technique requires that the solution of equation (3.42) be of the form $U(j,k)=\theta(j,t)cos\left[\gamma_j-\psi(j,t)\right]+\varepsilon_1\mu_1(j,t)+0\left(\varepsilon_1^{\ 2}\right)$ (3.56)

Where $\theta(j,t)$ and $\psi(j,t)$ are slowly time varying functions.

In order to obtain the modified frequency, equation (3.6) is substituted into the homogenous part of equation (.3.42). Thereafter, we extract only the variational part of the4 equation describing the behaviour of $\theta(j,t)$ and $\psi(j,t)$ during the motion of the mass. Thus, making this substitution and neglecting terms that do not contribute to variational equations yield.

$$\begin{aligned}&-2\gamma_j\dot{\theta}(j,t)sin\left[\gamma_j-\psi(j,t)\right]+2\theta(j.t)\dot{\psi}(j,t)\gamma_j\,cos\left[\gamma_j t-\psi(j,t)\right]\\&-2\gamma_j^{\ 2}\varepsilon_1\dot{\theta}(j,t)cos\left[\gamma_j t-\psi(j,t)\right]+\gamma_j^{\ 2}\varepsilon_1\theta(j,t)cos\,F_1J_0(G_1)cos\left[\gamma_j t-\psi(j,t)\right]\\&+2\varepsilon_1\theta(j,t)\frac{\beta\Lambda j\pi}{L}cos\left[\gamma_j t-\psi(j,t)\right]+\varepsilon_1\theta(j,t)\frac{\beta\Lambda j\pi}{L}cos\,F_1J_0(G_1)cos\left[\gamma_j t-\psi(j,t)\right]=0)\end{aligned} \qquad (3.57)$$

where $F_1=\frac{n\pi x_0}{L}$ and $G_1=\frac{n\pi\Lambda}{L}$. The variational equations of our problem are obtained by setting coefficients of $sin\left[\gamma_j t-\psi(j,t)\right]$ and $cos\left[\gamma_j t-\psi(j,t)\right]$ in equation (3.57) zero respectively. Thus, we have

$$-2\gamma_j\dot{\theta}(j,t)=0 \qquad (3.58)$$

and

$$\begin{aligned}&2\theta(j,t)\psi(j,t)\gamma_j-2\gamma_j^{\ 2}\varepsilon_1\theta(j,t)+\gamma_j^{\ 2}\varepsilon_1\theta(j,t)cos\;F_1J_0(G_1)\\&+2\varepsilon_1\theta(j,t)\frac{\beta\Lambda j\pi}{L}+\varepsilon_1\frac{\beta\Lambda j\pi}{L}\theta(j,t)cos\;F_1J_0(G_1)=0\end{aligned} \qquad (3.59)$$

Equation (3.58) and (3.59) respectively imply

$$\theta(j,t)=0 \qquad (3.60)$$

and

$$\psi(j,t)=\frac{\gamma_j\varepsilon_1}{2}\left[2-cos\,F_1J_0(G_1)-\frac{2\beta\Lambda j\pi}{L\gamma_j^{\ 2}}-\frac{\beta\Lambda j\pi}{L\gamma_1^{\ 2}}cos\,F_1J_0(G_1)\right] \qquad (3.61)$$

Thus, solving equation (3.60) and (3.61) one obtains

$$\theta(j,t)=C_j^{\ 0} \qquad (3.62)$$

where c_j^o is a constant and

$$\psi(j,t)=\frac{\gamma_j\varepsilon_1}{2}\left[2-cos\,F_1J_0(G_1)-\frac{2\beta\Lambda j\pi}{L\gamma_j^{\ 2}}-\frac{\beta\Lambda j\pi}{L\gamma_j^{\ 2}}cos\,F_1J_0(G_1)\right]t+\phi_j \qquad (3.63)$$

where ϕ_j is a constant and therefore, when the effect of the mass of the particle is considered, the first approximation to the homogenous system is $U(j,t)=C_0\,cos(\gamma_{jm}t-\phi_j)$ (3.64) where $\gamma_{jm} = \gamma_j \left\{ 1 - \frac{\varepsilon_1}{2} \left[2 - \cos F_1 J_0(G) - \frac{\beta \Lambda j \pi}{L \gamma_j^2} (2 + \cos F_1 J_0(G_1)) \right] \right\}$ (3.65)

is called the modified natural frequency representing the frequency of the free system due to the presence of the moving mass. Thus, the homogenous part of (3.45) can be written as:

$$\frac{d^2 U(j,t)}{dt^2} + \gamma_{jm}^2 U(j,t) = 0 \qquad (3.66)$$

and equation (3.45) then takes the form

$$\frac{d^2 U(j,t)}{dt^2} + \gamma_{jm}^2 U(j,t) = \varepsilon_1 g L \frac{\sin j\pi}{L} (x_0 + \Lambda \sin \beta t) \qquad (3.67)$$

