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On the theory of pre-p-*nil*-rings

Adewale Oladipo Oduwale
Department of Mathematics, University of Benin,
Benin City, Nigeria.
e-mail:askforadewale@yahoo.com

Abstract

Generalizing the concept of p-rings, Abian and Mcworter* [1] *call an associative and commutative ring $\Re$ with characteristic p a pre-p-ring if $xy^p = x^p y$ for every x and y in $\Re$. It was proved in* [1] *that every pre-p-ring $\Re$ is a direct sum R = B $\oplus$ N of a p-ring B and a nil ring N, where even $x^{p+2} = 0$ for every $x \in N$. It was also proved in* [1] *that N is the radical of R and hence N uniquely determined by R. Moreover, it is not difficult to show that B is also uniquely determined by R. A simple calculation shows that the converse that the direct sum R = B$\oplus$ N of a p-ring B and a pre-p-nil-ring N is a pre-p-ring. Since the structure of p-rings is known, there remains to investigate only the pre-p-nil-rings, which is the purpose of this paper.

Keywords: isomorphic, irreducibility, sub-directly, direct, annihilator

pp 273 - 276

1.0 Introduction

First, we determine the pre-*p*-nil-rings as certain algebras over the prime filed F_p of characteristic *p*. Many pre-*p*-nil-rings, which may be generated by one element, arise as semi group rings or as semi group rings with a factor system of F_p, but this is not true for all pre-*p*-nil-rings. In the second part, we deal with sub-direct decompositions of pre-*p*-nil-rings. We recall that every *p*-ring is sub-direct sum and in the finite case even a direct sum of prime fields F_p, [4], [5]; in particular, every sub-directly irreducible *p*-ring is isomorphic to F_p. Just so every pre-*p*-nil-ring is a sub-direct sum of sub-directly irreducible pre-*p*-nil-rings. But here we have for every characteristic *p* even an infinite number of sub-directly irreducible pre-*p*-nil-rings (likewise of finite as of infinite order), which are not isomorphic. Another contrast to the situation with *p*-rings is the existence of the finite pre-*p*-nil-rings, which are sub-directly reducible although they have no decomposition into a direct sum. Finally, for the sub-direct irreducibility of a pre-*p*-nil-ring *N*, it is necessary (Theorem 4) and in the finite case necessary and sufficient (Theorem 5), that the annihilator of *N* be a principal ideal different form zero (0).

2.0 An Algebra over F_p

By an algebra over F_p, we mean in the following a ring R (for our purposes always associative and commutative), which is also a vector space over the prime field F_p of characteristic *p*. Hence every element of R may be written uniquely in the form

$$\sum_{i \in I} a_i w_i, \; a_i \in F_p, \text{ only a finite number } a_i \neq 0 \tag{2.1}$$

by the help of a basis $\{w_i\} \subseteq R$, where *I* is an index system of an arbitrary cardinal number.

***Theorem* 1**

Let R be an algebra over F_p with a basis $\{w_i\}$ having the following properties:

(a) $w_i w_i^p = w_i^p w_j$ *for every* w_i *and* w_j

(b) *For every* w_i *there is a (minimal) natural number* $t = t(w_i) \geq 2$ *with* $w_i^t = 0$.
Then R is a pre-p-nil-ring and, conversely, every pre-p-nil-ring N is an algebra of this kind.

We note that (a) and (b) imply $t(w_i) \leq p+2$ for every w_i (see the proof of Lemma 1 in [1]), whereas conversely (a) is trivial if (b) always holds with $t(w_i) \leq p$.

Proof

By well-known rules for characteristics *p* we have for arbitrary elements (1) of the algebra R

$$\left(\sum a_i w_i\right)^{p2} = \sum a_i w_i^{p2} = 0 \,,$$

$$\left(\sum a_i w_i\right)\left(\sum b_j w_j\right)^p = \sum a_i b_j w_i w_j^p = \sum_j a_i b_j w_i^p w = \left(\sum a_i w_i\right)^p \left(\sum b_j w_j\right).$$

Hence *R* is a pre-*p*-nil-ring. Conversely, every pre-*p*-nil-ring *N* obviously is a unitary F_p-module and therefore a vector over F_p with a basis $\{w_i\} \subseteq N$. Now, (a) and (b) are true even for all elements of *N*.

