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Einstein's equations of motion in the gravitational field of an oblate spheroidal body

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Abstract

In an earlier paper we derived Einstein's geometrical gravitational field equations for the metric tensor due to an oblate spheroidal massive body. In this paper we derive the corresponding Einstein's equations of motion for a test particle of nonzero rest mass in the gravitational field exterior to a homogeneous oblate spheroidal massive body, expressed in oblate spheroidal coordinates convenient for mathematical investigation and hence physical interpretation and experimental investigation for bodies in the solar system.

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1.0 Introduction

It is well known how to formulate and solve Einstein's geometrical equations of motion for test particles of nonzero rest masses in the gravitational fields of massive spherical bodies. These equations, popularly known as the Schwarzchild's geodesic equations, are given in the spherical polar coordinates (r,θ,ϕ) with origin at the centre of the body, by

$$\ddot{t} + \frac{2k}{c^2 r^2}\left(1-\frac{2k}{c^2 r}\right)^{-1}\dot{r}\dot{t} = 0 \qquad (1.1$$

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0 \qquad (1.2)$$

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\,\dot{\theta}\dot{\phi} = 0 \qquad (1.3)$$

$$\left\{\ddot{r} + \frac{k}{r^2}\left(1-\frac{2k}{c^2 r}\right)\dot{t}^2 - \frac{k}{c^2 r^2}\left(1-\frac{2k}{c^2 r}\right)^{-1}\dot{r}^2 - r\left(1-\frac{2k}{c^2 r}\right)\dot{\theta}^2 - r\left(1-\frac{2k}{c^2 r}\right)\sin^2\theta\,\dot{\phi}^2\right\} = 0 \qquad (1.4)$$

where

$$k = GM_0 \qquad (1.5)$$

and M_o is the rest mass of the body and G is the universal gravitational constant. These equations have hitherto constituted the basis of the study of the motions of planets around their stars and artificial satellites and projectiles in the earth's atmosphere according to Einstein's geometrical theory of gravitation known as general relativity. But it is now well known experimentally that the earth and all the other planets as well as the sun are spheroidal in shape[4-9]. Therefore treating them as perfect spheres is at best an approximation for the sake of mathematical convenience. Moreover, spheroidal shape of a body will produce some corresponding non-spherical effects in the motions of test particles in its gravitational field. Therefore there has remained the need to extend the theory of motion in the Solar System from the fields of bodies of perfect spherical geometry to those of spheroidal geometry. And towards this goal we recently derived[1.] Einstein's geometrical field equations in the space-time due to an oblate spheroidal massive body. Therefore in this paper we derive and solve Einstein's geometrical equations of motion for test particles in general form in the field of a homogeneous oblate spheroidal massive body.

2.0 Equations of motion

In a recent paper we derived[1] Einstein's geometrical gravitation metric tensor exterior to a homogeneous oblate spheroidal body, in the oblate spheroidal coordinates (η,ξ,ϕ) with origin at the centre of the body as

$$g_{00}(\eta,\xi,\phi)=e^{-F} \qquad (2.1)$$

$$g_{11}(\eta,\xi,\phi)=-e^{-G} \qquad (2.2)$$

$$g_{22}(\eta,\xi,\phi)=-e^{-H} \qquad (2.3)$$

$$g_{33}(\eta,\xi,\phi)=-a^2(1-\eta^2)(1+\xi^2) \qquad (2.4)$$

where *F*, *G* and *H* are functions of the oblate spheroidal coordinates η and ξ only, and their equations are known[1]. But Einstein's geometrical equations of motion in gravitational fields are given by

$$\frac{d^2x^\alpha}{d\tau^2}+\Gamma^{\alpha}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=0 \qquad (2.5)$$

where τ is proper time and $\Gamma^{\alpha}_{\mu\nu}$ are the Riemann Christofel symbols[1], and x^α are the coordinates of space - time. In the oblate spheroidal coordinates with origin at the centre of the body

$$x^0=ct \qquad (2.6)$$

$$x^1=\eta \qquad (2.7)$$

$$x^2=\xi \qquad (2.8)$$

$$x^3=\phi \qquad (2.9)$$

where *t* is coordinate time and the connection coefficients are given in [1]` by

