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Q-Step methods for Newton-Jacobi operator equation.

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Abstract

The paper considers the Newton-Jacobi operator equation for the solution of nonlinear systems of equations. Special attention is paid to the computational part of this method with particular reference to the q-step methods.

Keywords: Q-Step Method, Newton-Jacobi Method, Systems of Nonlinear Equations, Gauss-Siedel Method.

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1.0 Introduction.

With a given approximate solution ξ to a nonlinear system of equations we can construct bounds around $\xi \in R^n$ for which it can be proved that there exists unique solution x^* such that $F(x*)=0$.

We assume that our hypotheses are motivated by differentiability conditions on F and that the Jacobian det $(F'(x*))\neq 0$. This means that $F'(x)$ is a continuously invertible linear operator for each x and
Sup $\left\{\left\|(F'(x))^{-1}\right\| : \|x\| \le r\right\} < \infty, \text{for } r > 0$.

In this paper we construct a family of continuous hybrid methods that will be referred to as the q-step methods for the solution of nonlinear systems of equations. Most nonlinear systems solvers use Newton methods, which can be written in the form

$$x^{(k+1)} = x^{(k)} \cap N(x^{(k)}),\ k = 0,1,\Lambda \qquad (1.1)$$

where the Newton operator N is defined, for instance by

$$N(x) = x - VF(x) \qquad (1.2)$$

and V denotes the Jacobian matrix $\left[F'(x^{(k)})^{-1}\right]$. More definitive in our approach is that such evaluation of a related operator $N(x) \subseteq x$ can be viewed as existence and uniqueness for $F(x)=0$ in the interior $x \in D_0$. Practically, we compute a vector d to contain the solution set to the linear system

$$Ad = -F(\xi) \qquad (1.3)$$

where A is usually a lipschitz matrix for F and $\xi \in x$ is a guess point. The linear system (1.3) is well posed in the sense of Hadamard in that

- the solution d exists for each $F(\xi)$,
- the solution is unique, and

- the solution depends continuously on the initial data.

Let us note in passing that the uniqueness and verification of method (1.1) depends on the regularity of V (see Ortega and Rheinboldt [4]). Fundamentally, we have

Definition *1*

(Ortega and Rheinboldt [4]). *The operator N in method (1.1) is called a P contraction if there exists a real matrix $P \in R^{n\times n}$, with spectral radius $\rho(P) < 1$ such that $q(N(x), N(y)) \leq pq(x,y)$ where $x,\ y$ are in the interior of $D_0 \in R^n$.*

A promising strategy for the convergence of (1.1) is that it be norm reducing in the sense that

$$\left\|F\left(x^{(k)}\right)\right\| \leq \left\|F\left(x^{k-1}\right)\right\| \tag{1.4}$$

holds in some norm.

Newton's method (1.1) can be modified for the linear system solvers (1.3). Such philosophical consideration

can be obtained by the use of Jacobi and gauss Siedel methods. This forms the plank of our consideration as discussed among others (Voigt [6, 7], More [3], Ortega and Rheinboldt [4] and Brent [1]).

To motivate our interest, we organized the remaining sections in this paper as follows. In section two, we reviewed the methods of linear iterative solvers -Jacobi, Gauss-Siedel as well as successive over relaxation parameter (S.O.R) methods. In particular, we paid special attentions to the computational aspect of the q-step method for the Newton-Jacobi operator equation since convergence of Gauss- Siedel or over relaxation parameter method is dependent on the convergence of Jacobi's method. In section three, we discussed our method with numerical example.

2.0 Construction of Linear Iterative Solvers

Let $A = F'(x)$ be a non-singular matrix, then a linear system of equation

$$Ad^{(m)} = -F\left(x^k\right),\ m = 0,1,\Lambda \tag{2.1}$$

with exact solution $d^{(m)} = A^{-1}\left(-F\left(x^k\right)\right)$ can be obtained. We assume that there is a non-singular matrix H such that the matrix A can admit the decomposition:

$$A = H - (H - A) \tag{2.2}$$

We introduce an iteration function of type

$$d^{(m+1)} = \Phi\left(d^m\right)\ \ k = 0,1,\ \Lambda\ ,m = 0,1\Lambda \tag{2.3}$$

from which we can write

$$Hd^{(m+1)} + (A - H)d^{(m)} = -F\left(x^{(k)}\right). \tag{2.4}$$

Thus there follows $d^{(m+1)} = d^{(m)} - H^{-1}\left(Ad^{(m)} + F\left(x^{(k)}\right)\right)$. That is,

$$d^{(m+1)} = \left(I - H^{-1}A\right)d^{(m)} - H^{-1}F\left(x^{(k)}\right) \tag{2.5}$$