3.0 Commutativity Semigroup with zero, 0

With the aid of this theorem many pre-*p*-nil-rings may easily be given. For example, one starts with a commutative semigroup $H = \{0, w_i\}, i \in I$ with zero 0, which satisfies (a) and (b), and takes $\{w_i\} = H \setminus \{0\}$ with the multiplication determined by *H* as basis of an algebra *R* over F_p. Such an algebra *R* is what we call here the *semigroup ring of H over* F_p. More precisely, in a first step one has to take also $0 \in H$ as a basis element of an algebra R^* over F_p, where $0 \neq 0^*$ for the zero 0^* of R^*. The considered ring *R* then arises by "identifying" of these zeros, that is, as residue class ring $\frac{R = R^*}{(0-0*)}$ of R^* modulo the ideal $(0-0^*)$; [6]. If we choose

$$H = \{w, w^2, \Lambda\ , w^{t-1}, w^t = w^{t+1} = \Lambda\ = 0\} \quad 2 \leq t \leq p+2 \qquad (3.1)$$

is the cyclic semigroup of order *t* and period 1 [2], (a) and (b) hold and we have

***Theorem* 2**

Every pre-p-nil-ring N, which is generated by one element, is a semgroup ring R of a cyclic semigroup H of period 1 and order t $(2 \leq t \leq p+2)$ *over* F_p *and conversely. Hence for a fixed characteristic p there exist exactly p + 1 pre-p-nil-rings (up to isomorphism), generated by one element.*

Proof

From the text above, it is clear that R is a pre-*p*-nil-ring, generated by the element *w*. Conversely, let *w* be a generating element of a pre-*p*-nil-ring *N*. Then all the powers of *w* form sub-semigroup (2) of *N*, and for the semigroup ring *R* of *H* over *Fp* we have $R \subseteq N$, from which by assumption *R* = *N*.

Let again $H = \{0, w_i\}$ be a (commutative) semigroup with zero 0. We say that *R* is a *semigroup ring of H over* F_p *with a factor system* $\{c_{w_i,w_j}\}$ if *R* is an algebra over F_p with the basis $\{w_i\}$, for which multiplication is defined by

$$w_i \circ w_j = c_{w_i,w_j} w_i w_j \,, \quad c_{w_i,w_j} \in F_p \qquad (3.2)$$

(Here we have also identified the zeros as explained in the paragraph preceding Theorem 2. Hence the values of the factors c_{w_i,w_j} for $w_i w_j = 0$ are irrelevant and may be chosen arbitrarily).

It is known that *R* is associative if and only if [6] $c_{w_i,w_j} c_{w_i w_j,w_k} = c_{w_i,w_j w_k} c_{w_j,w_k}$ for $w_i w_j w_k \neq 0$, whereas commutativity is preserved if and only if

$$c_{w_i,w_j} = c_{w_{ji} w_j} \text{ for } w_i w_j \neq 0 \,. \qquad (3.4)$$

The special case $c_{w_i,w_j} = 1 \in F_p$ for $w_i,\ w_j$ corresponds to the semigroup ring in the above sense.

Theorem 3

Let $H = \{0, w_i\}$ be a commutative semigroup with zero 0, which satisfies (a) and (b). Then every semigroup ring R of H over F_p with a factor system $\{c_{w_i,w_l}\}$ is a pre-p-nil-ring, if $c_{w_i,w_j} \neq 0$, for $w_i^2 \neq 0$.
(This condition is superfluous if $t(w_i) \leq p$ for every $w_i \in H$. On the other hand, it means no loss of generality, because all semigroup rings with factor systems (yet even all monomial algebras, [7]) are included, if one only considers factor systems with $c_{w_i,w_j} = 0 \Leftrightarrow w_i w_j = 0.$)