$$\Gamma^{0}_{01}=\Gamma^{0}_{01}=\frac{1}{2}F_\eta \qquad (2.10)$$

$$\Gamma^{0}_{02}=\Gamma^{0}_{20}=-\frac{1}{2}F_\xi \qquad (2.11)$$

$$\Gamma^{1}_{00}=-\frac{1}{2}e^{G-F}F_\eta \qquad (2.12)$$

$$\Gamma^{1}_{11}=-\frac{1}{2}G_\eta \qquad (2.13)$$

$$\Gamma^{1}_{12}=\Gamma^{1}_{21}=-\frac{1}{2}G_\xi \qquad (2.14)$$

$$\Gamma^{1}_{22}=\frac{1}{2}e^{G-H}H_\eta \qquad (2.15)$$

$$\Gamma^{1}_{33}=a^2\eta(1+\xi^2)e^{G} \qquad (2.16)$$

$$\Gamma^{2}_{00}=-\frac{1}{2}e^{H-F}F_\xi \qquad (2.17)$$

$$\Gamma^{2}_{11}=\frac{1}{2}e^{H-G}G_\xi \qquad (2.18)$$

$$\Gamma^{2}_{12}=\Gamma^{2}_{21}=-\frac{1}{2}H_\eta \qquad (2.19)$$

$$\Gamma^{2}_{22}=-\frac{1}{2}H_\xi \qquad (2.20)$$

$$\Gamma^{2}_{33}=-a^2\xi(1-\eta^2)e^{H} \qquad (2.21)$$

$$\Gamma^{3}_{13}=\Gamma^{3}_{31}=-\frac{\eta}{1-\eta^2} \qquad (2.22)$$

$$\Gamma_{2\,3}^{\;3} = \Gamma_{3\,2}^{\;3} = \frac{\xi}{1+\xi^2} \qquad (2.23)$$

$$\Gamma_{\mu\,\lambda}^{\;\sigma} = 0 \text{ , otherwise} \qquad (2.24)$$

It therefore follows from (2.5) and (2.10) – (1.24) that
$$0 = \ddot t + F_\eta \dot t \dot\eta + F_\xi \dot t \dot\xi \qquad (2.25)$$

and
$$0 = \ddot\eta - \frac{c^2}{2} e^{G-F} F_\eta - F_\xi \dot t^2 - \frac{1}{2} G_\eta \dot\eta^2 - G_\xi \dot\eta\dot\xi + a^2 \eta\left(1+\xi^2\right) e^G \dot\phi^2 \qquad (2.26)$$

and
$$0 = \ddot\xi - \frac{1}{2} c^2 e^{H-F} F_\xi \dot t^2 + \frac{1}{2} e^{H-G} G_\xi \dot\eta^2 - H_\eta \dot\eta\dot\xi - a^2\left(1+\eta^2\right)\xi e^H \dot\phi^2 \qquad (2.27)$$

and
$$0 = \ddot\phi - \frac{2\eta}{1-\eta^2} \dot\eta\dot\phi + \frac{2\xi}{1+\xi^2} \dot\xi\dot\phi \qquad (2.28)$$

The results (2.25) - (2.28) are the exact equations of motion for a test particle in the gravitational field of homogeneous oblate spheroidal body according to Einstein's geometrical laws of General Relativity.

3.0 General Azimuthal solution

Dividing through the azimuthal equation of motion (2.29) by $\dot\phi$ and integrating it follows that

$$\dot\phi(\eta,\xi,\phi) = \frac{\lambda}{\xi^2}\left(1-\eta^2\right)^{-1}\left(1+\frac{1}{\xi^2}\right)^{-1} \qquad (3.1)$$

where λ is a constant of the motion defining to the angular momentum per unit rest mass.

