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Further on stokes expansions for the finite amplitude water waves

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Abstract

An analytic method is applied to study the higher order approximate solution of Stokes waves. For comparison and effective analysis, solutions of first to fifth order approximations are studied. From the various expressions derived, the numerical computations and the graphical solutions, there is a gradual increase in the wave height from the first to fifth order and they follow one another very closely. However, there is a significant difference in that of fourth and fifth order compared with those of first to third. This analysis suggests that the wave height of fifth order is twice that of third order but with slight increase with respect to the fourth order solution. Fifth order Stokes waves may be peculiar to waves with unusual characters in view of the significant difference in the wave height when compared with those of lower order solution. However, dispersive phenomena among wave modes often limit the wave growth in practice.

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1.0 Introduction

Since the work of Stokes and some of his contemporaries such as Mitchell [1] in the nineteenth century, there have been remarkable advances in the mathematical and physical description of water waves. The detail analyses of the phenomena are readily available in such authoritative publications such as [2, 3, 4, 5, 6, and 7], just to mention a few.

Like most of the finite amplitude water waves with rigid profile, Stokes waves are characterized by non-vanishing divergence. This property had been exploited by a number of theorists in numerical and analytical study of certain geophysical processes. Hunt in [8] calculated the force and couple associated with Stokes waves on vertical piercing cylinder in both deep water and water of finite depth. This analysis explained the effects on oilrigs of the propagating ocean waves.

Further, Okeke in [9] obtained the solution of Stokes waves in the water of finite depth in form of solitary waves, which propagate into the adjoining estuary as bores.

This paper is thus aimed at the continuation of the analytical study of Stokes waves to the fifth order. This is a superposition of the waves modes from first to fourth. These lower modes had already been calculated in [2] and briefly reviewed in the earlier section of this study.

2.0 Review of earlier developments for stokes waves

The x-axis is perpendicular to the shoreline and z –axis perpendicular to the x-axis but pointing vertically towards the seabed in negative direction. The fluid medium is assumed to be irrotational but non-divergent. Let ϕ and ψ be the velocity potential and stream function respectively. Following [2], the following apply

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad (2.1)$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2} = 0 \qquad (2.2)$$

To solve for (2.2), the boundary conditions are: $\psi = 0 \quad \text{for} \quad z = \eta \qquad (2.3)$

$$\psi = k_1 \quad \text{for} \quad z = -h \qquad (2.4)$$

$$p = k_2 \quad \text{for} \quad z = \eta \qquad (2.5)$$

$\eta = \eta(x,t)$ is the wave profile. Dynamic boundary condition is

$$g\eta + \tfrac{1}{2}\left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right] = k_3 \quad \text{for} \quad z = \eta \qquad (2.6)$$

General solution for eqn (2.2) is $\Psi = c_1 z + c_2 + (c_3\cos kx + c_4\sin kx)(c_5 e^{kz} + c_6 e^{-kz}) \quad (2.7)$

c_i, $(i = 1,2,3,4,5)$ are constants. Since the fluid depth is assumed to be infinite $c_6 = 0$, and $c_1 = -c$

$$\therefore \Psi = -cz + c_2 + (c_3\cos kx + c_4\sin kx)c_5 e^{kz} \qquad (2.8)$$

A possible form of stream function is obtained if $c_4 = 0$ and $c_2 = 0$, therefore,

$\Psi = -cz + (c_3\cos kx)c_5 e^{kz}$

$$\Psi = -cz + c_3 c_5 \cos kx e^{kz}, \qquad (2.9)$$

on dividing through by c, we have $\dfrac{\Psi}{c} = -cz + \dfrac{c_3 c_5}{c}\cos kx e^{kz} \qquad (2.10)$

Therefore let, $\beta = \dfrac{c_3 c_5}{c}$, then $\dfrac{\Psi}{c} = -z + \beta e^{kz}\cos kx \qquad (2.11)$

But when $\psi = 0, z = \eta$, therefore $-\eta + \beta e^{k\eta}\cos kx = 0 \qquad (2.12)$

$$\eta = \beta e^{k\eta}\cos kx \qquad (2.13)$$

By perturbation methods involving $\eta(x,t)$, the following are obtained:

First order approximation $\eta(x) = -a\cos kx \qquad (2.14)$

Second order approximation $\eta(x) = -a\cos kx + \dfrac{1}{2}ka^2\cos 2kx \qquad (2.15)$

Third order approximation $\eta(x) = -a\cos kx + \dfrac{1}{2}ka^2\cos 2kx - \dfrac{3}{8}k^2a^3\cos 3kx \qquad (2.16)$

Fourth order approximation

$$\eta(x) = -a\cos kx + \left(\frac{1}{2}ka^2 + \frac{11}{6}k^3a^4\right)\cos 2kx - \frac{3}{8}a^3k^2\cos 3kx + \frac{1}{3}a^4k^3\cos^4 kx \qquad (2.17)$$

3.0 Derivation of fifth order approximation of Stokes waves

The fourth order solution of Stokes waves is

$$\frac{\psi}{c} = -z + \beta e^{kz}\cos kx + \gamma e^{2kz}\cos 2kx, \quad (\psi = 0 \ \text{at} \ z = \eta) \qquad (3.1)$$

Including the next term in the Fourier expansion for ψ and adjusting the coefficient, we have

$$\frac{\psi}{c} = -z + \beta \beta^{kz} \cos kx + \gamma \gamma^{2kz} \cos 2kx + \alpha \alpha^{3kx} \cos 3kx \qquad (3.2)$$

By assumption, $\alpha = 0(\beta^5)$. Dynamic boundary condition is

$$g\eta + \frac{1}{2}\left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right] = k_3 \quad at \quad z = \eta \qquad (3.3)$$

$$\psi = -c\left[z - \beta e^{kz} \cos kx - \gamma e^{2kz} \cos 2kx - \alpha e^{3kx} \cos 3kx\right]$$

$$\frac{\partial\psi}{\partial x} = -c\left[k\beta e^{kz} \sin kx + 2k\gamma e^{2kz} \sin 2kx + 3k\alpha e^{3kx} \sin 3kx\right] \qquad (3.4)$$

$$\left(\frac{\partial\psi}{\partial x}\right)^2 = c^2\left[k\beta k^{kz} \sin kx + 2k\gamma^{2kz} \sin 2kx + 3k\alpha\alpha^{3kx} \sin 3kx\right]^2$$

$$\frac{1}{c^2}\left(\frac{\partial\psi}{\partial x}\right)^2 = k^2\beta^2 e^{2kz} \sin^2 kx + 4k^2\gamma^2 e^{4kz} \sin^2 2kx + 9k^2\alpha^2 e^{6kx} \sin^2 3kx \qquad (3.5)$$

$$+4k^2\beta\gamma e^{3kz} \sin kx \sin 2kx + 6k^2\beta\alpha e^{4kz} \sin kx \sin 3kx + 12k^2\gamma\alpha e^{5kz} \sin 2kx \sin 3kx$$