Proof

From $w_i^t = 0$, it follows immediately that $\underbrace{w_i \circ \ldots \circ w_i}_{t} = c^{t-1}_{w_i w_i} w_i^t = 0$. Moreover, the property (a) of the multiplication $(\cdot)$ of the w_i carries over the multiplication $\circ$ according to

$$w_i \circ \underbrace{w_j \circ \ldots \circ w_j}_{p} = c_{w_i w_j} c^{p-1}_{w_j,w_j} w_i w_j^p = c_{w_i,w_j} c^{p-1}_{w_i,w_i} w_i^p w_j = \underbrace{w_i \circ \ldots \circ w_i}_{p} \circ w_j$$

This is not trivial only in the case $w_i w_j^p = w_i^p w_j \neq 0$ and $c_{w_i w_j} = 0$. But then, by assumption, $c_{w_i w_i} \neq 0$ and $c_{w_j w_j} \neq 0$; hence the $(p-1)-th$ powers of these factors are equal to $1 \in F_p$. Therefore *R* is a pre-*p*-nil ring by Theorem 1.

We note at once that not all pre-*p*-nil-rings, not even those, which are finite or sub-directly irreducible, arise as semigroup rings with a factor system according to Theorem 3 (hence also not as monomial algebras, [7]). We restrict ourselves to a simple counter-example for characteristic *p* = 2 and consider the algebra *N* = *R* over F_2 with the basis $\{u, u^2, v, v^2, q\}$ and the multiplication table

	u	u^2	v	v^2	q
u	u^2	0	u^2+v^2	q	0
u^2	0	0	q	0	0
v	u^2+v^2	q	v^2	0	0
v^2	q	0	0	0	0
q	0	0	0	0	0

It is easy to see that by Theorem 1 *R* is a pre-2-nil-ring. But it is impossible to choose a basis $\{w_1, \ldots, w_5\} \subseteq R$ in such a way that for every w_i, w_j the product $w_i w_j$ is a multiple of one of the elements of this basis (that is, here of course equal to zero or to the basis element itself, since we have F_2 as operator domain). In order to prove this assertion, we observe first that in each basis there must be two elements w_1 and w_2 such that the terms *u* or *u* + *v* occur in w_1 and *v* or *u* + *v* occur in w_2 (and of course not *u* + *v* in both). Then we get from

$$w_1^2, w_2^2, w_1 w_2, w_1 w_2^2 = w_1^2 w_2 \qquad (3.5)$$

up to the order of succession the four linearly dependent elements u^2 or $u^2 + q, v^2$ or $v^2 + q, u^2 + q, u^2 + v^2$ or $u^2 + v^2 + q, q$ which cannot be obtained in the basis $\{w_i\}$.

4.0 Decomposition of pre-*p*-nil rings

The considerations of this section concerning the decomposition of pre-*p*-nil-rings as sub-direct sums (for he concepts and properties of sums, used in the following, [4]) are based on the following

Preposition 1

Every pre-p-nil-ring N is isomorphic to a sub-direct sum of sub-directly irreducible pre-p-nil-rings N_i, and conversely every sub-direct sum of pre-p-nil-rings is again a pre-p-nil ring.

The first part is an immediate consequence of the fact that a homomorphic image of a pre-*p*-nil-ring is also one. For the second, it suffices to show that the direct sum of pre-*p*-nil-rings N_i is also a pre-*p*-nil ring, which follows essentially from $x_i^{p+2} = 0$ for every x_i of each *pre-p*-nil-ring N_i.

Therefore our main interest is the study of sub-directly irreducible pre-*p*-nil-rings. As already mentioned in the introduction, the situation is here much more complicated than for *p*-rings, but, some information is given in what follows, where the annihilator *Q* of *N*, that is, the ideal of all elements $q \in N$ with $qx = 0$ for every $x \in N$, is of some importance.

Theorem 4

Every sub-directly irreducible pre-p-nil-ring N has an annihilator $Q \neq (0)$, which is a principal ideal of N and hence it consists of exactly p elements $q, 2q, \Lambda, pq = 0$

This assertion is even true for nilpotent rings with characteristic *p* and is a special case of Theorem 2 of [3].