Evidently, this equation is analogues to equation (3.35). Applying the initial yield conditions and inverting yield;

$$U(x,t) = \frac{2}{L} \sum_{j=1}^{\infty} \frac{\varepsilon_1 g L}{\gamma_{jm}} \left\{ \frac{\sin F J_0(G)}{b_0} \gamma_{jm} \left[\cos(\gamma_{jm} - b_0) - \cos \gamma_{jm} t \right] \right.$$
$$+ \gamma_{jm} t \sin F \sum_{k=1}^{\infty} J_{2k}(G) \left[\frac{\cos(\gamma_{jm} - b_1) - \cos \gamma_{jm} t}{b_1} \right] + \left[\frac{\cos(\gamma_{jm} - b_2) t - \cos \gamma_{jm} t}{b_2} \right] + \cos F \sum_{k=0}^{\infty} J_{2k+1} \qquad (3.68)$$
$$\left. \left| \gamma_{jm} \sin(\gamma_{jm} - b_4) t - (\gamma_{jm} - b_4) \sin \gamma_{jm} t \left[\frac{\gamma_{jm} \sin(\gamma_{jm} - b_3) - (\gamma_{jm} - b_3) t \sin \gamma_{jm} t}{b_3} \right] \right\} \sin \frac{j\pi x}{L}$$

4.0 Analysis of closed form solutions

The transverse displacement of an elastic beam may increase without bound. Thus one is interested in the resonance conditions. Equation (3.41) clearly depicts that the elastic beam resting on and elastic foundation and by a moving force will grow without bound whenever.

$$\gamma_j = 2k\beta \qquad and \qquad \gamma_j = (2k+1)\beta \qquad (4.1)$$

While equation (3.68) shows that the same beam under the action of a moving mass experiences resonance when

$$\gamma_{jm} = 2k\beta \text{ and } \gamma_{jm} = (2k+1)\beta \qquad (4.2)$$

But

$$\gamma_{jm} = \gamma_j \left\{ 1 - \frac{\varepsilon_1}{2} \left[\left(2 - \cos F_1 J_0(G) - \frac{\beta \Lambda j \pi}{L \gamma_j^2} \right) [2 + \cos F_1 J_0(G_1)] \right] \right. \qquad (4.3)$$

which implies $\gamma_j = \dfrac{2k\beta}{1 - \frac{\varepsilon_1}{2} \left[\left(2 - \cos F_1 J_0(G) - \frac{\beta \Lambda j \pi}{L \gamma_j^2} \right) [2 + \cos F_1 J_0(G] \right]}$

Therefore, it is evident from (4.2) and (4.4) that for same natural frequency, the critical speed for the system consisting of an elastic beam resting on an elastic foundation and traversed by moving force with variable speed is smaller than that of moving mass problem. Hence resonance is reached earlier in the former

5.0 Remarks on numerical results

In order to illustrate the foregoing analysis, the uniform beam of length 12.20m considered. Furthermore, $\frac{EJ}{\mu_b} = 2200\, m^4 / s^2, \Lambda = 2x10^{-4} m, \beta = \frac{3}{4}\pi, x_0 = \frac{1}{20}$ and the ratio of the mass of the load to the beam is 0.25. The transverse deflections of the beam are calculated and plotted against time for various values of foundation constants (moduli) and axial force. Values of K between 0 N/m^3 and 400,000 N/m^3 were used while the values of N were varied between N=0 and N = 2,000,000.

Figure 5.1 displays the deflection profile of the simply supported beam under the action of forces moving at variable speeds for various values of foundation moduli K for N = 200,000. The figure shows that as K increases the deflection of the uniform beam decreases. The same results obtain when the simply supported beam is traversed by a concentrated mass moving at variable speed as shown in Figure 5.4. Also for various time t, the displacement of the beam for fixed K for various values of N are shown in Figure 5.2. It is shown that higher values of axial force reduce the displacement amplitudes of the beam. The same behavior characterizes the deflection profile of the simply supported beam under concentrated masses moving at variable speed for various values of axial force N for fixed foundation moduli as shown in figure 5.5.

Finally, Figure 5.3 depicts the comparison of the traversed displacement of moving force and moving mass cases for simply supported beam traversed by a load moving at variable speed for N = 20,000 and K = 4000. Clearly, the response amplitudes of moving mass is higher than that of the moving force. This important result has also been reported in [1, 2, and 5] for cases when the traveling load is moving at constant speed.

