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Buys-Ballot estimates when stochastic trend is quadratic

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Abstract

The Buys-Ballot estimation procedure developed by **[3]** *for time series decomposition when trend-cycle component is linear is extended to cases where trend-cycle component is quadratic. Estimates are derived for the additive and multiplicative models. A simulated example is used to illustrate the methods developed. Buys-ballot estimates are compared with the least squares estimates.*

Key Words: Additive Model, Multiplicative Model, Quadratic Trend, Least Squares Estimates, and Buys-Ballot Estimates

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1.0 Introduction

As noted by Wei in [4], the analysis of seasonal time series with periodicity s (length of the periodic interval) requires the arrangement of the series in a two-dimensional table called Buys-Ballot table (Table 1). Table 1 shows the within-periods and between-periods relationships. Within-periods relationships represent the correlation among, $\Lambda, X_{t-2}, X_{t-1}, X_t, X_{t+2}, X_{t+2}, \Lambda$ and the between-periods relationships represent the correlation among $\Lambda, X_{t-2s}, X_{t-s}, X_t, X_{t+s}, X_{t+2s}, \Lambda$. In general, the within-periods relationships represent the non-seasonal part of the series while the between-periods relationships represent the seasonal part.

Iwueze and Nwogu (2004) [3] have developed an estimation procedure based on the row and column averages of the Buys-Ballot table for the parameters of the trend-cycle component and the seasonal indices. Their two alternative methods are (*i*) the Chain Base Estimation (CBE) method (computes slope from the relative periodic average changes) and (*ii*) the Fixed Base Estimation (FBE) method (computes slope using the first period as the base period for the periodic average changes). For the linear trend-cycle component,

$$M_f = a + bt,\ t = 1,2,\Lambda, n \qquad (1.1)$$

they have shown that the estimation of the parameters is the same for the additive and multiplicative models.

In their summary, [3] have shown that the estimation of the slope of the line is easily computed as a weighted average of the periodic averages. That is,

$$\hat{b} = \frac{1}{(n-s)}\left(\bar{X}_m - \bar{X}_1\right) \qquad \textbf{(1.2)}$$

for the CBE method, and

$$\hat{b} = \frac{1}{(n-s)}\sum_{i=2}^{m}\left\{\frac{\bar{X}_i - \bar{X}_1}{i-1}\right\} \qquad \textbf{(1.3)}$$

for the FBE method. After the estimation of the slope, the Buys-Ballot estimate of the intercept is

Table 1:Buys-Ballot table for a seasonal time series.

PERIOD	SEASON						TOTAL	AVERAGE
	1	**2**	…	***j***	…	***s***		
1	X_1	X_2	…	X_j	…	X_s	$T_{1.}$	$\bar{X}_1$
2	X_{s+1}	X_{s+2}	…	X_{s+j}	…	X_{2s}	$T_{2.}$	$\bar{X}_2$
$\vdots$	$\vdots$	$\vdots$	…	$\vdots$	…	$\vdots$	$\vdots$	$\vdots$
i	$X_{(i-1)s+1}$	$X_{(i-1)s+2}$	…	$X_{(i-1)s+j}$	…	X_{is}	$T_{i.}$	$\bar{X}_i$
$\vdots$	$\vdots$	$\vdots$	…	$\vdots$	…	$\vdots$	$\vdots$	$\vdots$
m	$X_{(m-1)s+1}$	$X_{(m-1)s+2}$	…	$X_{(m-1)s+j}$	…	X_{ms}	$T_{m.}$	$\bar{X}_m$
TOTAL	$T_{.1}$	$T_{.2}$	…	$T_{,j}$	…	$T_{.s}$	T	
AVERAGE	$\bar{X}_{.1}$	$\bar{X}_{.2}$	…	$\bar{X}_{.j}$	…	$\bar{X}_{.s}$		$\bar{X}_{\Lambda}$

where m = number of periods and s = length of the periodic interval (or length of periodicity), $n = ms$ = length of series.

$$\hat{a} = \overline{X} - \frac{\hat{b}}{2}(n+1) \quad (1.4)$$

Having obtained the Buys-Ballot estimates of the slope and intercept, estimates of the seasonal indices are given by

$$\hat{S}_j = \overline{X}_{.j} - \overline{X}_{\Lambda} - \frac{\hat{b}}{2}(2j-s-1), \quad j = 1,2,\Lambda\ ,s \quad (1.5)$$

for the additive model, and

$$\hat{S}_j = \overline{X}_{.j} \Big/ \left\{ \overline{X}_{\Lambda} + \frac{\hat{b}}{2}(2j-s-1) \right\}, \quad j = 1,2,\Lambda\ ,s \quad (1.6)$$

for the multiplicative model.