$$Similarly,\ \frac{\partial\psi}{\partial z} = c\left[-1 + k\beta e^{kz} \cos kx + 2k\gamma^{2kz} \cos 2kx + 3k e^{3kx} \cos 3kx\right] \qquad (3.6)$$

$$\left(\frac{\partial\psi}{\partial z}\right)^2 = c^2\left[-1 + k\beta e^{kz} \cos kx + 2k\gamma e^{2kz} \cos 2kx + 3k\alpha e^{3kx} \cos 3kx\right]^2$$

$$\frac{1}{c^2}\left(\frac{\partial\psi}{\partial z}\right)^2 = 1 + k^2\beta^2 e^{2kz} \cos^2 kx + 4k^2\gamma^2 e^{4kz} \cos^2 2kx + 9k^2\alpha^2 e^{6kx} \cos^2 3kx$$

$$+ 4k^2\beta\gamma e^{3kz} \cos kx \cos 2kx + 6k^2\beta\alpha e^{4kz} \cos kx \cos 3kx + 12k^2\gamma\alpha e^{5kz} \cos 2kx \cos 3kx - \qquad (3.7))$$

$$2k\beta e^{kz} \cos kx - 4k\gamma e^{2kz} \cos 2kx - 6k\alpha e^{3kx} \cos 3kx$$

Adding equations (3.5) and (3.7) gives

$$\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2 = c^2[\, 1 + k^2\beta^2 e^{2kz} + 4k^2\gamma^2 e^{4kz} + 9k^2\alpha^2 e^{6kx} + 4k^2\beta\gamma e^{3kz} \cos kx$$

$$+ 6k^2\beta\alpha e^{4kz} \cos 2kx + 12k^2\gamma\alpha e^{5kz} \cos kx - 2k\beta e^{kz} \cos kx - 4k\gamma e^{2kz} \cos 2kx \qquad (3.8)$$

$$- 6k\alpha e^{3kx} \cos 3kx\,]$$

$g\eta + \frac{1}{2}\left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right] = k_3$, now becomes

$$\frac{2g\eta}{c^2} + 1 + k^2\beta^2 e^{2kz} + 4k^2\gamma^2 e^{4kz} + 9k^2\alpha^2 e^{6kx} + 4k^2\beta\gamma e^{3kz} \cos kx + 6k^2\beta\alpha e^{4kz} \cos 2kx \qquad (3.9)$$

$$+ 12k^2\gamma\alpha e^{5kz} \cos kx - 2k\beta e^{kz} \cos kx - 4k\gamma e^{2kz} \cos 2kx - 6k\alpha e^{3kx} \cos 3kx = k_4 \quad at \quad z = \eta$$

$$\frac{2g\eta}{c^2} + k^2\beta^2 e^{2kz} + 4k^2\gamma^2 e^{4kz} + 9k^2\alpha^2 e^{6kx} + 4k^2\beta\gamma e^{3kz} \cos kx$$

$$+ 6k^2\beta\alpha e^{4kz} \cos 2kx + 12k^2\gamma\alpha e^{5kz} \cos kx - 2k\beta\beta^{kz} \cos kx$$

$$- 4k\gamma\gamma^{2kz} \cos 2kx - 6k\alpha\alpha^{3kx} \cos 3kx = k_5 \qquad (3.10)$$

$$But \quad \eta = \beta e^{k\eta} \cos kx + \gamma e^{2k\eta} \cos 2kx + \alpha e^{3k\eta} \cos 3kx \qquad (3.11)$$

(Putting $z = \eta$ *when* $\psi = 0$ *in equation (3.2))*

$$\frac{2g\eta}{c^2}+k^2\beta^2 e^{2k\eta}+4k^2\gamma^2 e^{4k\eta}+9k^2\alpha^2 e^{6k\eta}+4k^2\beta\gamma e^{3k\eta}\cos kx+6k^2\beta\alpha e^{4k\eta}\cos 2kx+12k^2\gamma\alpha e^{5k\eta}\cos kx$$
$$-2k\left[\beta e^{k\eta}\cos kx+\gamma e^{2k\eta}\cos 2kx+\alpha e^{3k\eta}\cos 3kx\right]-2k\gamma e^{2k\eta}\cos 2kx-4k\alpha e^{3k\eta}\cos 3kx=k_5 \qquad (3.12)$$

Substituting for $\eta=\beta e^{k\eta}\cos kx+\gamma e^{2k\eta}\cos 2kx+\alpha e^{3k\eta}\cos 3kx$, *we have*

$$\frac{2g\eta}{c^2}+k^2\beta^2 e^{2k\eta}+4k^2\gamma^2 e^{4k\eta}+9k^2\alpha^2 e^{6k\eta}+4k^2\beta\gamma e^{3k\eta}\cos kx+6k^2\beta\alpha e^{4k\eta}\cos 2kx$$
$$+12k^2\gamma\alpha e^{5k\eta}\cos kx-2k\eta\, 2k\gamma e^{2k\eta}\cos 2kx-4k\alpha e^{3k\eta}\cos 3kx=k_5 \qquad (3.13)$$

$$\frac{2g\eta}{C^2}+k^2\beta^2 e^{2k\eta}-2k\eta-2k\gamma e^{2k\eta}\cos 2kx-4k\alpha e^{3k\eta}\cos 3kx\approx 0 \quad (\text{neglecting } 0(\beta^5) \text{ and above}) \qquad (3.14)$$

Recall from (3.11) $\eta=\beta e^{k\eta}\cos kx+\gamma\gamma^{2k\eta}\cos 2kx+\alpha e^{3k\eta}\cos 3kx$. *Making coskx the subject of the formula,*

