Journal of the Nigerian Association of Mathematical Physics,
***Volume 8* (*November 2004*)**

Complete Schwarzschild's planetary equation.

S. X. K. Howusu and **D.D Bakwa**
Department of physics,
University of Jos, Jos, Nigeria
e-mail:howusus@yahoo.co.uk

Abstract

In this paper we derive the complete planetary equation from the Schwarzschild's equations of motion and compare it with the corresponding well-known planetary equation from the Schwarzschild's line element.

pp 265 -268

1.0 Introduction

In the year 1916, K. Schwarzschild published [K. Schwarzschild, S.B. Preuss. Akad. Wiss., 189 (1916)] the solution of the Einstein's geometrical gravitational field equations due to a homogeneous spherical massive body of radius R and rest mass M_0. That solution paved the way for the derivation of the Einstein's geometrical equations of motion for test particles of nonzero rest masses in the gravitational fields exterior to a homogeneous spherical body, which are given in the Spherical polar coordinates (t, r, θ, ϕ) with origin at the centre of the body as

$$\ddot{t} + \frac{2k}{c^2 r^2}\left(1 - \frac{2k}{c^2 r}\right)^{-1} \dot{t}\dot{r} = 0 \qquad (1.1)$$

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0 \qquad (1.2)$$

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\,\dot{\theta}\dot{\phi} = 0 \qquad (1.3)$$

$$\left\{\ddot{r} + \frac{k}{r^2}\left(1 - \frac{2k}{c^2 r}\right)\dot{t}^2 - \frac{k}{c^2 r^2}\left(1 - \frac{2k}{c^2 r}\right)^{-1}\dot{r}^2 - r\left(1 - \frac{2k}{c^2 r}\right)\dot{\theta}^2 - r\left(1 - \frac{2k}{c^2 r}\right)\sin^2\theta\,\dot{\phi}^2\right\} = 0 \qquad (1.4)$$

where

$$k = GM_0 \qquad (1.5)$$

and G is the universal gravitational constant, c is the speed of light in vacuo and a dot denotes one differentiation with respect to proper time, τ. These four equations are henceforth called the Schwarzchild's equations of motion. They also paved the way for the derivation of the corresponding, Einstein's line element in the gravitational field exterior to a homogeneous spherical body as

$$ds^2 = c^2 d\tau^2 = \left\{c^2\left(1 - \frac{2k}{c^2 r}\right)dt^2 - \left(1 - \frac{2k}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2\right\} \qquad (1.6)$$

called the Schwarzschild's line element.

According to the relativity authorities the Schwarzs child's equation of motion (1.1) – (1.4) form a test particle confined to the equatorial plane in the gravitational field exterior to a homogenous spherical body are " complicated " [1, 2, 3]. Therefore they normally resort to the analysis or manipulation of the Schwarzschild's line element as an equation of motion. The results are the well-known Einstein's equations of motion in the equatorial plane

$$c^2 = \left\{c^2\left(1-\frac{2k}{c^2 r}\right)\dot{t}^2 - \left(1-\frac{2k}{c^2 r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2\right\} \qquad (1.7)$$

And the corresponding radial equation

$$\ddot{r} = -\frac{k}{r^2}+\frac{l^2}{r^3}-\frac{3\,l^2 k}{c^2 r^4} \qquad (1.8)$$

And corresponding planetary equation

$$\frac{d^2u}{d\phi^2}+u=\frac{k}{l^2}+3k\,u^2 \qquad (1.9)$$

In this paper we analyse the Schwarzschils's equations of motion (1.1) - (1.4) and deduce the corresponding equation of motion in the equatorial plane and corresponding radial equation and corresponding planetary equation for comparison with the well equations known the line element and application to the motions of test particles such as planets, comets and asteroids in the solar system.

2.0 Analysis

Consider a particle moving in the equatorial plane exterior to a homogenous spherical body. For this test particle

$$\theta \equiv \pi/2 \qquad (2.1)$$

Hence the Schwazschild's equation of motion in the time direction (1.1) integrates exactly to yield:

$$\dot{t} = A\left(1-\frac{2k}{c^2 r}\right)^{-1} \qquad (2.2)$$

Hence under the experimentally well established condition that $t=\tau, \quad r=\infty$ (2.3)

where τ is proper time, it follows that

$$\dot{t} = \frac{dt}{d\tau} = \left(1-\frac{2k}{c^2 r}\right)^{-1} \qquad (2.4)$$

This expression also corresponds to time dilation in the gravitational field exterior to a homogeneous spherical body. It is therefore henceforth called the expression for gravitational time dilation according to the Schwarzschild's equation of motion.

The Schwarzschild's equation of motion in the azimuthal angular direction (1.2) integrates exactly to yield:

$$\dot{\phi} = \frac{l}{r^2} \qquad (2.5)$$

where l is the constant angular momentum per unit mass. This is the same as the pure Newtonian/angular momentum per unit mass.