The main objective of this paper is to obtain Buys-Ballot estimates of the trend line equation and the seasonal indices for both the additive and multiplicative models when the trend-cycle component is quadratic. That is,

$$M_t = a + bt + ct^2, \quad t = 1,2,\Lambda\ ,n \quad (1.7)$$

We would want to see if the Buys-Ballot estimates for the linear trend-cycle component discussed by [3] are obtained from the Buys-Ballot estimates of the quadratic trend-cycle component when $c = 0$.

2.0 Buys-Ballot estimates for additive model

The additive model is given by

$$X_t = M_t + S_t + e_t, \quad t = 1,2,\Lambda\ ,n \quad (2.1)$$

where S_t is the seasonal component whose sum over a complete period is zero $\left(\sum_{j=1}^{s} S_{t+j} = 0\right)$; e_t is the irregular component which for our discussion is the Gaussian $N(0, \sigma_1^2)$ white noise and M_t given by equation (1.7) is the trend-cycle component. Ignoring the irregular component, we obtain the following row, column and overall totals and averages.

$$T_i = \sum_{j=1}^{s} X_{(i-1)s+j}, \quad i = 1,2,\Lambda\ ,m. \quad (2.2)$$

$$= \left\{a + b[(i-1)s+1] + c[(i-1)s+1]^2 + S_1\right\} + \left\{a + b[(i-1)s+2] + c[(i-1)s+2]^2 + S_2\right\} + \Lambda\ \{a + b[(i-1)s+s] + c[(i-1)s + 1]^2 + S_s\} = as + b(1+2+3+\Lambda\ +s) + b(i-1)s^2 + c(i-1)^2 s^3 + 2c(i-1)s(1+2+3+\Lambda\ +s) + c(1+2^2+3^2+s^2) = as + \frac{bs(s+1)}{2} + b(i-1)s^2 + c(i-1)^2 s^3 + c(i-1)s^2(s+1) + \frac{cs(s+1)(s+1)(2s+1)}{6}$$

$$= as + \frac{bs}{2}\{(2i-1)s+1\} + cs^2(i-1)(is+1) + \frac{cs}{6}(s+1)(2s+1) \quad \textbf{(2.3)}$$

$$\overline{X}_i = \frac{T_i}{s},\ i = 1,2,\Lambda\ ,m \quad \textbf{(2.4)}$$

$$= a + \frac{bs}{2}\{(2i-1)s+1\} + cs(i-1)(is+1) + \frac{c}{6}(s+1)(2s+1) \quad \textbf{(2.5)}$$

$$T._j = \sum_{i=1}^{m-1} X_{(i-1)s+j},\quad j = 1,2,\Lambda\ ,s. \quad \textbf{(2.6)}$$

$$= \left\{a + bj + cj^2 + S_j\right\} + \left\{a + b(sd+j) + c(s+j)^2 + S_j\right\} + \left\{a + b(2s+j) + c(2s+j)^2 + S_j\right\} + \Lambda + \left\{a + b[(m-1)s+j] + c[(m-1)s+j]^2 + S_j\right\} = ma + mbj + bs[1+2+\Lambda\ +(m-1)] + mcj^2 + cs^2\left[1+2^2+3^2+\Lambda\ +(m-1)^2\right] + 2csj[1+2+\Lambda\ +(m-1)] + mS_j = ma + mbj - mcj^2 + \frac{mbs}{2}(m-1) + m(m-1)csj + \frac{m(m-1)(2m-1)cs^2}{6} + mS_j \quad \textbf{(2.7)}$$

$$\overline{X}_i = \frac{T_i}{m},\ j = 1,2,\Lambda\ ,s \quad \textbf{(2.8)}$$

$$= a + \frac{b}{2}(n-s+2j) + cj(n-s+j) + \frac{(n-s)(2n-s)c}{6} + S_j \quad \textbf{(2.9)}$$

$$T = \sum_{i=1}^{m} T_{i.} = \sum_{j=1}^{s} T_{.j} \quad \textbf{(2.10)}$$

$$= na + \frac{nb(n+1)}{2} + \frac{n(n+1)(2n+1)c}{6} \quad \textbf{(2.11)}$$

$$\overline{X}_\Lambda = \frac{T}{n} \quad \textbf{(2.12)}$$

$$= a + \frac{b(n+1)}{2} + \frac{c(n+1)(2n+1)}{6} \quad \textbf{(2.13)}$$

Now let $$Y_i = \Delta \overline{X}_i = \overline{X}_{(i+1)} - \overline{X}_i\,,\ \ i = 1,2,\Lambda\ ,m-1 \qquad \textbf{(2.14)}$$