$$\beta e^{k\eta}\cos kx=\eta-\gamma e^{2k\eta}\cos 2kx-\alpha e^{3k\eta}\cos 3kx \qquad (3.15)$$
$$\cos kx=\beta^{-1}e^{-k\eta}(\eta-\gamma e^{2k\eta}\cos 2kx-\alpha e^{3k\eta}\cos 3kx) \qquad (3.16)$$
$$\cos 2kx=2\cos^2 kx-1 \qquad (3.17)$$
$$\cos 3kx=4\cos^3 kx-3\cos kx \qquad (3.18)$$
$$\cos 2kx=2\left[\beta^{-1}e^{-k\eta}(\eta-\gamma e^{2k\eta}\cos 2kx-\alpha e^{3k\eta}\cos 3kx\right]^2-1 \qquad (3.19)$$
$$=2\beta^{-2}e^{-2k\eta}(\eta^2+\gamma^2 e^{4k\eta}\cos^2 2kx+\alpha^2 e^{6k\eta}\cos^2 3kx-2\eta e^{2k\eta}\cos 2kx-2\eta\alpha e^{3k\eta}\cos 3kx+2\gamma\alpha e^{5k\eta}\cos 2kx\cos 3kx-1$$
$$=2\beta^{-2}\eta^2 e^{-2k\eta}+2\beta^{-2}\gamma^2 e^{2k\eta}\cos^2 2kx+2\beta^{-2}\alpha^2 e^{4k\eta}\cos^2 3kx$$
$$-4\beta^{-2}\gamma\gamma\eta^{2k\eta}\cos 2kx-4\beta^{-2}\alpha\alpha\eta^{k\eta}\cos 3kx+4\beta^{-2}\gamma\gamma\alpha^{3k\eta}\cos 2kx\cos 3kx-1 \qquad (3.20$$
$$\Rightarrow -2k\gamma e^{2k\eta}\cos 2kx=-2k\gamma e^{2k\eta}(2\beta^{-2}\eta^2 e^{-2k\eta}+2\beta^{-2}\gamma^2 e^{2k\eta}\cos^2 2kx+$$
$$2\beta^{-2}\alpha^2 e^{4k\eta}\cos^2 3kx-4\beta^{-2}\gamma\eta e^{2k\eta}\cos 2kx- \qquad (3.21)$$
$$4\beta^{-2}\alpha\eta e^{k\eta}\cos 3kx+4\beta^{-2}\gamma\alpha e^{3k\eta}\cos 2kx\cos 3kx+2k\gamma e^{2k\eta}\cos 2kx)$$

$$-2k\gamma e^{2k\eta}\cos 2kx=-4k\beta^{-2}\gamma\eta^2-4k\beta^{-2}\gamma^3 e^{4k\eta}\cos^2 2kx-$$
$$4k\beta^{-2}\gamma\alpha^2 e^{6k\eta}\cos^2 3kx+8k\beta^{-2}\gamma^2\eta e^{2k\eta}\cos 2kx+8k\beta^{-2}\gamma\alpha\eta e^{3k\eta}\cos 3kx- \qquad (3.22)$$
$$8k\beta^{-2}\gamma^2\alpha e^{5k\eta}\cos 2kx\cos 3kx-4k^2\gamma^2 e^{4k\eta}\cos 2kx)$$
$$=-4k\beta^{-2}\gamma\eta^2 \qquad (\text{neglecting terms of } o(\beta^5) \text{ and above}) \qquad (3.23)$$

Similarly, $\cos^3 kx=\left[\left(\beta^{-1}e^{-k\eta}(\eta-\gamma e^{2k\eta}\cos 2kx-\alpha e^{3k\eta}\cos 3kx\right)\right]^3 \qquad (3.24)$

$$=\beta^{-3}e^{-3k\eta}(\eta^2+\gamma^2 e^{4k\eta}\cos^2 2kx+\alpha^2 e^{6k\eta}\cos^2 3kx-2\eta\eta\gamma^{2k\eta}\cos 2kx$$
$$-2\eta\alpha e^{3k\eta}\cos 3kx+2\gamma\alpha e^{5k\eta}\cos 2kx\cos 3kx)(\eta-\gamma e^{2k\eta}\cos 2kx-\alpha e^{3k\eta}\cos 3kx \qquad (3.25)$$

$$4\cos^3 k=4\beta^{-3}e^{-3k\eta}[(\eta^3+\eta\gamma^2 e^{4k\eta}\cos^2 2kx+\eta\alpha^2 e^{6k\eta}\cos^2 3kx-2\eta^2\gamma\gamma^{2k\eta}\cos 2kx$$
$$-2\eta^2\alpha e^{k\eta}\cos 3kx+2\gamma\alpha\eta^{5k\eta}\cos 2kx\cos 3kx)+(-\eta^2\gamma e^{2k\eta}\cos 2kx$$
$$-\gamma^3 e^{6k\eta}\cos 2kx\cos^2 2kx-\gamma\alpha^2 e^{8k\eta}\cos 2kx\cos^2 3kx+2\eta\gamma^2 e^{4k\eta}\cos^2 2k$$
$$+2\eta\gamma\alpha e^{5k\eta}\cos 2kx\cos 3kx-2\gamma^2\alpha e^{7k\eta}\cos^2 2kx\cos 3kx)+(-\eta^2\alpha e^{3k\eta}\cos 3kx \qquad (3.26)$$
$$-\gamma^2\alpha e^{7k\eta}\cos^2 2kx\cos 3kx-\alpha^3 e^{9k\eta}\cos^3 3kx+2\eta\gamma\alpha^{5k\eta}\cos 2kx\cos 3kx$$
$$+2\eta\alpha^2 e^{6k\eta}\cos^2 3kx-2\gamma\alpha e^{8k\eta}\cos 2kx\cos^2 3kx)]$$

From equation (3.18) $\cos 3kx=4\cos^3 kx-3\cos kx$, substituting for $4\cos^3 kx$ in equation (3.18) gives

$$cos3kx \;=\; 4\beta^{-3}\eta^{3}\, e^{-3k\eta} \;+\; 4\eta\beta^{-3}\,\gamma^{2}\, e^{k\eta}\; cos^{2}2kx \;+\; 4\eta\,\beta^{-3}\,\alpha^{2}\, e^{3k\eta}\; cos^{2}3kx$$
$$-8\,\eta^{2}\,\beta^{-3}\gamma e^{-k\eta}\; cos2kx \;-8\,\eta^{2}\,\beta^{-3}\alpha e^{k\eta}\; cos3kx \;+\; 8\,\beta^{-3}\gamma\alpha\eta e^{2k\eta}\; cos2kx\; cos3kx$$
$$+\;[-4\eta^{2}\beta^{-3}\gamma\, e^{-k\eta}\; cos2kx - 4\,\beta^{-3}\,\gamma^{3}\, e^{3k\eta}\; cos2kxcos^{2}2kx - 4\,\beta^{-3}\gamma\alpha^{2}\, e^{5k\eta} cos2kx\, cos^{2}3kx$$
$$+8\eta\beta^{-3}\,\gamma^{2}\, e^{k\eta} cos^{2}2k + 8\eta\beta^{-3}\gamma\alpha e^{2k\eta}\; cos2kx\; cos3kx\; -8\,\beta^{-3}\,\gamma^{2}\,\alpha\alpha^{4k\eta}\;\; cos^{2}2kx\; cos3kx\;] \qquad (3.27)$$
$$+\,[-\,4\,\beta^{-3}\,\eta^{2}\alpha\;\; cos3kx\;\; -4\,\beta^{-3}\,\gamma^{2}\alpha\; e^{4k\eta}\; cos^{2}2kxcos3kx\; -4\,\beta^{-3}\,\alpha^{3}\, e^{6k\eta}\; cos^{3}3kx$$
$$+8\eta\beta^{-3}\gamma\alpha^{2k\eta}\; cos2kxcos3kx + 8\eta\beta^{-3}\,\alpha^{2}\, e^{3k\eta}\; cos^{2}3kx$$
$$-\;8\,\beta^{-3}\gamma\,\alpha\, e^{5k\eta} cos2kx\, cos^{2}3kx\;]\; -3coskx$$
$$=\;\; 4\beta^{-3}\eta^{3}\; e^{-3k\eta}\;\; -\;8\,\eta^{2}\,\beta^{-3}\gamma\, e^{-k\eta}\;\; cos\,2kx\;\; -\,4\eta^{2}\,\beta^{-3}\gamma\, e^{-k\eta}\;\; cos\,2kx -\;\; 3\,cos\,kx$$
$$(neglecting\;\; 0(\,\beta^{5}\,)\;\; and\;\; above\,) \qquad (3.28)$$