6.0 **Conclusion**

A closed form solution is presented for the displacement response of a uniform beam under the actions of a concentrated mass moving with variable velocity. The solution technique, is based on integral transformation, the expansion of the Dirac Delta function in Series form, a modification of Struble's asymptotic method and the use of the generating function of the Bessel function. Numerical analysis is also carried out and the results show the following:

a) For the moving force and moving mass problems the response amplitudes of the beam traversed by a load moving with variable speed decrease with an increase in the foundation constant K.

b) The critical speed for the system traversed by moving force is smaller than that under the influence of moving mass.

c) Higher values of axial force N reduce the response amplitudes of both moving force and moving mass models.

d) The problem of a uniform beam under the actions of a load moving with variable velocity, the responses amplitude of the moving force problem is smaller than that of the moving mass. This shows that resonance is reached earlier in moving mass problem.

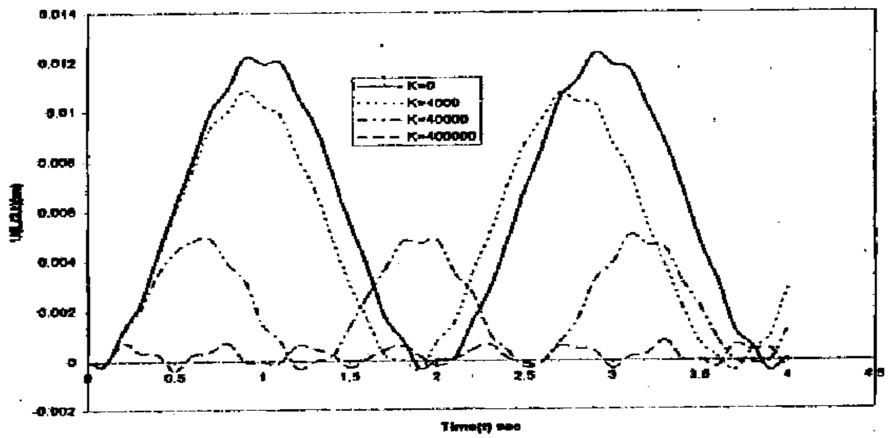

Fig 5.1 Deflection profile of the simply supported beam under the action of forces moving at variable speeds for various values of foundation moduli K for fixed N (20000)

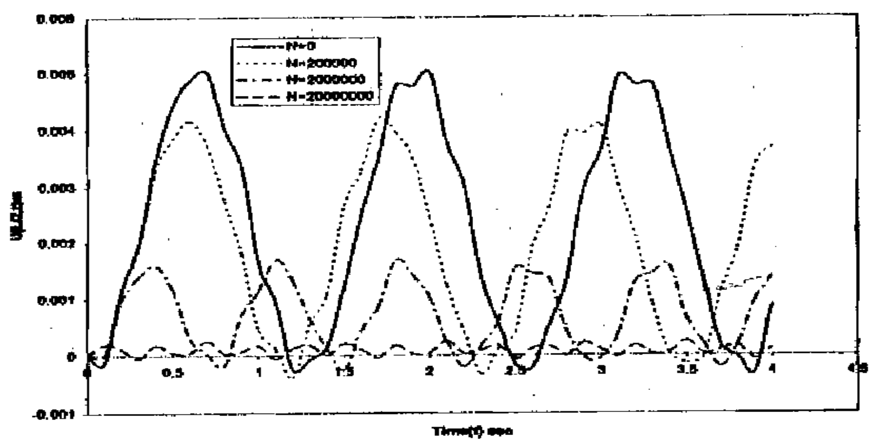

Fig 5.2 Transverse displacement of the simply supported beam under forces moving at variable speeds for various values of axial force for fixed foundation moduli K(40000)

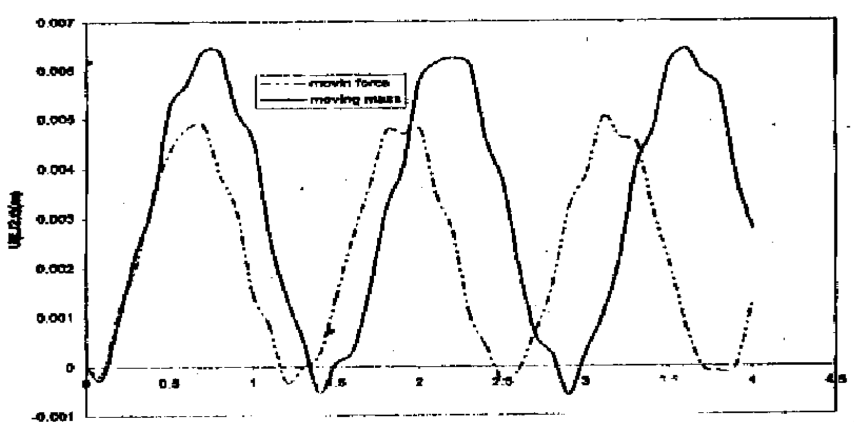

Fig 5.3 Comparison of the displacement response of Moving force and Moving mass cases for simply supported beam for N=20000 and K=40000