It is worthwhile to note the following.

$$Y_i = \Delta \overline{X}_i = bs + cs(2is+1) \qquad \textbf{(2.15)}$$

$$Z_i = \Delta^2 \overline{X}_i = \overline{X}_{(i+2)} - 2\overline{X}_{(i+1)} + \overline{X}_i = 2cs^2\,,\ \ i = 1,2,\Lambda\ ,m-2 \qquad \textbf{(2.16)}$$

$$W_i = Y_{i+1} - Y_1 = 2cs^2 i,\ i = 1,2,\Lambda\ ,m-2 \qquad \textbf{(2.17)}$$

$$\sum_{i=1}^{m-1} Y_i = \sum_{i=1}^{m-1}\left[\overline{X}_{(i+1)} - \overline{X}_{i\cdot}\right] = \sum_{i=1}^{m-1}\left[bs + cs(2is+1)\right]$$

Now $$\sum_{i=1}^{m-1}\left[\overline{X}_{(i+1)} - \overline{X}_i\right] = \overline{X}_m - \overline{X}_1 \qquad \textbf{(2.18)}$$

And $$\sum_{i=1}^{m-1}\left[bs + cs(2is+1)\right] = b(n-s) + c(n-s)(n+1) = (n-s)\left[b + c(n+1)\right] \qquad \textbf{(2.19)}$$

Thus $$\overline{X}_m - \overline{X}_1 = (n-s)\left[b + c(n+1)\right] \qquad \textbf{(2.20)}$$

The computation of '*c*' is done by expressing the changes in the first order differences of the periodic averages as differences with reference to the differenced average value at some earlier period. Like in Iwueze and [3], two alternatives are possible: (*i*) Equation (2.16) computes '*c*' from the second differences of the periodic averages (Chain Base Estimation (CBE) method) and (*ii*) Equation (2.17) computes '*c*' using the first differences (Y_1) of the first order differences as the earlier period (Fixed Base Estimation (FBE) method. The two possibilities will each give rise to (*m* - 2) different estimates of '*c*'. The average of these (*m* - 2) different estimates will be taken as the Buys-Ballot estimate of '*c*'. Having estimated '*c*', we use equation (2.15) to find (*m* - 1) different estimates of '*b*'. Again, the average of these (m - 1) different values of '*b*' will be taken as the Buys-Ballot estimate of '*b*'. Of course, from equation (2.20)

$$\hat{b} = \left\{\frac{\overline{X}_{m.} - \overline{X}_{1.}}{n-s}\right\} - \hat{c}(n+1) \qquad \textbf{(2.21)}$$

Finally, after the estimation of '*c*' and '*b*', we use equation (2.5) to find m different estimates of *'a'.* Again the average of these *m* different values of *'a'* will be taken as the

Buys-Ballot estimate of *'a'*. That is, using equation (2.13)

$$\hat{a} = \overline{X}_{\Lambda}\ \frac{\hat{b}(n+1)}{2} - \frac{\hat{c}(n+1)(2n+1)}{6} \qquad \textbf{(2.22)}$$

The seasonal indices are thereafter obtained from equation (2.9). That is,

$$\begin{aligned}\hat{S}_j &= \overline{X}_j - \hat{a} - \frac{\hat{b}(n-s+2j)}{2} - \hat{c}j(n-s+j) - \frac{(n-s)(2n-s)\hat{c}}{6} \\ &= \hat{X}_j = \overline{X}_{\Lambda} - \frac{\hat{b}(2j-s-1)}{2} + \frac{\hat{c}[(3n-s+1)(s+1)-6j(n-s+j)]c}{6}\end{aligned} \qquad \textbf{(2.23)}$$