The last term in equation (3.14) is

$$-4k\alpha\alpha^{3k\eta}\, cos\,3kx \;=\; -4k\alpha\, e^{3k\eta}\; [\,4\beta^{-3}\eta^{3}\; e^{-3k\eta} -\; 8\;\eta^{2}\;\beta^{-3}\gamma\, e^{-k\eta}\;\; cos\,2kx\; -\, 4\eta^{2}\beta^{-3}\gamma\; e^{-k\eta}\;\; cos\,2kx$$
$$-\;3\,cos\,kx\;] \qquad (3.29)$$
$$=-\;\; 16k\alpha\beta^{-3}\eta^{3} + 32\;k\,\eta^{2}\;\;\beta^{-3}\gamma\alpha\, e^{2k\eta}\; cos\,2kx + 16\;\, k\eta^{2}\,\beta^{-3}\gamma\;\alpha\; e^{-k\eta}\; cos\,2kx$$
$$+12k\alpha\, e^{3k\eta}\;\, cos\,kx\;] \qquad (3.30)$$
$$=\;\; -\;16\,\mathrm{k}\;\alpha\beta^{-3}\eta^{3}\quad \text{(neglecting}\;\; 0(\beta^{5}\,)\;\text{and above.} \qquad (3.31)$$

Recall that from equation (3.13)

$\dfrac{2\mathrm{g}\eta}{\mathrm{c}^{2}} \;+\; \mathrm{k}^{2}\beta^{2}\; \mathrm{e}^{2\mathrm{k}\eta}\; -\; 2\mathrm{k}\eta\; -\; 2\mathrm{k}\;\gamma\,\mathrm{e}^{2\mathrm{k}\eta}\; \cos2\mathrm{kx} -\; 4\mathrm{k}\alpha\,\mathrm{e}^{3\mathrm{k}\eta}\; \cos3\mathrm{kx}\; \approx\; 0\,.$ Replacing $-4k\alpha e^{3k\eta}\; cos\,3kx$ in equation (3.13) with $-16k\alpha B^{-3}\eta^{3}$ gives

$$\frac{2\mathrm{g}\eta}{\mathrm{c}^{2}} \;+\; \mathrm{k}^{2}\beta^{2}\; \mathrm{e}^{2\mathrm{k}\eta}\; -2\mathrm{k}\eta\, -\; 4\mathrm{k}\beta^{-2}\;\gamma\eta^{2} -\; 16\,\mathrm{k}\;\alpha\beta^{-3}\eta^{3} \approx\;\; 0\;\;.. \qquad (3.32)$$

Applying Taylor's series for $e^{2k\eta}$ in equation (3.32) gives

$$\frac{2g\eta}{c^{2}} \;\;+\;\; k^{2}\beta^{2}\left[1 \;+\; 2k\eta + \frac{4k^{2}\;\eta^{2}}{2} \;+\; \frac{8k^{3}\;\eta^{3}}{6} \;+\; 0(\eta^{4}\,)\;\right]$$
$$-\;2k\eta - 4k\beta^{-2}\gamma\eta^{2} \;-\; 16\;k\alpha\beta s^{-3}\eta^{3}\;\;\approx\;\; 0 \qquad (3.33)$$
$$\frac{2g\eta}{c^{2}} + k^{2}\beta^{2} + 2k^{3}\;\beta^{2}\eta\; + 2k^{4}\;\beta^{2}\;\eta^{2} + \frac{8k^{5}\;\beta^{2}\;\;\eta^{3}}{6} -\, 2k\eta - 4k\beta^{-2}\;\gamma\eta^{2} - 16k\alpha\beta^{-3}\eta^{3}\;\;\approx 0 \qquad (3.34)$$
$$\left(\frac{2\mathrm{g}}{\mathrm{c}^{2}} + 2\mathrm{k}^{3}\;\beta^{2}\; -2\mathrm{k}\;\right)\eta + \left(2\mathrm{k}^{4}\;\beta^{2}\; -\; 4\mathrm{k}\beta^{-2}\;\gamma\,\right)\eta^{2} + \mathrm{k}^{2}\beta^{2} + \left(\;\frac{8\mathrm{k}^{5}\;\beta^{2}}{6} - 16\,\mathrm{k}\;\alpha\beta^{-3}\right)\eta^{3} = 0 \qquad (3.35)$$

Equating coefficients of η, η^{2} and η^{3} in equation (3.35), we have

$$\eta : \frac{2\mathrm{g}}{\mathrm{c}^{2}} \;+\; 2\mathrm{k}^{3}\;\beta^{2}\; -2\mathrm{k} = 0 \qquad (3.36)$$
$$\eta^{2}\;:\; 2k^{4}\;\beta^{2}\; -\; 4k\beta^{-2}\;\gamma \;=\; 0,\, 2k^{4}\beta^{2}\;\;=\; 4k\beta^{-2}\;\gamma \qquad (3.37)$$
$$\Rightarrow\;\; \gamma\;\; =\; \frac{1}{2}\beta^{4}\;\; \mathrm{k}^{3} \qquad (3.38)$$
$$\eta^{3}\;:\; \frac{8k^{5}\;\beta^{2}}{6} \;-\; 16\;k\;\alpha\beta^{-3} \;=\; 0\;,\;\; \frac{8k^{5}\;\beta^{2}}{6} = 16\;k\;\alpha\beta^{-3} \qquad (3.39)$$
$$\Rightarrow\;\alpha\;=\;\frac{1}{12}\beta^{5}\mathrm{k}^{4} \qquad (3.40)$$

$\dfrac{2g}{\mathrm{c}^{2}} = 2\mathrm{k}\; -2\beta^{2}\mathrm{k}^{3} \Rightarrow\; \dfrac{2g}{\mathrm{c}^{2}} = 2\mathrm{k}(1\; -\;\; \beta^{2}\mathrm{k}^{2}\;) \Rightarrow g = \mathrm{c}^{2}\mathrm{k}(1 - \beta^{2}\mathrm{k}^{2}\;).$ Therefore $\dfrac{g}{k}\,\dfrac{1}{(1\;\; -\;\; \beta^{2}\mathrm{k}^{2}\;)} \;=\; \mathrm{c}^{2}$

$$= \frac{g}{k}(1+\beta^2 k^2+\beta^4 k^4 \quad +\Lambda\ .) \qquad (3.41)$$

From equation (3.11), $\eta = \beta e^{k\eta} \cos kx + \gamma e^{2k\eta} \cos 2kx + \alpha e^{3k\eta} \cos 3kx$ where γ and α are given by equation