Inview of (2.5), method (1.1) will be written as

$$x^{(k+1)} = x^{(k)} + \lambda^{(k)}d^{(m)},\ k = 0,1,\Lambda \tag{2.6}$$

where $\lambda^{(k)}$ is a descent direction, and is the spectral radius of the matrix whose

$$W = \left[\begin{array}{c|c} 1 & 0 \\ \hline H^{-1}F & 1 - H^{-1}A \end{array}\right] \tag{2.7}$$

dominant eigenvalue is 1. Usually, we take $\lambda = 1$ an indication of the Brezis-Browder principle that provides abstract Newton Kantorovich scheme for solving system of nonlinear equations (of Ortega and Rheinboldt [4]).
Different choices of H will lead to different iterative methods. Interestingly, cases occur if we take H to be the diagonal or lower triangular part of A. Thus we will be led to the familiar Newton-Jacobi or Newton-Gauss-Siedel methods:

$$d_i^{(m+1)} = \frac{1}{a_{ij}}\left[-F\left(x^{(k)}\right) - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} d_j^{(m)} \right], \ m = 0,1,\Lambda \qquad (2.8)$$

where $a_{ij} \in A_{ij}$ and

$$d_i^{(m+1)} = \frac{1}{a_{ij}}\left[-F\left(x^{(k)}\right) - \sum_{j=1}^{i-1} a_{ij} d_j^{(m+1)} - \sum_{j=i+1}^{n} d_{ij} d_j^{(m)} \right], \ m = 0,1,\Lambda \qquad (2.9)$$

Using methods (2.8) and (2.9) the Newton operator expressed in (1.1) will now be written thus

$$x^{(k+1)} = x^{(k)} + \frac{1}{a_{ij}}\left[-F\left(x^{(k)}\right) - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} d_j^{(m)} \right] \qquad (2.10)$$

(Newton-Jacobi method) and

$$x^{(k+1)} = x^{(k)} + \frac{1}{a_{ij}}\left[-F\left(x^{(k)}\right) - \sum_{j=1}^{i-1} a_{ij} d_j^{(m+1)} - \sum_{j=i+1}^{n} a_{ij} d_j^{(m)} \right], \ k = 0,1,\Lambda \ , \ m = 0,1,\Lambda \qquad (2.11$$

(Newton-Gauss-Siedel Method).

Suppose instead we start the linear system (1.3) directly with the Gauss-Siedel method (2.11) and introduce the first components of auxiliary Vector iterates $d^{(m)}$ in the form

$$a_{ii}\, \bar{d}^{(m+1)} = -\sum_{j=1}^{i-1} a_{ij} d_j^{(m+1)} - \sum_{j=i+1}^{n} a_{ij} d_j^{(m)} - F\left(x^{(k)}\right) \qquad (2.12)$$

then the exact components $d_i^{(m+1)}$ can be written in the form:

$$d_i^{(m+1)} = d_j^{(m)} + \omega\left\{\bar{d}_i^{(m+1)} - d_i^{(m)}\right\} = (1-\omega) d_i^{(m)} + \omega \bar{d}_i^{(m+1)} \qquad (2.13)$$

We call ω the relaxation parameter. If we plug equations (2.12) and (2.13) together, we have

$$d_i^{(m+1)} = d_i^{(m)} + \frac{\omega}{a_{ij}}\left\{ -\sum_{j=1}^{i-1} a_{ij} d_j^{(m+1)} - \sum_{j=i+1}^{n} a_{ij} d_j^{(m)} - a_{ij} d_i^{(m)} - F\left(x^{(k)}\right) \right\} \qquad (2.14)$$