Note **that when there is no trend (*b* = 0, *c* = 0), it is clear from equation (2.23) that**

$$\hat{S}_j = \overline{X}_j - \overline{X} \qquad \textbf{(2.24)}$$

3.0 Buys-Ballot estimates for multiplicative model

The multiplicative model is given by

$$X_t = M_t * S_t * e_t,\ \ t = 1, 2, \Lambda\ , n \qquad \textbf{(3.1)}$$

where S_t is the seasonal component whose sum over a complete period is s $\left(\sum_{j=1}^{s} S_{t+j} = 0\right)$; e_t is the regular component which for our discussion is the Gaussian N(1, σ_2^2) white noise and M_t given by equation (1.7) the trend-cycle component. Ignoring the irregular component, we obtain the following row, column and overall totals and averages

$$
\begin{aligned}
T_{i.} &= \left\{a+b[(i-1)s+1]+c[(i-1)s+1]^2\right\}S_1+\left\{a+b[(i-1)s+2]+c[(i-1)s+2]^2\right\}S_2 \\
&+\left\{a+b[(i-1)s+3]+c[(i-1)s+3]^2\right\}S_3+\Lambda\ \left\{a+b[(i-1)s+s]+c[(i-1)s+s]^2\right\}S_s \\
&= a(S_1+S_2+S_3+\Lambda+S_s)+b(i-1)s(S_1+S_2+S_3+\Lambda+S_s)+b(S_1+2S_2+3S_3+\Lambda+sS_s) \quad \textbf{(3.2)} \\
&+c(i-1)^2 s^2(S_1+S_2+S_3+\Lambda+S_s)+2c(i-1)s(S_1+2S_2+3S_3+\Lambda+sS_s) \\
&+c(i-1)s+c(S_1+2^2S_2+3^2S_3+\Lambda+s^2S_s)
\end{aligned}
$$

Since for the multiplicative model we expect $S_t \approx 1.0$ [1] it follows that

$$S_1+2S_2+3S_3+\Lambda+sS_s \approx \frac{s(s+1)}{2} \tag{3.3}$$

$$S_1+2^2S_2+3^3S_3+\Lambda+s^2S_s \approx \frac{s(s+1)(2s+1)}{6} \tag{3.4}$$

Thus,

$$
\begin{aligned}
T_i &= as+b(i-1)s^2+\frac{bs(s+1)}{2}+c(i-1)^2s^3+2c(i-1)s\left[\frac{s(s+1)}{2}\right]+\frac{cs(s+1)(2s+1)}{6} \\
&= as+\frac{bs[(2i-1)s+1]}{2}+cs^2(i-1)(is-1)+\frac{cs(s+1)(2s+1)}{6},\ i=1,2,\Lambda\ ,m
\end{aligned} \tag{3.5}
$$

$$\bar{X}_i = a+\frac{b}{2}[(2i-1)s+1]+cs(i-1)(is+1)+\frac{c(s+1)(2s+1)}{6},\ i=1,2,\Lambda\ ,m \tag{3.6}$$

results, totally in agreement with the additive case.

$$
\begin{aligned}
T_{.j} &= \left\{a+bj+cj^2\right\}S_j+\left\{a+b(s+j)+c(s+j)^2\right\}S_j+\left\{a+b(2s+j)+c(2s+j)^2\right\}S_j+\Lambda\ + \\
&\left\{a+b[(m-1)s+j]+c[(m-1)s+j]^2\right\}S_j=\{a(ma+mbj+bs[1+2+\Lambda+(m-1)])+mcj^2+cs^2[1+ \\
&2^2+\Lambda+(m-1)^2]+2csj[1+2+\Lambda+(m-1)]\}S_j=\left\{ma+mbj+mcj^2+\frac{mbs(m-1)}{2}\right. \\
&\left.+m(m-1)csj+\frac{cs^2m(m-1)(2m-1)}{6}\right\}S_j,\ \ j=1,2,\Lambda\ s
\end{aligned} \tag{3.7}
$$

$$\bar{X}_j=\left\{a+\frac{b}{2}(n-s+2j)+cj(n-s+j)+\frac{(n-s)(2n-s)c}{6}\right\}S_j,\ j=1,2,\Lambda\ ,s \tag{3.8}$$

Finally,

$$T=na+\frac{nb(n+1)}{2}+\frac{n(n+1)(2n+1)c}{6} \tag{3.9}$$

And

$$\bar{X}=a+\frac{b(n+1)}{2}+\frac{(n+1)(2n+1)c}{6} \tag{3.10}$$

Equations (3.9) and (3.10**)** are in agreement with the additive case. Since the row averages, which are functions of *a*, *b***,** and *c***,** are the same for both additive and multiplicative models, Buys-Ballot estimates of the parameters of the trend-cycle component are the same for both models. However, we use equation (3.8) to estimate the seasonal indices of the multiplicative model. That is,