(3.38) and (3.39) respectively. Hence

$$\eta = \beta e^{k\eta} \cos kx + \frac{1}{2}\beta^4 k^3 e^{2k\eta} \cos 2kx + \frac{1}{12}\beta^5 k^4 e^{3k\eta} \cos 3kx \qquad (3.42)$$

Let $$\eta = \eta_0\beta + \eta_1\beta^2 + \eta_2\beta^3 + \eta_3\beta^4 + \eta_4\beta^5 + \Lambda \qquad (3.43)$$

Utilising Taylor's series for $e^{k\eta}, e^{2l\eta}$ and $e^{3k\eta}$ in equation (3.42) and equating to equation (3.43) gives

$$\eta_0\beta + \eta_1\beta^2 + \eta_2\beta^3 + \eta_3\beta^4 + \eta_4\beta^5 = \beta\left(1 + k\eta + \frac{1}{2}k^2\eta^2 + \frac{k^3\eta^3}{6} + \frac{k^4\eta^4}{24} + \Lambda\right)\cos kx$$
$$+\frac{1}{2}\beta^4 k^3\left(1 + 2k\eta + \frac{4k^2\eta^2}{2} + \frac{8k^3\eta^3}{6} + \Lambda\right)\cos 2kx + \frac{1}{12}\beta^5 k^4\left(1 + 3k\eta + \frac{9k^2\eta^2}{2}\right. \qquad (3.44)$$
$$\left. + \frac{27k^3\eta^3}{6} + \Lambda\right)\cos 3kx$$

$$= \beta\,[\,1 + k(\eta_0\beta + \eta_1\beta^2 + \eta_2\beta^3 + \eta_3\beta^4 + \eta_4\beta^5) + \frac{k^2}{2}(\eta_0\beta + \eta_1\beta^2 + \eta_2\beta^3 + \eta_3\beta^4 + \eta_4\beta^5)^2$$
$$+ \frac{k^3}{6}(\eta_0\beta + \eta_1\beta^2 + \eta_2\beta^3 + \Lambda)^3\,] \cos kx + \frac{1}{2}\beta^4 k^3\,[\,1 + 2k(\eta_0\beta + \eta_1\beta^2 + \eta_2\beta^3 + \eta_3\beta^4 + \eta_4\beta^5)$$
$$+ 2k^2(\eta_0\beta + \eta_1\beta^2 + \eta_2\beta^3 + \eta_3\beta^4 + \eta_4\beta^5)^2 + \frac{8k^3}{6}(\eta_0\beta + \eta_1\beta^2 + \eta_2\beta^3 + \Lambda)^3\,] \cos 2kx$$
$$+ \frac{1}{12}\beta^5 k^4\,[\,1 + 3k(\eta_0\beta + \eta_1\beta^2 + \eta_2\beta^3 + \eta_3\beta^4 + \eta_4\beta^5) + \frac{9k^2}{2}(\eta_0\beta + \eta_1\beta^2 + \eta_2\beta^3 + \eta_3\beta^4 + \eta_4\beta^5)^2$$
$$+ \frac{9k^3}{2}(\eta_0\beta + \eta_1\beta^2 + \eta_2\beta^3 + \Lambda)^3\,] \cos 3kx \qquad (3.45)$$

Equating coefficients of β, β^2, β^3, β^4 and β^5, we have'

$$\beta: \quad \eta_0 = \cos kx \qquad (3.46)$$

$$\beta^2: \quad \eta_1 = k\eta_0 \cos kx = k \cos^2 kx \qquad (3.47)$$

$$\beta^3: \quad \eta_2 = (k\eta_1 + \frac{k^2}{2}\eta_0) \cos kx = (k(k \cos^2 kx) + \frac{k^2}{2}\cos kx) \cos kx \qquad (3.48)$$

$$\eta_2 = k^2 \cos^3 kx + \frac{k^2}{2}\cos^2 kx \qquad (3.49)$$

$$\beta^4: \quad \eta_3 = (k\eta_2 + k^2\eta_0\eta_1 + \frac{k^3}{6}\eta_0^3)\cos kx + \frac{1}{2}k^3 \cos 2kx \qquad (3.50)$$

$$\eta_3 = k\eta_2 \cos kx + k^2\eta_0\eta_1 \cos kx + \frac{k^3}{6}\eta_0^3 \cos kx + \frac{1}{2}k^3 \cos 2kx$$

$$= k \cos kx\,(k^2 \cos^3 kx + \frac{k^2}{2}\cos^2 kx) + k^2(\cos kx)(k \cos^2 kx)(\cos kx)$$
$$+ \frac{k^3}{6}(\cos kx)^3 \cos kx + \frac{1}{2}k^3 \cos 2kx \qquad (3.51)$$

$$= k^3 \cos^4 kx + \frac{k^3}{2}\cos^3 kx + k^3\cos^4 kx + \frac{k^3}{6}\cos^4 kx + \frac{1}{2}k^3 \cos 2kx \tag{3.52}$$

$$\eta_3 = \frac{13}{6}k^3 \cos^4 kx + \frac{k^3}{2}\cos^3 kx + \frac{1}{2}k^3 \cos 2kx \tag{3.53}$$

$$\beta^5 : \eta_4 = (k\eta_3 + \frac{k^2}{2}\eta_1{}^2 + k^2\eta_0\eta_2 + \frac{k^3}{2}\eta_0{}^2\eta_1)\cos kx$$
$$+ k^4\eta_0 \cos 2kx + \frac{k^4}{12}\cos 3kx \tag{3.54}$$

$$\eta_4 = k\cos kx[\frac{13}{6}k^3\cos^4 kx + \frac{k^3}{2}\cos^3 kx + \frac{1}{2}k^3 \cos 2kx] + \frac{k^2}{2}\cos kx[k\cos^2 kx]^2$$
$$+ k^2\cos kx[(\cos kx)(k^2\cos^3 kx + \frac{k^2}{2}\cos^2 kx)]$$
$$+ \frac{k^3}{2}\cos kx(\cos^2 kx)(k\cos^2 kx) + k^4(\cos kx)\cos 2kx + \frac{k^4}{12}\cos 3kx \tag{3.55}$$

$$\eta_4 = \frac{13}{6}k^4\cos^5 kx + \frac{k^4}{2}\cos^4 kx + \frac{k^4}{2}\cos 2kx \cos kx + \frac{k^4}{2}\cos^5 kx + k^4\cos^5 kx$$
$$+ \frac{k^4}{2}\cos^4 kx + \frac{k^4}{2}\cos^5 kx + k^4\cos 2kx \cos kx + \frac{k^4}{12}\cos 3kx \tag{3.56}$$

$$= (\frac{13}{6}k^4 + \frac{k^4}{2} + \frac{k^4}{2} + k^4)\cos^5 kx + (\frac{k^4}{2} + \frac{k^4}{2})\cos^4 kx$$
$$+ (\frac{k^4}{2} + k^4)\cos 2kx \cos kx + \frac{k^4}{12}\cos 3kx \tag{3.57}$$

$$\eta_4 = \frac{25}{6}k^4\cos^5 kx + k^4\cos^4 kx + \frac{3}{2}k^4\cos 2kx \cos kx + \frac{k^4}{12}\cos 3kx \tag{3.58}$$