The Schwarzschild's equation of motion in the radial direction (1.4) is given by

$$\ddot{r} - \frac{k}{c^2 r^2}\dot{r}^2 = \left\{ -\frac{k}{r^2}\left(1-\frac{2k}{c^2 r}\right)\dot{t}^2 + r\left(1-\frac{2k}{c^2 r}\right)\dot{\phi}^2 \right\} \qquad (2.6)$$

It therefore follows from the expression for gravitational time dilation according to the Schwarzschild's equations of motion (3.4) and the azimuthal angular speed (3.5) that

$$\ddot{r} - \frac{k}{c^2 r^2}\left(1-\frac{2k}{c^2 r}\right)^{-1}\dot{r}^2 = -\frac{k}{r^2}\left(1-\frac{2k}{c^2 r}\right)^{-1} + \frac{l^2}{r^3}\left(1-\frac{2k}{c^2 r}\right) \qquad (2.7)$$

Hence by the transformation $\dot{r}(r) = \omega(r)$ (2.8)

(2.7) becomes

$$\omega\frac{d\omega}{dr} - \frac{k}{c^2 r^2}\left(1-\frac{2k}{c^2 r}\right)^{-1}\omega^2 = \left\{ -\frac{k}{r^2}\left(1-\frac{2k}{c^2 r}\right)^{-1} + \frac{l^2}{r^3}\left(1-\frac{2k}{c^2 r}\right) \right\} \qquad (2.9)$$

This is Bernoulli's differential equation. Hence by the transformation

$$z(r) = \omega^2(r) \qquad (2.10)$$

it becomes

$$\frac{dz}{dr} + P(r)z = Q(r) \qquad (2.11)$$

where

$$P(r) = -\frac{2k}{c^2 r^2}\left(1-\frac{2k}{c^2 r}\right)^{-1} \qquad (2.12)$$

and

$$Q(r) = \left\{ -\frac{2k}{r^2}\left(1-\frac{2k}{c^2 r}\right)^{-1} + \frac{2\ l^2}{r^3}\left(1-\frac{2k}{c^2 r}\right) \right\} \qquad (2.13)$$

Equation (2.11) is a linear second order non homogeneous equation and hence has general solution

$$\dot{r}^2(r) = c^2 + \left(B - \frac{l^2}{r^2}\right)\left(1-\frac{2k}{c^2 r}\right) \qquad (2.14)$$

where B is an arbitrary constant. Let r_i be any apsidal of the motion $\dot{r} = 0;\quad r = r_i$ (2.15)

Then (2.14) becomes, after some rearrangement:

$$\dot{r}^2(r) = \left\{ c^2\left[1-\left(1-\frac{2k}{c^2 r_i}\right)^{-1}\left(1-\frac{2k}{c^2 r}\right)\right] - l^2\left(\frac{1}{r^2}-\frac{1}{r_i^2}\right)\left(1-\frac{2k}{c^2 r}\right) \right\} \qquad (2.16)$$

This is the exact expression for the instantaneous radial speed in terms of the radial coordinate, from the Schwarzschild's equations of motion. It therefore follows from (2.16) and (2.7) and some manipulation that

$$\ddot{r}(r) = \left\{ -\frac{k}{r^2}\left[\left(1-\frac{2k}{c^2 r_i}\right)^{-1} - \frac{l^2}{c^2 r_i^2}\right] + \frac{l^2}{r^3} - \frac{3kl^2}{c^2 r^4} \right\} \qquad (2.17)$$

This is the exact expression for the instantaneous radial acceleration in terms of the radial coordinate from the Schwarzschild's equations of motion. Finally it follows by the transformation

$$u(r) = \frac{1}{r} \qquad (2.18)$$

and the angular solution (2.5) and some manipulation that

$$\frac{d^2 u}{d\phi^2} + u = \left\{ \frac{k}{l^2}\left[\left(1-\frac{2k}{c^2 r_i}\right)^{-1} - \frac{l^2}{c^2 r_i^2}\right] + \frac{3k}{c^2}u^2 \right\} \qquad (2.19)$$

This is the exact planetary equation from the Schwarzschild's equations of motions.

3.0 Summary and conclusion

In this paper we derived Schwarzschild's equations of motion in the equatorial plane exterior to a homogeneous spherical body as equations (2.4), (2.5), (2.16), (2.17) and (2.19).

In the first place it may be noted that the gravitational time dilation expression from the Schwarzschild' s equations of motion in this paper (2.4) is not the same as the well-known gravitational time dilation expression from the Schwarzschild's line element given by

$$\frac{dt}{d\tau} = \dot{t} = \left(1 - \frac{2k}{c^2 r}\right)^{-\frac{1}{2}} \qquad (3.1)$$

In the second place it may be noted that the radial equation of motion in the equatorial plane exterior to a spherical body from the Schwarzschild's equation in this paper (2.5) is not the same as the corresponding well-known equation from the Schwarzschild's line element (1.7).

In the third place it may be noted that the instantaneous radial acceleration in the equatorial plane exterior to a spherical body from the Schwarzschild's equations in this paper (2.19) is not the same as the corresponding well-known equation from the Schwarzschild's line element (1.8).

In the fourth place it may be noted that the planetary equation in the equatorial plane exterior to a spherical body from the Schwarzschild's equations in this paper (2.19) is not the same as the corresponding well-known equation from the Schwarzschild's line element (1.9).

Finally, the door is hence forth opened for the solution of the planetary equation for a test particle in the equatorial plane exterior to a homogeneous spherical body (2.19) obtained from the Schwarzschild's equations in this paper for comparison with the corresponding well known planetary equation from the Schwarzschild's line element as well as experimental astronomical measurements.

References

[1] Anderson, J. L.(1967): Principle of Relativity Physics, Newyork:Academic press, pp 372-402.

[2] Anderson, J. L.(1967): Principle of Relativity Physics, Newyork:Academic press, pp 403-421.

[3] Arfhen, G. (1968): Mathematical Methods for Physicists, Newyork: Academic press, p 210.

[4] Weinberg, S. (1972): Gravitation and Cosmology: Principle and Application of the General Theory of Relativity, [5] Newyork: J. Willey and Sons, pp 152-172.

[6] Weinberg, S. (1972): Gravitation and Cosmology: Principle and Application of the General Theory of Relativity, [7] Newyork: J. Willey and Sons, pp 175-210.

[8] Foster, J. and Nigtingale, J. D. (1979): Consequences of the Schwarzschild's Metric. In: A Short Course in General Relativity, Newyork: Longman, pp 60-72