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Fractional Duffing Oscillator

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Abstract

We solve the linearly damped fractional mechanical oscillator equation using the Laplace transform method. The solution is expressed in terms of the two-parameter Mittag-Leffler function. We considered the effect of damping and of the fractional order on the response function. We observe a significant influence of the fractional derivative on degree of damping.

Keywords: Laplace transforms, Rieman-Liouville fractional integral, Caputo's fractional derivative, Mittag-Leffler function, Fractional differential equation, Damping.

1.0 Introduction

The theory of fractional calculus is concerned with the generalization of the concepts of differentiation and integration to arbitrary orders. It is an outgrowth of the traditional definitions of the derivative and integral operations in much the same way as the fractional exponent is the natural extension of exponents with integer values **[1]**.It developed simultaneously with the theory of integer order calculus. Unlike many branches of mathematics and other disciplines whose exact origins are not clear, we can point to a particular date when fractional calculus was born. This interesting field of study was initiated in a correspondence **[2]** between L'Hopital and Leibniz, who co-invented the calculus with Newton. In a letter dated 30th September, 1695, L'Hopital had asked Leibniz the meaning of the notation the latter had used for the nth derivative $\frac{d^n x}{dx^n}$ *of the linear function* $f(x) = x$ in his publication. L'Hopital posed the question what would the result be if $n = \frac{1}{2}$? Leibniz replied: **"An apparent paradox from which one day useful consequences will be drawn."** Later, this little conversation between these two mathematical giants caught the attention of other prominent mathematicians like Lacroix, Abel, Euler, Liouville and Riemann e.t.c. Each of these researchers shaped the evolution of the fractional calculus in their own ways.

The utility of the fractional order calculus is not in doubt judging from recent and current findings among researchers in biological, physical sciences and engineering. Fractional differentiation has been used in models to study speech signals **[3],** astronomical image processing **[4, 5, 6, 7],** earthquakes [**8, 9**] and visco-elasticity**[10,11]** .An enthusiastic reader can quickly browse through a catalogue of the applications of fractional order calculus in[**12, 13**] . So far, we have studied physical systems in terms of integer order calculus. Intuitively, one can argue that fractional calculus is more harmonious with the real world and in fact it has been shown that fractional order models capture phenomena and properties that classical integer order simply neglects **[14, 15.]**. Nature, we all know, does not always obey the integers. Little wonder fractional calculus has generated so much interest across the mathematical world. Researches that are based on the theory of fractional calculus are ongoing and it is among the expanding and promising frontiers of mathematics. It is obvious that greater applications of calculus to human problems in the future will likely depend on fractional calculus.

This research was inspired by three journal articles **[16, 17, and 18].** Using analytical techniques of classical (discrete) calculus Oyesanya **[16]** treated the nonlinear Duffing oscillator

$$\ddot{q} + \delta\dot{q} - \alpha q + \beta q^3 = \gamma\cos(\omega t) \quad (1)$$

and applied the results to the phenomenon of earthquake prediction.

Later, Naber **[17]** treated the fractional oscillator equation considering only the case where the fractional derivative is on the damping term:

$$D^2 x + \lambda D^\alpha x + \omega^2 x = 0 \quad 0 < \alpha < 1 \quad (2)$$

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Using Laplace transform technique, he approached the problem through contour integration and found that there are nine distinct cases as opposed to the usual three cases for integer calculus.

The equation

$$D^{\alpha}x - x + x^3 = F\sin\nu t \qquad 1<\alpha<2 \tag{3}$$

Was considered by Zaslavsky et al in [18] with very interesting results.

The above three equations are analogues of the Duffing Oscillator. This work solves the linear and unforced analogues of (1)) and (3) where the derivatives are all fractional. Thus we consider linear fractionally damped Duffing oscillator dealing squarely with the extension of (2) by making all derivatives fractional but differs from it by expressing the solution in terms of the Mittag-leffler function; both, however, explored the use of Laplace Transform Method. In a nutshell, we seek to investigate the solution of fractionally damped linear oscillator.

2.0 Process

We shall devote this section to the development of the mathematical formulations which form the framework of fractional calculus beginning with the definition of vital concepts.

2.1 The Mittag-Leffler function

The one-parameter Mittag-Leffler function is defined by **[19, 20, and 21]**

$$E_{\alpha}(t) = \sum_{k=o}^{\infty}\frac{t^k}{\Gamma(\alpha k+1)} \quad \alpha > 0 \tag{3}$$

The generalized form or two-parameter type of this function is given as

$$E_{\alpha,\beta}(\lambda t^{\alpha}) = \sum_{k=0}^{\infty}\frac{\lambda^k t^{\alpha k}}{\Gamma(\alpha k+\beta)} \quad \alpha,\beta > 0 \tag{4}$$

This function is credited to the Swedish mathematician, **Magnus Gosta Mittag-Leffler (1846 -1927).** His works and contributions to mathematical analysis are available in **[22].**

2.2 Rieman-liouville Fractional Integral

Let $Re\,\nu > o$ and f be piecewise continuous on $I' = (0,\ \infty)$ and integrable on any finite subinterval of $I = [0,\infty)$. Then for $t > 0$,

$$0D_t^{-\nu}f(t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-x)^{\nu-1}f(x)dx \tag{5}$$

is called the fractional integral of f of order ν. This can simply be denoted as $D^{-\nu}$ or J^{ν}.