From equation (3.43) $\eta = \beta\eta_0 + \beta^2\eta_1 + \beta^3\eta_2 + \beta^4\eta_3 + \beta^5\eta_4$

Substituting for $\eta_0, \eta_1, \eta_2, \eta_3$ *and* η_4 *in equation (*3.43*) gives*

$$\eta = \beta\cos kx + \beta^2 k\cos^2 kx + \beta^3(k^2\cos^3 kx + \frac{k^2}{2}\cos^2 kx)$$
$$+ \beta^4(\frac{13}{6}k^3\cos^4 kx + \frac{k^3}{2}\cos^3 kx + \frac{1}{2}k^3\cos 2kx) + \beta^5(\frac{25}{6}k^4\cos^5 kx \tag{3.59}$$
$$+ k^4\cos^4 kx + \frac{3}{2}k^4\cos 2kx\cos kx + \frac{k^4}{12}\cos 3kx)$$

$$\eta = \beta\cos kx + \beta^2 k\cos^2 kx + \beta^3 k^2\cos^3 kx + \frac{1}{2}\beta^3 k^2\cos^2 kx$$
$$+ \frac{13}{6}\beta^4 k^3\cos^4 kx + \frac{1}{2}\beta^4 k^3\cos^3 kx + \frac{1}{2}\beta^4 k^3\cos 2kx + \frac{25}{6}\beta^5 k^4\cos^5 kx \tag{3.60}$$
$$+ \beta^5 k^4\cos^4 kx + \frac{3}{2}\beta^5 k^4\cos 2kx\cos kx + \frac{1}{12}\beta^5 k^4\cos 3kx)$$

Rearranging the terms, we have $\eta = \beta \cos kx + (\beta^2 k + \frac{1}{2}\beta^3 k^2) \cos^2 kx$

$$+\frac{1}{2}\beta^4 k^3 \cos 2kx + (\frac{1}{2}\beta^4 k^3 + \beta^3 k^2) \cos^3 kx + \frac{1}{12}\beta^5 k^4 \cos 3kx$$
$$+\frac{3}{2}\beta^5 k^4 \cos 2kx \cos kx + (\frac{13}{6}\beta^4 k^3 + \beta^5 k^4)\cos^4 kx + \frac{25}{6}\beta^5 k^4 \cos^5 kx \qquad (3.61)$$

The following relations apply

$$\cos^2 kx = \frac{1}{2}(\cos 2kx + 1) \qquad (3.62)$$

$$\cos^3 kx = \frac{1}{4}(\cos 3kx + 3\cos kx) \qquad (3.63)$$

$$\cos^4 kx = \frac{1}{8}(\cos 4kx + 4\cos 2kx + 3) \qquad (3.64)$$

$$\cos^5 kx = \frac{1}{16}(\cos 5kx + 5\cos 3kx + 10\cos kx) \qquad (3.65)$$

$$\cos 2kx \cos kx = \frac{1}{2}(\cos 3kx + \cos kx) \qquad (3.66)$$

Substituting for the identities, we have

$$\eta = \beta \cos kx + (\beta^2 k + \frac{1}{2}\beta^3 k^2)(\frac{1}{2}\cos 2kx + \frac{1}{2}) + \frac{1}{2}\beta^4 k^3 \cos 2kx$$
$$+ (\frac{1}{2}\beta^4 k^3 + \beta^3 k^2)(\frac{1}{4}\cos 3kx + \frac{3}{4}\cos kx) + \frac{1}{12}\beta^5 k^4 \cos 3kx$$
$$+\frac{3}{2}\beta^5 k^4 (\frac{1}{2}\cos 3kx + \frac{1}{2}\cos kx) + (\frac{13}{6}\beta^4 k^3 + \beta^5 k^4)(\frac{1}{8}\cos 4kx \qquad (3.67)$$
$$+ \frac{4}{8}\cos 2kx + \frac{3}{8}) + \frac{25}{6}\beta^5 k^4 (\frac{1}{16}\cos 5kx + \frac{5}{16}\cos 3kx + \frac{10}{16}\cos kx)$$

$$\eta = \beta \cos kx + \frac{1}{2}\beta^2 k \cos 2kx + \frac{1}{2}\beta^2 k + \frac{1}{4}\beta^3 k^2 \cos 2kx + \frac{1}{4}\beta^3 k^2$$
$$+\frac{1}{2}\beta^4 k^3 \cos 2kx + \frac{1}{8}\beta^4 k^3 \cos 3kx + \frac{3}{8}\beta^4 k^3 \cos kx + \frac{1}{4}\beta^3 k^2 \cos 3kx$$
$$+\frac{3}{4}\beta^3 k^2 \cos kx + \frac{1}{12}\beta^5 k^4 \cos 3kx + \frac{3}{4}\beta^5 k^4 \cos 3kx + \frac{3}{4}\beta^5 k^4 \cos kx$$
$$+\frac{13}{48}\beta^4 k^3 \cos 4kx + \frac{13}{12}\beta^4 k^3 \cos 2kx + \frac{39}{48}\beta^4 k^3 + \frac{1}{8}\beta^5 k^4 \cos 4kx \qquad (3.68)$$
$$+\frac{1}{2}\beta^5 k^4 \cos 2kx + \frac{3}{8}\beta^5 k^4 + \frac{25}{96}\beta^5 k^4 \cos 5kx + \frac{125}{96}\beta^5 k^4 \cos 3kx$$
$$+\frac{250}{96}\beta^5 k^4 \cos kx$$

$$\eta = (\beta(+\frac{3}{4}\beta^3 k^2 + \frac{3}{8}\beta^4 k^3 + \frac{3}{4}\beta^5 k^4 + \frac{125}{48}\beta^5 k^4) \cos kx$$

$$+ (\frac{1}{2}\beta^2 k + \frac{1}{4}\beta^3 k^2 + \frac{1}{2}\beta^4 k^3 + \frac{13}{12}\beta^4 k^3 + \frac{1}{2}\beta^5 k^4) \cos 2kx$$