$$\hat{S}_j=\bar{X}_j/d_j,\ \ j=1,2,\Lambda\ ,s \tag{3.11}$$

where

$$
\begin{aligned}
d_j &= \hat{a}+\frac{\hat{b}(n-s+s+2j)}{2}+\hat{c}j(n-s+j)+\frac{(n-s)(2n-s)\hat{c}}{6}=\bar{X}+\frac{\hat{b}(2j-s-1)}{2} \\
&-\frac{\hat{c}[(3n-s+1)(s+1)-6j(n-s+j)]}{6},\ j=1,2,\Lambda\ ,s
\end{aligned} \tag{3.12}
$$

when there is no trend (b = 0, c = 0), it is clear from equation (3.11) that

$$\hat{S}_j=\bar{X}_{.j}/\bar{X}_{..},\ \ j=1,2,\Lambda\ ,s \tag{3.14}$$

4.0 An empirical example.

We have shown in Sections 2 and 3 that the Buys-Ballot estimates of *a*, *b* and *c* are the same for both the additive and multiplicative models. The estimates of the seasonal indices are however different. In summary, for $j = 1.2, \Lambda\ s$, $\hat{S}_j = \overline{X}_{.j} - dj$ for the additive model, while $\hat{S}_j = \overline{X}_{.j} / dj$ for the multiplicative model, where

d_j is given by equation (3.12). Based on the aforementioned reason, we will use the additive model to illustrate the methods described in Sections 2 and 3. This example shows a simulation of 100 values from an additive model

$$X_t = a + bt + ct^2 + S_t + e_t \qquad (4.1)$$

with $a = 180,\ b = -0.35,\ c = 0.35,\ S_1 = -50,\ S_2 = 30,\ S_3 = 80,\ S_4 = -60$ and e_t being Guassian N(0,1) white noise. The series is listed in Table 2 with its row and column averages and standard deviations. As shown in Figures 1,2 and 3, it is clearly seasonal with a quadratic trend and stable variance that mimics the quadratic trend. There is a reasonably stable seasonal pattern (upsurge in the second and third quarters and a sharp drop in the first and fourth quarters) over the periods suggesting the additive model.

4.1 **Least squares estimates**

First, the parameters of the quadratic trend curve were obtained by least squares estimation

Table 2: Simulated data from $X_t = a + bt + ct^2 + S_t + e_t$ with $s = 4,\ a = 180,\ b = -0.35,\ c = 0.35,\ S_1 = -50,\ S_2 = 30,\ S_3 = 80,\ S_4 - 60,\ e_t \sim N(0,1)$

PERIOD	SEASON I	II	III	IV	TOTAL	AVERAGE	STANDARD DEVIATION
1	130.68	210.71	261.29	123.38	725.96	181.4900	66.3085
2	136.44	219.54	276.20	138.94	771.12	192.7800	67.6952
3	154.26	240.72	297.51	166.11	858.60	214.6500	67.2025
4	184.37	273.32	333.99	204.70	996.38	249.0950	68.2013
5	224.70	316.65	380.89	253.45	1175.69	293.9225	69.5453
6	277.39	371.55	436.60	313.11	1398.65	349.6625	69.7534
7	341.41	437.02	506.16	385.88	1670.47	417.6175	70.7839
8	414.09	513.32	585.65	468.23	1981.29	495.3225	72.6078
9	500.00	601.49	677.19	561.28	2339.96	584.9900	74.2935
10	595.01	699.60	778.16	667.01	2739.78	684.9450	75.9688
11	703.77	812.68	893.28	781.20	3190.93	797.7325	78.4326
12	824.04	935.03	1017.31	910.23	3686.61	921.6525	79.5571
13	954.47	1064.27	1151.25	1048.28	4218.27	1054.5675	80.6240
14	1094.08	1210.97	1299.87	1198.12	4803.04	1200.7600	84.2904
15	1247.21	1367.06	1458.19	1357.76	5430.22	1357.5550	86.3979
16	1410.60	1534.87	1627.00	1532.14	6104.61	1526.1525	88.7589
17	1586.61	1713.00	1809.17	1713.67	6822.45	1705.6125	91.2967
18	1772.82	1900.19	1999.87	1908.47	7581.35	1895.3375	93.3338
19	1968.59	2101.05	2202.95	2115.88	8388.47	2097.1175	96.7597
20	2177.23	2313.51	2418.18	2331.59	9240.51	2310.1275	99.6814
21	2400.82	2536.97	2642.19	2560.90	10140.88	2535.2200	100.2804
22	2628.46	2768.24	2879.40	2798.82	11074.92	2768.7300	104.6091
23	2872.34	3013.04	3126.17	3049.86	12061.41	3015.3525	106.3472
24	3125.02	3270.05	3385.56	3309.11	13089.74	3272.4350	109.3606
25	3389.65	3536.68	3654.66	3582.68	14163.67	3540.9175	111.9245
TOTAL	31113.96	33961.53	36098.69	33480.80	134654.98	-	-
AVERAGE	1244.5584	1358.4612	1443.9476	1339.2320	-	1346.5498	-
STADARD DEVIATION	1034.2460	1054.4218	1073.8778	1093.3739	-	-	-