4.0 Solutions in the equatorial plane

For motion confined to the equatorial plane of the body,

$$\eta \equiv 0 \qquad (4.1)$$

Consequently the time equation (2.25) reduces to

$$0 = \ddot t - f'(\xi)\dot\xi \dot t \qquad (4.2)$$

where
$$f(\xi) = F(\eta,\xi)\big|_{\eta=0} \; . \qquad (4.3)$$

This equation integrates exactly to yield the solution

$$\dot t = A\, exp\{f(\xi)\} \qquad (4.4)$$

where A is an arbitrary constant. But by definition

$$\dot t \to 1 \;\; as \;\; \xi \to \infty \qquad (4.5)$$

and hence
$$A = exp\{-f(\xi)\}\big|_{\xi=\infty} \qquad (4.6)$$

In the second place in the equatorial plane the equation of motion in the $\hat\eta$ direction (2.26) reduces to

$$F_\eta(\eta,\xi)\big|_{\eta=0} \equiv 0 \qquad (4.7)$$

In the third place in the equatorial plane the azimuthal solution (3.1) reduces to

$$\dot\phi = \frac{\lambda}{\xi^2}\left(1+\frac{1}{\xi^2}\right)^{-1} \qquad (4.8)$$

In the fourth place in the equatorial plane the equation of motion in the $\hat\xi$ direction (2.27) subject to (4.4) and (4.8) yields
$$\ddot\xi = \frac{a}{\xi^3}\left(1+\frac{1}{\xi^2}\right)^{-2} exp\{h(\xi)\} + \frac{1}{2}c^2 A^2 f(\xi) exp\left[f(\xi)+h(\xi)\right] \qquad (4.9)$$

where
$$h(\xi) = H(\eta,\xi)\big|_{\eta=0} \qquad (4.10)$$

In principle the results (4.9), (4.8) and (4.4) constitute a sufficient solution of motion in the equatorial plane in terms of the radial coordinate ξ. But it may be most interesting and instructive to express the motion in terms of the angular coordinate ϕ. Towards this goal let u be the reciprocal "distance" defined by

$$u(\phi) = \frac{1}{\xi(\phi)} \tag{4.11}$$

Then it follows from (4.9) that

$$\ddot{\xi} = -l^2 u^2 \left(1+u^2\right)^{-1} \frac{d}{d\phi}\left\{\left(1+u^2\right)^{-1}\frac{du}{d\phi}\right\} \tag{4.12}$$

Consequently, (4.9) becomes

$$u^2\left(1+u^2\right)^{-2}\frac{d}{d\phi}\left\{\left(1+u^2\right)^{-1}\frac{du}{d\phi}\right\} = -\frac{a}{\lambda^2}u^3\left(1+u^2\right)^{-2} exp\left\{f\left(\frac{1}{u}\right)\right\} + \frac{1}{2}c^2A^2 f\left(\frac{1}{u}\right) exp\left\{f\left(\frac{1}{u}\right)+h\left(\frac{1}{u}\right)\right\} \tag{4.13}$$

or equivalently,

$$\frac{d^2u}{d\phi^2} - 2u\left(1+u^2\right)^{-2}\left(\frac{du}{d\phi}\right)^2 = -\frac{a}{\lambda^2}u\, exp\left\{h\left(\frac{1}{u}\right)\right\} + \frac{1}{2}c^2A^2\left(1+\frac{1}{u^2}\right) f\left(\frac{1}{u}\right) exp\left\{f\left(\frac{1}{u}\right)+h\left(\frac{1}{u}\right)\right\} \tag{4.14}$$

This is the planetary equation in the equatorial plane of a homogeneous oblate spheroidal body according to Eisnstein's geometrical laws of General Relativity. It is therefore now opened up for comparison with the well-known Einstein's planetary equation in the field of a homogeneous spherical massive body.

5.0 Summary and conclusion

In this paper we derived Einstein's geometrical equation of motion for a test particle in the gravitational field of a homogeneous oblate spheroidal massive body. Then we showed how they may be solved exactly and analytically for motions confined to the equatorial plane.

It is most interesting and instructive to note that the planetary equation (4.14) contains at least one term of order u^3 corresponding to orbital precession.

It is most interesting and instructive to note that Einstein's planetary equation (4.14) derived in this paper contains infinitely many pure spheroidal (non-spherical) terms and hence effects which are henceforth opened up for mathematical analysis and physical experimental and investigation in the motions of the planets, comets and asteroids in the Solar System.

The door is now opened for the mathematical analysis of Einstein's equations of motion for a test particle in the gravitational field exterior to a homogeneous oblate spheroidal massive body for more general motions other than the one confined to the equatorial plane of the body.

Finally it may be noted that Einstein's geometrical equations of motion for test particles in the gravitational field of a homogeneous oblate spheroidal body derived in this paper may now be compared with the corresponding Newton's dynamical equations of motion derived by us in a recent paper[10].

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