As a follow up in this consideration, we wish to construct q-step-methods for Newton-Jacobi operator equations. The relevant details can be obtained from the pioneering work of Wang and WU [8] see also [5]. Hence we have the algorithm in the form

$$d_i^{\left(m+\frac{\lambda+1}{q}\right)} = \frac{1}{a_{ii}}\left[-F\left(x^{(k)}\right) - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} d_j^{\left(m+\frac{\lambda}{q}\right)} \right] \qquad (2.15)$$

and

$$d_i^{\left(m+\frac{\lambda+1}{q}\right)} = \frac{1}{a_{ii}}\left[-F\left(x^{(k)}\right) - \sum_{j-1}^{i-1} a_{ij} d_j^{\left(m+\frac{\lambda+1}{q}\right)} - \sum_{j=i+1}^{n} a_{ij} d_j^{\left(m+\frac{\lambda}{q}\right)} \right] \qquad (2.16)$$

$$\left(q = 2,\Lambda \ , \ \lambda = 0,\ 1,\Lambda \ , q-1, \ \ m = 0,\ 1,\Lambda \quad k = 0,\ 1\Lambda \ ,\right).$$

The most favourable choice for the value of q is 2.

3.0 **Numerical results.**

Consider the test problem 1 (Burden and Fairs [2])

$3x_1 - cos(x_2 x_3) - \frac{1}{2} = 0$, $x_1^2 - 81(x_2 + 0.1)^2 + sinx_3 + 1.06 = 0$, $e^{-x_1 x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0$, Then

$x^{(0)} = (0.1, 0.1, -0.1)$. $J(x) = \begin{bmatrix} 3 & x_3 \sin(x_2 x_3) & x_2 \sin(x_2 x_3) \\ 2x & -162(x_2 + 0.1) & \cos x_3 \\ -x_2 e^{-x_1 x_2} & -x_1 e^{-x_1 x_2} & 20 \end{bmatrix}$.

Table 1 shows the numerical result for the methods.

No. of iterations *k*	Newton-Jacobi Operator (2.10)	*q*-step Newton-Jacobi Operator (2.15)	*q*-Step-Newton-Gauss-Siedel Operator (2.16)
0	(0.1, 0.1, -0.1)	(0.1, 0.1, -0.1)	(0.1, 0.1, -0.1)
1	0.500002898 0.022430579 -0.521505141	0.499998194 0.022532911 -0.521504680	0.499998194 0.022430546 -0.521505164
2	0.499999836 0.026805394 -0.523076442	0.499999865 0.026775562 -0.522933728	0.499999853 0.026779265 -0.522933639
3	0.499999993 0.026736944 -0.522905029	0.499999991 0.026773150 -0.522933907	0.599999991 0.026773133 -0.522933907
4	0.499999972 0.026727274 -0.522934809		
5	0.4499998442 0.026728716 -0.522935075		

Table 2 Shows numerical values of $F(x^{(k)})$

No. of iterations (k)	Newton-Jacobi Operator (2.10) $F(x^{(k)})$	q-Step Newton-Jacobi Operator (2.15) $F(x^{(k)})$	q-Step-Newton-Gauss-Siedel Operator (2.16) $F(x^{(k)})$
0	-1.199999985 -2.171745328 8.462025346	-1.199999985 -2.171745328 8.0462025346	-1.199999985 -2.171745328 8.462025346
1	0.000008714 0.086772061 0.030719995	-0.0000054 0.085647052 0.030678724	-0.000005401 0.087678175 0.030719655
2	-0.000000462 -0.001577687 -0.003430003	-0.000000380 0.000049759 0.000002391	-0.000000416 -0.000125822 0.000002345
3	0.000000062 0.000168785 0.000595421	-0.000000002 -0.000000099 -0.000000001	0.000000002 0.000000249 0.000000006
4	-0.000000054 0.000029205 0.000004591		

4.0 **Conclusion.**

From Table 1, it is evident that the numerical results obtained from problem 1 are reasonably good enough to the requested zeros for $F(x)$. This is corroborated from Table 2. It is quite glaring to note that the q-step Newton-Jacobi's operator equation (2.15) performed substantially better than the Classical Newton Jacobi's method (2.10). It is also not known from the computational experience if the q-step Newton-Gauss-Siedel method (2.16) has any computational advantage over the q-step Newton-Jacobi's method.

The great consolation we have is potentiated with the fact that both q-step methods of Newton-Jacobi and Newton-Gauss-Siedel performed acceleratively better than the classical Newton Jacobi's method and they mapped the convex, compact set

$x \in R^n$ into itself as evidenced by the Schauder fixed point theorem (or a similar weaker assumption such as Miranker's theorem). The continuity of these methods follows from that of F. Because of the non-singularity of the Jacobian matrix, a fixed point of $N(x)$ in method (1.1) is a solution of $F(x) = 0$.

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