Procedure (using MINITAB) to be

$$\hat{M}_T = \underset{(18.0100)}{179.2300} - \underset{(0.8232)}{0.3138}T + \underset{(0.0079)}{0.3497}t^2 \qquad (4.2)$$

where values in parentheses below the parameter estimates are the associated standard errors. The associated seasonal analysis procedure estimates the seasonal indices as: $\hat{S}_1 = -49.8342$, $\hat{S}_2 = 29.7638$, $\hat{S}_3 = 80.2452$, $\hat{S}_4 = -60.1748$.

The residuals (which we estimate by subtracting both the trend line and seasonal effects) indicate no model inadequacy with respect to the autocorrelation function (ACF). The residual mean = -0.0430 while the residual variance, $\hat{\sigma}_1^2 = 0.9734$.

4.0 **Buys-Ballot estimates**

The computational procedure for the Buys-Ballot estimates for the quadratic trend is laid out in Table 3. The seasonal indices are estimated using

$$\hat{S}_j = \overline{X}_j - d_j \qquad (4.3)$$

which is the same as equation (2.23). The computational procedure for estimating the seasonal indices is laid out in Table 4.

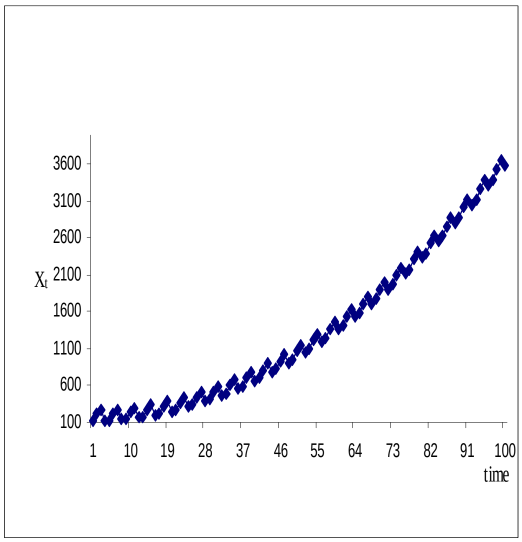
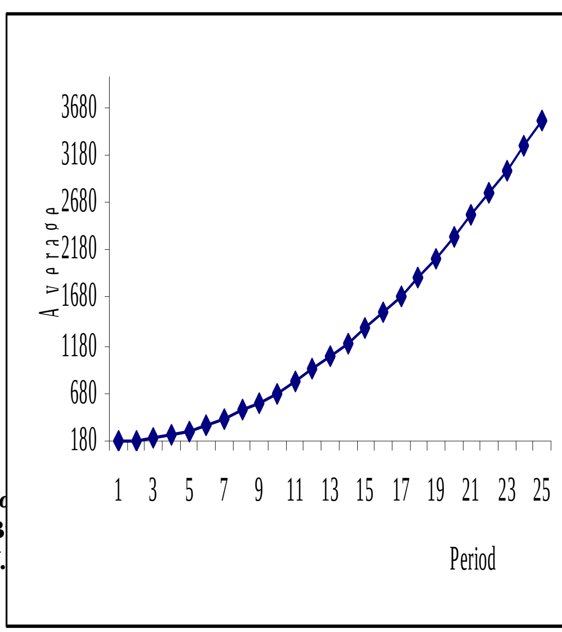


Figure 1: A simulated additive series: $X_t = 180 - 0.35t + 0.35t2 + S_t$, with $S_1 = -0.50$, $S_2 = 30$, $S_3 = 80$, $S_4 = -60$, $e_t \sim N(0,1)$.