$$+ (\frac{1}{4}\beta^3 k^2 + \frac{1}{8}\beta^4 k^3 + \frac{1}{12}\beta^5 k^4 + \frac{3}{4}\beta^5 k^4 + \frac{125}{96}\beta^5 k^4) \cos 3kx \quad (3.69)$$

$$+ (\frac{13}{48}\beta^4 k^3 + \frac{1}{8}\beta^5 k^4) \cos 4kx + \frac{25}{96}\beta^5 k^4 \cos 5kx$$

$$+ (\frac{1}{2}\beta^2 k + \frac{1}{4}\beta^3 k^2 + \frac{13}{16}\beta^4 k^3 + \frac{3}{8}\beta^5 k^4)$$

$$\eta = (\beta + \frac{3}{4}\beta^3 k^2 + \frac{3}{8}\beta^4 k^3 + \frac{161}{48}\beta^5 k^4) \cos kx + (\frac{1}{2}\beta^2 k + \frac{1}{4}\beta^3 k^2$$

$$+ \frac{19}{12}\beta^4 k^3 + \frac{1}{2}\beta^5 k^4) \cos 2kx + (\frac{1}{4}\beta^3 k^2 + \frac{1}{8}\beta^4 k^3 + \frac{205}{96}\beta^5 k^4) \cos 3kx$$

$$+ (\frac{13}{48}\beta^4 k^3 + \frac{1}{8}\beta^5 k^4) \cos 4kx + \frac{25}{96}\beta^5 k^4 \cos 5kx + (\frac{1}{2}\beta^2 k + \frac{1}{4}\beta^3 k^2 \quad (3.70)$$

$$+ \frac{13}{16}\beta^4 k^3 + \frac{3}{8}\beta^5 k^4)$$

Let $a = (\beta + \frac{3}{4}\beta^3 k^2 + \frac{3}{8}\beta^4 k^3 + \frac{161}{48}\beta^5 k^4)$ (3.71)

Let $\beta = c_1 a + c_2 a^2 + c_3 a^3 + c_4 a^4 + c_5 a^5$ (3.72)

$$a + c_1 a + c_2 a^2 + c_3 a^3 + c_4 a^4 + c_5 a^5 + \frac{3}{4} k^2 (c_1 a + c_2 a^2 + c_3 a^3 + c_4 a^4 + c_5 a^5)^3$$

$$+ \frac{3}{8} k^3 (c_1 a + c_2 a^2 + c_3 a^3 + c_4 a^4 + c_5 a^5)^4 + \frac{3}{4} k^4 (c_1 a + c_2 a^2 + c_3 a^3 + c_4 a^4 + c_5 a^5)^5 \quad (3.73)$$

$\beta \cong$ a and $\beta^2 \cong$ a^2

$$\eta = (a + \frac{3}{4}a^3 k^2 + \frac{3}{8}a^4 k^3 + \frac{161}{48}a^5 k^4) \cos kx + (\frac{1}{2}a^2 k + \frac{1}{4}a^3 k^2 + \frac{19}{12}a^4 k^3$$

$$+ \frac{1}{2}a^5 k^4) \cos 2kx + (\frac{1}{4}a^3 k^2 + \frac{1}{8}a^4 k^3 + \frac{205}{96}a^5 k^4) \cos 3kx + (\frac{13}{48}a^4 k^3 \quad (3.74)$$

$$+ \frac{1}{8}a^5 k^4) \cos 4kx + \frac{25}{96}a^5 k^4 \cos 5kx + (\frac{1}{2}a^2 k + \frac{1}{4}a^3 k^2 + \frac{13}{16}a^4 k^3 + \frac{3}{8}a^5 k^4)$$

Shifting the origin, we have

$$\eta = (- a - \frac{3}{4}a^3 k^2 + \frac{3}{8}a^4 k^3 - \frac{161}{48}a^5 k^4) \cos kx + (\frac{1}{2}a^2 k - \frac{1}{4}a^3 k^2 + \frac{19}{12}a^4 k^3$$

$$- \frac{1}{2}a^5 k^4) \cos 2kx + (- \frac{1}{4}a^3 k^2 + \frac{1}{8}a^4 k^3 - \frac{205}{96}a^5 k^4) \cos 3kx + (\frac{13}{48}a^4 k^3 \quad (3.75)$$

$$- \frac{1}{8}a^5 k^4) \cos 4kx - \frac{25}{96}a^5 k^4 \cos 5kx + (\frac{1}{2}a^2 k - \frac{1}{4}a^3 k^2 + \frac{13}{16}a^4 k^3 - \frac{3}{8}a^5 k^4)$$

We ignore the last term, that is, $(\frac{1}{2}a^2 k - \frac{1}{4}a^3 k^2 + \frac{13}{16}a^4 k^3 - \frac{3}{8}a^5 k^4)$ to agree with the observed wave from

$$\eta = - a \cos kx + (\frac{1}{2}a^2 k - \frac{1}{4}a^3 k^2 + \frac{19}{12}a^4 k^3 - \frac{1}{2}a^5 k^4) \cos 2kx + (- \frac{1}{4}a^3 k^2$$

$$+ \frac{1}{8}a^4 k^3 - \frac{205}{96}a^5 k^4) \cos 3kx + (\frac{13}{48}a^4 k^3 - \frac{1}{8}a^5 k^4) \cos 4kx - \frac{25}{96}a^5 k^4 \cos 5kx \quad (3.76)$$

while the term $-\frac{3}{4}a^3 k^2 + \frac{3}{8}a^4 k^3 - \frac{161}{48}a^5 k^4 = 0$ because, a linear wave term cannot contain terms

involving the product of wave amplitude. Equation (3.76) is the desired fifth order Stokes waves. It is a superposition of the first order to fourth order in addition to extra term having wave number five times that of the linear solution. The numerical calculations involving the foregoing developments are detailed in the appendix.

4.0 Discussion of results

For a given wavelength of 99.82m, the fifth order has a wave height of about 54m above sea level while the wave height of fourth order is about 45m for same amplitude as shown in Figure 1. The approximate wave height for third order is 25m, second order is 20m and first order is 14m as shown in Figure 2. Hence, the higher the order of Stokes waves, the higher the different wave heights above sea level obtained as it is observed from first to fifth order approximations. However, there is a significant difference with order four and order five when compared with order one to three in terms of wave height.

5.0 Conclusion

From the analysis, it is observed that the approximate solution of fifth order Stokes waves has similar wave profile with those of the first to fourth order. The analysis also suggests that there is a gradual increase in the wave height beginning from the first to the fifth order.

The fifth order Stokes waves approximation may be applicable in deep waters to determine the wave heights of water waves that exhibit unusual characteristics such as rigid profile and speed greater than those of linear waves.

The wave heights predicted by present numerical calculations may appear higher than those from observation. This may be explained by the fact that the present calculations ignored some of the physical forces such as windstorms over ocean surfaces. These forces may reduce wave heights usually observes. However, the order of magnitude in the calculations is sufficiently near to reality.

