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Deformation of an elastic crust

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Abstract

The crust is modelled as an elastic body, which is subjected to a vertical load. The resulting deformation is analyzed. Finally, an attempt is made to use the result to determine the time scale for isostatic adjustments for Hudson's Bay ice sheet load.

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1.0 Introduction

One of the most interesting aspects of geophysics, and also unfortunately one of the least susceptible to theoretical description has to do with the deformation of the earth's crust and upper mantle through geologic time. The surficial features of these deformations and related processes are readily apparent in continents and ocean basins, island arcs and deep sea trenches, mountain ranges, volcanoes, mid-ocean ridges and the like. Field geological and geophysical information adds a great deal more as to the structure, composition and geologic history of these features.

Unfortunately, however, these deformations are not particularly susceptible to explanation by the present methods of theoretical physics. The earth beneath the crust exhibits a complex response to impressed stresses. For, example, the theory based on perfect elasticity adequately predicts observed effects. In postglacial rise of areas, the effect is continuous long after the removal of the ice. The explanation is that the material of the mantle reacts in different ways, depending upon the time duration of the applied stress and also upon the magnitude of the stress [1].

The purpose of this paper is to investigate the deformation produced in the earth's crust under a vertical load. In this paper, the crust is considered as an elastic body. The result is applied to a crust, which has a size of the Hudson's bay ice sheet.

2.0 Mathematical formulation

An incompressible half-space ($K = \infty$) of uniform density ρ and rigidity μ is subjected to a pressure $p(exp\, ikx)$ at the top surface. We want to investigate the resulting deformation. We shall designate the displacement components in the *X, Y,* and *Z* directions by *u, v, w* and we shall take the *Z* axis as vertical downward. The problem will be restricted to a two-dimensional case, so that $v = 0$, and $\frac{\partial v}{\partial y} = 0$.

3.0 Theoretical analysis

If *E* denotes Eulerian and *L* Lagrangian, then the equation of static equilibrium within an elastic plate subject to a uniform, vertical externally applied gravitational field and pre-stressed in a hydrostatic state can be expressed compactly as

$$P_1^E = P_1^L - \rho_0 g_0 w \tag{3.1}$$

where P_1 is the perturbation pressure, ρ_0 is the density, g_0 is gravity and *w* is displacement.

Since the analysis is restricted to an incompressible plate, the momentum balance equation is Eulerian, that is

$$-\nabla P_1^E + \mu \nabla^2 S = 0 \tag{3.2}$$

where ∇ is the vector differential operator or del, V^2 is the Laplacian and S is the displacement vector, that is; $S = (u, 0, w)$ .and μ is the rigidity and it is the Lame constant. The boundary conditions on $z = 0$ are

(a) when there is no shear stress; $$\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}=0, \quad (3.3)$$

and

(b) the Lagrangian normal stress $$P_1^L-\frac{2\mu\,\partial w}{\partial z}=Pe^{ikx}, \quad (3.4)$$

Since the elastic plate is incompressible, its constitutive equations are [2].

$$\sigma_{xx}=-P+\frac{2\mu\,\partial u}{\partial x},\quad \sigma_{xx}=-P+\frac{2\mu\,\partial w}{\partial z},\quad \sigma_{xz}=\mu\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right) \quad (3.5)$$

Here u and w denote displacement components in the x and z directions, respectively. P is the elastic perturbation pressure. For an incompressible elastic solid, it is defined by [3].

$$P=\lim_{\substack{\lambda\to\infty\\ \Delta\to 0}}(\lambda\,\Delta) \quad (3.6)$$

where $\Delta=\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}$ is the dilation [3], λ is Lame constant. Substituting (3.6) into (3.5), the constitutive equations become $\sigma_{xx}=(\lambda+2\mu)\frac{\partial u}{\partial x}+\lambda\frac{\partial w}{\partial x}$, $\sigma_{xz}=\sigma_{zx}=\mu\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)$, $\sigma_{zz}=\lambda\frac{\partial u}{\partial x}+(\lambda+2\mu)\frac{\partial w}{\partial z}$ (3.7)

Following [4] we introduce a displacement potential X such that;

$$\sigma_{xx}=\frac{\partial^2 X}{\partial z^2},\quad \sigma_{xz}=\sigma_{zx}=-\frac{\partial^2 X}{\partial x^2\partial z^2},\quad \sigma_{zz}=\frac{\partial^2 X}{\partial x^2} \quad (3.8)$$