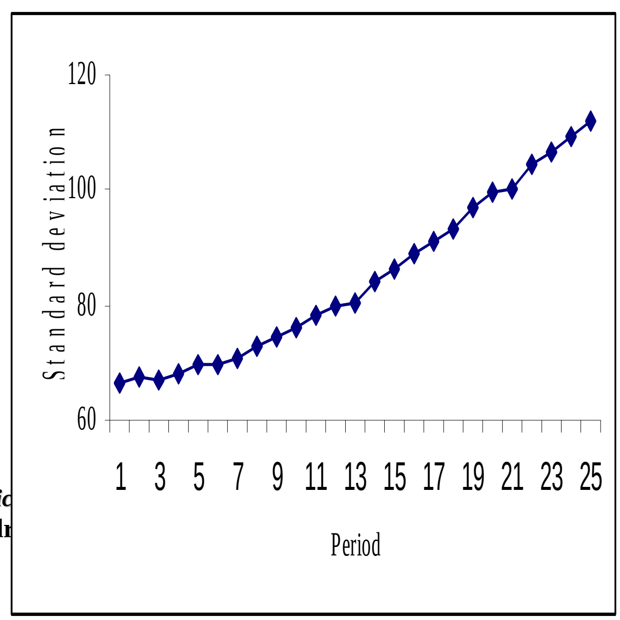


Figure 2: Periodic averages of data of Table *Figure 3: Periodic standard deviations of data of Table 2*

The CBE and FBE estimates can each be used to obtain component analysis tables after which the irregular components obtained can then be checked for randomness. Only the residual ACF of the FBE method indicate model inadequacy. The estimates and the corresponding error means and variances are shown in Table 5.

Table 5: Summary of estimates (additive model with quadratic trend)

PARAMETER	ACTUAL VALUE	LSE	CBE	FBE
a	180.0000	179.2300 (18.0100)	179.3514 (0.6256)	179.1369 (0.6731)
b	-0.3500	-0.3138 (0.8232)	-0.3001 (0.1793)	-0.2875 (0.1798)
c	0.3500	0.3497 (0.0079)	0.3495 (0.0392)	0.3493 (0.0052)
S_1	-50.0000	-49.8342	-49.8497	-49.8498
S_2	30.0000	29.7638	29.7579	29.7577
S_3	80.0000	80.2452	80.2502	80.2501
S_4	-60.0000	-60.1748	-60.1584	-60.1582
Error Mean	0.0000	-0.0430	-0.1796	0.0754
Error STANDARD DEVIATION (σ_1)	1.0000	0.9735	0.9894	1.0694

Table 3: Buys-Ballot estimates for the parameters of the quadratic trend curve

PERIOD	AVERAGE $\overline{X}_i$	$Y_I = \nabla\overline{X}$	CBE				FBE			
			$Z_I = \nabla^2 \overline{X}_i$	$\hat{c}$	$\hat{b}$	$\hat{a}$	$W_i=Y_{i+1}-Y_i$	$\hat{c}$	$\hat{b}$	$\hat{a}$
1	181.4900	11.2900	10.5800	0.33063	-0.32252	179.61942	10.5800	0.33063	-0.32140	179.58880
2	192.7800	21.8700	12.5750	0.39297	-0.47310	179.52977	23.1550	0.36180	-0.47097	179.45316
3	214.6500	34.4450	10.3825	0.32445	-0.12493	178.83782	33.5375	0.34935	-0.12180	178.71921
4	249.0950	44.8275	10.9125	0.34102	-0.32488	179.53855	44.4500	0.34727	-0.32075	179.38196
5	293.9225	55.7400	12.2150	0.38172	-0.39233	179.43949	56.6650	0.35416	-0.38720	179.24890
6	349.6625	67.9550	9.7500	0.30469	-0.13415	179.07062	66.4150	0.34591	-0.12803	178.85004
7	417.6175	77.7050	11.9625	0.37383	-0.49223	179.73446	78.3775	0.34990	-0.48510	179.48788
8	495.3225	89.6675	10.2875	0.32148	-0.29718	178.96598	88.6650	0.34635	-0.28906	178.69741
9	584.9900	99.9550	12.8325	0.40102	-0.52088	178.97769	101.4975	0.35242	-0.51176	178.69114
10	684.9450	112.7875	11.1325	0.34789	-0.10833	178.09461	112.6300	0.35197	-0.09821	177.79406
11	797.7325	123.9200	8.9950	0.28109	-0.12078	178.86173	121.6250	0.34553	-0.10966	178.55118
12	921.6525	132.9150	13.2775	0.41492	-0.66761	179.57904	134.9025	0.35131	-0.65548	179.26250
13	1054.5675	146.1925	10.6025	0.33133	-0.14381	178.10903	145.5050	0.34977	-0.13069	177.79051
14	1200.7600	156.7950	11.8025	0.36883	-0.28876	178.73424	157.3075	0.35113	-0.27464	178.41772
15	1357.5550	168.5975	10.8625	0.33945	-0.13371	178.77963	168.1700	0.35035	-0.11859	178.46912
16	1526.1525	179.4600	10.2650	0.32078	-0.21366	179.44523	178.4350	0.34851	-0.19754	179.14472
17	1705.6125	189.7250	12.0550	0.37672	-0.44299	179.79101	190.4900	0.35017	-0.42586	179.50452
18	1895.3375	201.7800	11.2300	0.35094	-0.22482	179.21950	201.7200	0.35021	-0.20669	178.95101
19	2097.1175	213.0100	12.0825	0.37758	-0.21289	179.52068	213.8025	0.35165	-0.19377	179.27420
20	2310.1275	225.0925	8.4175	0.26305	0.01216	179.86956	222.2200	0.34722	0.03228	179.64908
21	2535.2200	233.5100	13.1125	0.40977	-0.67904	181.11863	235.3325	0.35020	-0.65792	180.92816
22	2768.7300	246.6225	10.4600	0.32688	-0.19649	179.60290	245.7925	0.34914	-0.17437	179.44644
23	3015.3525	257.0825	11.4000	0.35625	-0.37707	180.01735	257.1925	0.34945	-0.35395	179.89934
24	3272.4350	268.4825	-	-	-0.32265	179.70852	-	-	-0.29852	179.63308
25	3540.9175	-	-	-	-	179.61938	-	-	-	179.58894
TOTAL	33663.7450	-	-	8.03729	-7.20265	4483.78585	-	8.03440	-6.89968	4478.42308
AVERAGE	1346.7450	-	-	0.34945	-0.30011	179.35143	-	0.34932	-0.28749	179.13692
STANDARD DEVIATION	-	-	-	0.03920	0.17926	0.62560	-	0.00525	0.17981	0.67311