Further, the fifth order is a superposition of all orders to the fourth. This increase with increasing order will not continue due to inherent damping and waves dispersion usually inherent in the system.
However, the outcome of the investigation is quite interesting in that it suggests the geophysical realities if wave modes are increased to the extent allowed by analytical difficulties.

Figure 1: Graph of 4th and 5th order (Under Stormy Weather)

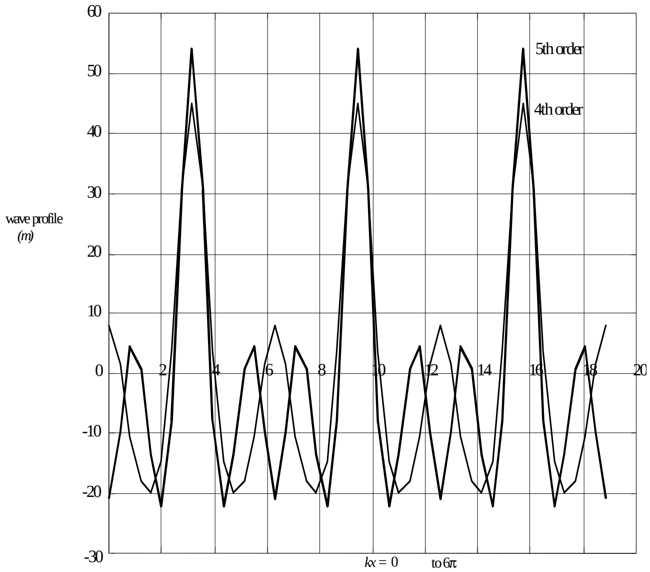
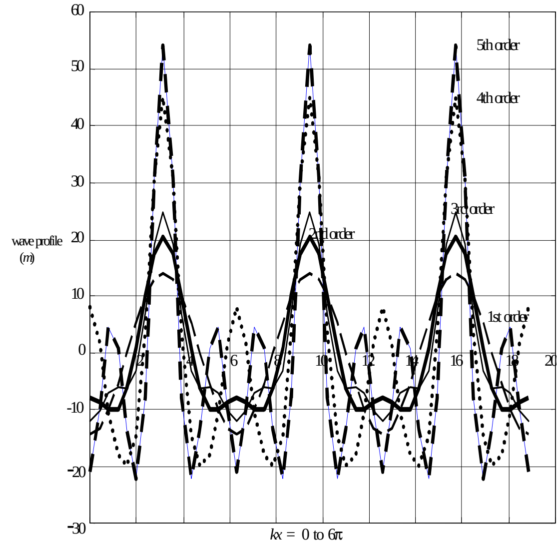


Figure 2: Graph of 1st, 2nd, 3rd, 4th and 5th order (Under Stormy Weather)



APPENDIX

NUMERICAL COMPUTATION OF FIFTH ORDER

The input data are obtained from deep water finite amplitude waves.
They are as follows:
L = 99.8200m
a = 14.1744m
k = 0.0629
kx = 0.0629x

FIFTH ORDER APPROXIMATION

```
» y1 = -a*cos(kx);
» y2 = [(0.5)*(a^2)*k - (1/4)*(a^3)*(k^2) + (19/12)*(a^4)*(k^3) –
        (0.5)*(a^5)*(k^4)]*cos(2*(kx));
» y3 = [(-0.25)*(a^3)*(k^2) + (1/8)*(a^4)*(k^3) - (205/96)*(a^5)*(k^4)]*cos(3*(kx));
» y4 = [(13/48)*(a^4)*(k^3) - (1/8)*(a^5)*(k^4)]*cos(4*(kx));
» y5 = (-25/96)*(a^5)*(k^4)*cos(5*(kx));
» y = y1 + y2 + y3 + y4 + y5
y =
 Columns 1 through 7
 -20.7016   -9.5663    4.6949    1.0066  -13.3474  -22.1508   -7.9025
 Columns 8 through 14
  30.7104   53.8115   30.7104   -7.9025  -22.1508  -13.3474    1.0066
 Columns 15 through 21
  4.6949   -9.5663  -20.7016   -9.5663    4.6949    1.0066  -13.3474
 Columns 22 through 28
 -22.1508   -7.9025   30.7104   53.8115   30.7104   -7.9025  -22.1508
 Columns 29 through 35
 -13.3474    1.0066    4.6949   -9.5663  -20.7016   -9.5663    4.6949
 Columns 36 through 42
   1.0066  -13.3474  -22.1508   -7.9025   30.7104   53.8115   30.7104
 Columns 43 through 49
  -7.9025  -22.1508  -13.3474    1.0066    4.6949   -9.5663  -20.7016
» plot (kx,y)
» xlabel('kx = 0 to 6π);
» ylabel('wave profile');
» title ('GRAPH OF FIFTH ORDER');
» grid on
```

COMPUTATION TO PLOT MULTIPLE GRAPHS

```
» y1 =  -a*cos(kx);
» y2 =  -a*cos(kx) + 0.5*k*(a^2)*cos(2*(kx));
» y3 =  -a*cos(kx) + 0.5*k*(a^2)*cos(2*(kx)) - (3/8)*(k^2)8(a^3)*cos(3*(kx));
```

```
» y3 = -a*cos(kx) + 0.5*k*(a^2)*cos(2*(kx)) - (3/8)*(k^2)*(a^3)*cos(3*(kx));
» y4 = -a*cos(kx) + [0.5*k*(a^2) + (11/6)*(k^3)*(a^4)]*cos(2*(kx)) - (3/8)*(k^2)*(a^3)*cos(3*(kx)) +
(1/3)*(k^3)*(a^4)*cos(4*(kx));
»
» y11 =  -a*cos(kx);
» y22 =  [0.5*(a^2)*k - (1/4)*(a^3)*(k^2) + (19/12)*(a^4)*(k^3) -          0.5*(a^5)*(k^4)]*cos(2*(kx));
» y33 =  [-0.25*(a^3)*(k^2) + (1/8)*(a^4)*(k^3) - (205/96)*(a^5)*(k^4)]*cos(3*(kx));
» y44 =  [(13/48)*(a^4)*(k^3) - (1/8)*(a^5)*(k^4)]*cos(4*(kx));
» y55 =  (-25/96)*(a^5)*(k^4)*cos(5*(kx));
» y5  =  y11 +  y22  +  y33  +  y44  +  y55;
»
» plot (kx,y4,kx,y5)
» xlabel ('kx = 0 to 6π');
» ylabel ('wave profile');
» grid on
» title ('GRAPH OF FOURTH AND FIFTH ORDER');
»
» plot (kx,y1,kx,y2,kx,y3,kx,y4,kx,y5);
» xlabel ('kx = 0 to 6π');
» ylabel ('wave profile');
» title ('GRAPH OF FIRST,SECOND,THIRD, FOURTH AND FIFTH ORDER');
» grid on
```

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