From the first and third of (3.7) we have solving for $\frac{\partial u}{\partial x}$ and $\frac{\partial w}{\partial z}$ the following relations

$$\frac{\partial u}{\partial x}=\frac{1}{4\mu(\lambda+\mu)}\{(\lambda+2\mu)\sigma_{xx}-\lambda\sigma_{zz}\} \text{ and } \frac{\partial w}{\partial x}=\frac{1}{4\mu(\lambda+\mu)}\{(\lambda+2\mu)\sigma_{zz}-\lambda\sigma_{xx}\} \quad (3.9)$$

We derive from (3.8) and (3.7) using (3.9) the following results:

$$-\frac{\partial^2 X}{\partial x^2\partial z^2}=\frac{\partial^2\sigma_{xz}}{\partial x\partial z}=\mu\left(\frac{\partial^3 u}{\partial x\partial z^2}+\frac{\partial^3 w}{\partial x^2\partial z}\right)=\frac{1}{4(\lambda+\mu)}\left\{\frac{\partial^2\left[(\lambda+2\mu)\partial^2 X-\lambda\,\partial^2 X\right]}{\partial z^2\partial x^2}\right.$$
$$\left.+\frac{\partial^2\left[(\lambda+2\mu)\frac{\partial^2 X}{\partial x^2}-\frac{\lambda\,\partial^2 X}{\partial z^2}\right]}{\partial x^2}\right\}=\frac{1}{4(\lambda+\mu)}\left[(\lambda+2\mu)\frac{\partial^4 X}{\partial x^4}-\frac{2\lambda\,\partial^4 X}{\partial x^2\partial z^2}+(\lambda+2\mu)\frac{\partial^4 X}{\partial z^2}\right] \quad (3.10)$$

Let $$S=\left(\frac{\partial X}{\partial z},0,-\frac{\partial X}{\partial x}\right) \quad (3.11)$$

then the dot product $\nabla\cdot S=0$ satisfies the condition of incompressibility. Therefore the left side of (3.10) reduces to zero. In many problems in elasticity, it is convenient to assume that $\lambda=\mu$. This is the Poisson or Cauchy relation. In our case here we shall take $\lambda=-1$, and $\mu=1$ so as to obtain an analytic solution for the deformation. Hence (3.10) becomes

$$\frac{\partial^4 X}{\partial x^2}+2\frac{\partial^4 X}{\partial x^2\partial z^2}+\frac{\partial^4 X}{\partial z^2} \quad (3.12)$$

(3.12) yields the bi-harmonic equation, $$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)^2 X=0 \quad (3.13)$$

In (3.13) we try the solution, $$X=(A+B\,z)\left(e^{-ikx}\right)^2 \quad (3.14)$$

where A and B are constants determined to satisfy the boundary conditions in the usual way and k is the wave number. After some algebraic manipulations we obtain using (3.3), $B = kA$ so that $X=A(1+k\,z)e^{-kz}e^{-ikx}$. Using (3.4) $A=\frac{P}{ik(2\mu k+\rho_0 g_0)}$. We want $w=-\frac{\partial X}{\partial x}$ on $z = 0$. $w(0)=-ikA=\frac{Pe^{ikx}}{2\mu k+\rho_0 g_0}$ (3.15)

The elastic deformation of the surface is given by (3.15)

4.0 Numerical example

The above result will now be applied to the time scale for isosatic adjustment for a Hudson's Bay ice sheet load for typical values of ρ_0, μ, g_0 and ν. For Hudson's Bay, $k \approx \frac{2\pi}{5000km} \approx 10^{-6} cm^{-1}$, $\frac{\rho_o g_0}{2\mu k} = \frac{3 \cdot 10^3}{2 \cdot 10^{-8} \cdot 5 \cdot 10^{11}} \approx \frac{1}{3}$ which is small compared to 1. So for Hudson's Bay, the period τ is $\tau \approx \frac{2\nu k}{\rho_0 g_0}$, which is controlled only by viscosity ν. If ν = 10 [22] poise, then, $\tau \approx \frac{2 \cdot 10^{22} \cdot 10^{-8}}{3 \cdot 10^3} \approx 6 \cdot 10^{10}\ sec = 2{,}000$ years for 10^{22} poise. The dependence on viscosity ν is linear.

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