Table 4: Buys-Ballot estimates of the seasonal indices (additive model with quadratic trend)

j	$X_{.j}$	CBE			FBE		
		d_j	$\hat{S}_j = \overline{X}_{.j} - d_j$	Adjusted $\hat{S}_j$	d_j	$\hat{S}_j = \overline{X}_{.j} - d_j$	Adjusted $\hat{S}_j$
1	1244.5584	1294.4094	-49.8510	-49.8497	1294.4073	-49.8489	-49.8498
2	1358.4612	1328.7046	29.7566	29.7579	1328.7026	29.7586	29.7577
3	1443.9476	1363.6987	80.2489	80.2502	1363.6966	80.2510	80.2501
4	1339.2320	1399.3917	-60.1597	-60.1584	1399.3893	-60.1573	-60.1582

5.0 Conclusion

By deriving several estimates of each parameter of the trend equation, we have provided empirical estimates of the standard errors of such parameter estimates. We have not succeeded, however, in deriving analytical estimates of the standard errors as is done in least squares regression method [2]. However, the analytical derivations should be interpreted with caution since normality and independence assumptions are not strictly valid.

The difference in method lies in the computation of " c " which is easily computed from differences in the periodic averages. The computation reduces to

$$\hat{c} = \frac{1}{2(m-2)s^2}\sum_{I=1}^{m-2}\nabla^2 X_i = \frac{1}{2(m-2)s^2}\left(\overline{X}_1 - \overline{X}_2 - \overline{X}_{(m-1)} + \overline{X}_m\right) \tag{5.1}$$

for the CBE method and

$$\hat{c} = \frac{1}{2(m-2)s^2}\sum_{I=1}^{m-2}(W_i\,|1) = \frac{1}{2(m-2)s^2}\left\{\sum_{i=1}^{m-2}\left[\frac{\overline{X}_{(i+2)} - \overline{X}_{(i+1)}}{i}\right]\right\} - \left(\overline{X}_2 - \overline{X}_1\right)\sum_{I=1}^{m=2}(1/i) \tag{5.2}$$

for the FBE method.

The CBE method takes into consideration only the first two and the last two periodic averages, while the FBE method takes into consideration all the periodic averages. We adopt the recommendation of [3] that the FBE method should be used when it leads to adequate fit in terms of the randomness of the residuals.

Finally, equation (2.21) reduces to equation (1.2) while equation (2.22) reduces to equation (1.4) when c = 0. It is then clear that the Buys-Ballot estimates for the linear trend-cycle component discussed by [3] are obtainable from our Buys-Ballot estimates when the trend-cycle component is quadratic by putting c = 0 in equation (1.7).

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