We give first an example of pre-*p*-nil-ring without an annihilator $(\neq 0)$, which is then sub-directly reducible by Theorem 4. Let *H* be the commutative semigroup with zero 0, generated by the (countable) infinite set of elements u_1, u_2, Λ, with the defining relations

$$u_i^2 = 0 \quad \text{for every } i. \tag{4.1}$$

Then every nonzero element of *H* has a unique representation

$$u_{k_1} u_{k_2} \Lambda\ u_{k_m} \text{ with } k_1 \pi\ k_2\ \pi \Lambda\ \pi\ k_m \tag{4.2}$$

Now we regard the semigroup ring *N* of *H* over F_p (with arbitrary characteristic *p*), which is according to Section 1 a pre-*p*-nil-ring. This ring *N* has only the zero as annihilator; hence it is a ring of the desired kind. Indeed, every element $x \neq 0$ of *N* is a linear combination of a finite number of elements (4.2), and we have $xu_j \neq 0$ for every *j* greater than all the indices k_m of the factors u_{k_m}, which occur in the terms of *x*.

By the help of this example, we see that the converse of Theorem 4 is false. For this purpose we take the direct sum $N \oplus Q$ of *N* with a zero-ring *Q* of *p* elements. Then $N \oplus Q$ is a pre-*p*-nil-ring with principal ideal $Q \neq (0)$ as annihilator, though it is subdirectly reducible. But we have

Theorem 5

A finite pre-p-nil-ring N always has an annihilator $Q \neq (0)$ and it is subdirectly irreducible if and only if Q is a principal ideal.

Proof

For the first assertion we take an arbitrary element $x_1 \neq 0$ of *N*. Neither we have $x_1 x_2 \neq 0$ for every $x_2 \in N$, or there is an element $x_2 \in N$ with $x_1 x_2 \neq 0$. In the same manner we regard $(x_1 x_2) x_3$ and so on. Since *N* is finite and every element of *N* is nilpotent, we obtain in this way an annihilator element $q = x_1 x_2 \Lambda\ x_n \neq 0$. Hence $Q \neq (0)$. Moreover, it follows from this consideration, that every ideal $A \neq (0)$ of *N* contains at least one annihilator element $q \neq 0$. Therefore the intersection of all ideals $A \neq (0)$ of *N* contains *Q* and hence *N* is subdirectly irreducible. If the latter is true, then *Q* is a principal ideal by Theorem 4.

Corollary 1

Every pre-p-nil-ring N, generated by one element w, is subdirectly irreducible.

Proof

According to Theorem 2, *N* is the semigroup ring of the semigroup $w, w^2, \Lambda, w^{t(w)-1}$ and the annihilator of *N* is the principal ideal $Q = (w^{t(w)-1})$.

5.0 Conclusion

We note that subrings of pre-*p*-nil-ring *N* throughout may be subdirectly reducible. For example, suppose that $t(w) \geq 4$, and let *U* be the subring of *N* which is generated by the elements $w^2, \Lambda, w^{t(w)-1}$. Then $w^{t(w)-2} \neq 0$ and $w^{t(w)-1} \neq 0$ are elements of the annihilator *Q* of *U*, which is therefore not a principal ideal. Hence *U* is subdirectly reducible by Theorem 5. Contrasting the situation with finite *p*-rings (cf. the introduction), *U* is not directly reducible if $t(w) = 6$. In order to prove this, let us assume that $U = U_1 \oplus U_2$. Then at least in one of the U_i say in U_1, there must occur an element $x = cw^2 + \Lambda$ with $cw^2 \neq 0$, perhaps among other terms. Suitable multiplications with w^2 and w^3 show that $cw^{t(w)-1} \in U_1$, then $cw^{t(w)-2} \in U_1$ and therefore $Q \in U_1$. As we must have $Q \cap U_2 \neq 0$ (cf. the proof of Theorem 5), this contradicts $U_1 \cap U_2 = (0)$.

Moreover, besides the pre-*p*-nil rings generated by one element, there are many others, which are likewise subdirectly irreducible (for example, the ring presented at the end of Section 1 with the annihilator $Q = (q)$, or those subrings of the pre-*p*-nil-ring *N* without annihilator constructed after Theorem 4, which are generated by the elements

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