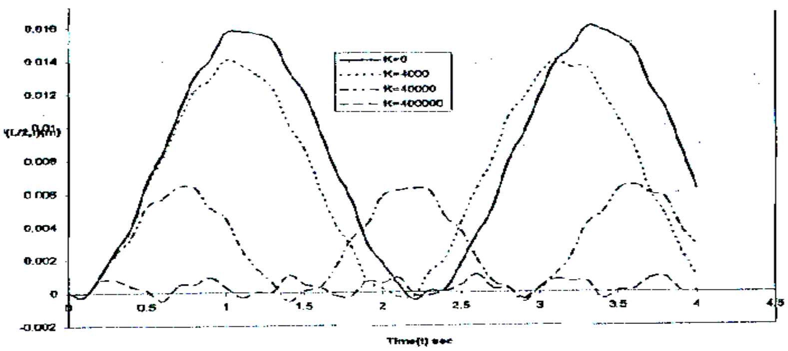

Fig 5.4 Deflection profile of the simply supported beam under the action of concentrated masses moving at variable speed for various values of foundation moduli K for fixed axial force N (20000)

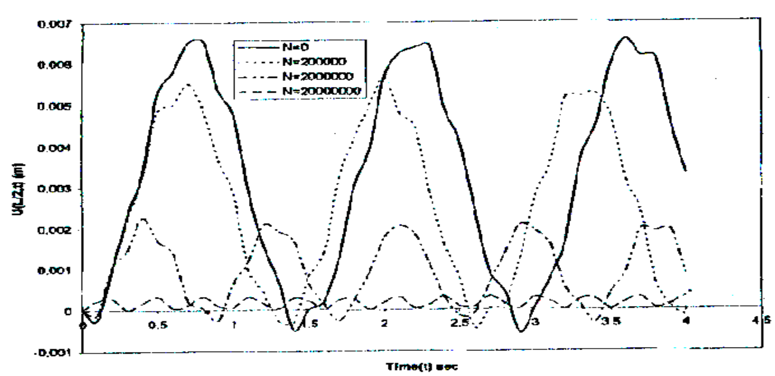

Fig 5.5 Transverse displacement of the simply supported beam under concentrated masses moving at variable speeds for various values of axial force N for fixed foundation moduli K (40000)

References

[1] Statistic, M.M, Euler, J.A; and Montgomery, S.T: On a theory concerning dynamical behaviors of structures carrying moving masses. Ing. Archiv, 43. pp 295-305, 1974.

[2] Milormir et al : On the response of beam to an arbitrary number of concentrated moving masses. Journal of the frankline, Institute, Vol. 287, No 2, February 1960.

[3] Sadiku, S and Leipholz, H.H.E: On the dynamics of elastic systems with moving concentrated masses Ing. Archive. 223-242 1981.

[4] Oni S.T.: On the dynamic response of elastic structures to moving multi mass system. PhD thesis. University of Ilorin 1991.

[5] Oni S.T.: Flexural vibrations under moving load of isotropic rectangular plates on a non-Winkler elastic foundation. Journal of the Nigerian Society of Engineers Vol. 35, No 1 Pp18-27, 40-41, 2000.

[6] Gbadeyan, J.A. and Oni S.T.: Dynamic behaviors of beams rectangular plates under moving loads Journal of sound and vibration 182(5), 677-6995.

[7] Fryba, L., Vibration of Solids and Structures under moving loads. Groningen: Noordhoff 1972.

[8] Lowan, A.N : On transverse oscillations of beam under the of moving variable loads; phil. Mag Ser. 7, Vol. 19,No 127, Pp708-715 1935.

[9] Kokhmanyuk S.S, Filippov A.P: Dynamic effects on a beam of a load moving at variable speed. Stroitel' n mekhanka i raschet so-oruzhenii, 9,No 2, 36-39, 1967.

[10] Gbadeyan, J. A and Aiyesimi. Y.M: Response of an elastic beam resting on viscoelastic foundation to a load moving at non-uniform speed. Nigerian Journal of Mathematics and Applications Vol. 3, Pp73-90, 1990.