2.3 Formulations of Fractional Derivative

Consider a differentiation of an arbitrary order α, we choose the smallest integer m such that

$$\boldsymbol{m-1 < \alpha < m}$$

This leaves us with two options as to how to define the fractional derivative of a function of an arbitrary order α. One option leads to the Rieman-Liouville's formulation while the other generates Caputo's fractional derivative.

2.3.1 Rieman-liouville Fractional Derivative

Let $m-\alpha = \nu > 0, \Rightarrow \alpha = m-\nu$

Hence $D^{\alpha}f(t) = D^{m-\nu}f(t) = D^m D^{-\nu}f(t)$ [**Integrating before differentiation**]

$$= D^m\left[\frac{1}{\Gamma(\nu)}\int_0^t (t-\tau)^{\nu-1}f(\tau)d\tau\right]$$

Reintroducing $\nu = m-\alpha$ gives

$$D^{\alpha}f(t) = \begin{cases}\frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_0^t (t-\tau)^{m-\alpha-1}f(\tau)d\tau & m-1<\alpha<m \\ D^m f(t) & m=\alpha\end{cases} \tag{6}$$

2.3.2 Caputo's Fractional Derivative

From the operation $D^{\alpha}f(t) = D^{m-\nu}f(t) = D^m D^{-\nu}f(t)$

$D^{\alpha}f(t) = D^{m-\nu}f(t) = D^{-\nu}f^{(m)}(t)$ [**Differentiating before fractional integration**]

$$D^{\alpha}f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}f^{(m)}(\tau)d\tau \quad m-1<\alpha<m \tag{7}$$

We remark that there are some slight differences between Riemann and Caputo's definitions of the fractional derivative. For our purpose, we point out the obvious which is that Riemann's formulation requires fractional order initial conditions. Unfortunately, as at now, we have no physical interpretation of fractional initial conditions. Generally, physical interpretation of fractional calculus is yet to be fully conceptualized **[12].** This makes Caputo's formulation more popular in terms of applications since it simply requires integer-order initial conditions like our discrete calculus. Consequently, it shall be used for the rest of this work.

2.4 Laplace Transform

Let a function f be of an exponential order. Then the Laplace transform of f is defined as

$$L[f(t)] = \int_0^\infty f(t)e^{-st}\,dt = F(s) \tag{8}$$

The inversion integral is given by

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}ds \tag{9}$$

2.5 The Convolution Theorem

If $F(s) = L[f(t)]$ *and* $G(s) = L[g(t)]$ and both exist for $s > a \geq 0$, then

$$H(s) = F(s)G(s) = L[h(t)] \tag{10}$$

Where h is known as the convolution of f and g and defined as

$$h(t) = \int_0^t f(t-x)g(x)dx = \int_0^t g(t-x)f(x)dx$$

2.6 Laplace Transform of the Fractional Integral

We recall that the fractional integral is defined as

$$D^{-\nu}f(t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-x)^{\nu-1}f(x)dx$$

So $L[D^{-\nu}f(t)] = L\left\{\frac{1}{\Gamma(\nu)}\int_0^t (t-x)^{\nu-1}f(x)dx\right\}$

$$= \frac{1}{\Gamma(\nu)}L[t^{\nu-1}] * L[f(x)] \quad \textit{By convolution theorem}$$

$$L[D^{-\nu}f(t)] = s^{-\nu}F(s) \tag{11}$$

2.7 Laplace Transform of Caputo's Fractional Derivative

It is easy to see that the Caputo derivative given by

$$D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}f^{(m)}(\tau)d\tau \quad m-1 < \alpha < m$$

is a convolution of two functions. Applying the convolution property of the Laplace transform gives

$$L[D^\alpha f(t)] = L\left[\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}f^{(m)}(\tau)d\tau\right]$$

$$= \frac{1}{\Gamma(m-\alpha)}L[t^{m-\alpha-1}]L\left[f^{(m)}(\tau)\right]$$

$$= \frac{1}{s^{m-\alpha}}L\left[f^{(m)}(\tau)\right] = S^{-(m-\alpha)}L\left[f^{(m)}(\tau)\right]$$

Recall that, from integer-order calculus, the Laplace transform of the m[th] derivative $f^{(m)}$ of f is

$$L\left[f^{(m)}\right] = s^m F(s) - \sum_{k=0}^{m-1} s^{m-k-1}f^{(k)}(0) \tag{12}$$

Putting this into the above convolution integral gives

$$L[D^\alpha f(t)] = s^{-(m-\alpha)}\left[s^m F(s) - \sum_{k=0}^{m-1} s^{m-k-1}f^{(k)}(0)\right]$$

$$L[D^\alpha f(t)] = s^\alpha F(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1}f^{(k)}(o) \tag{13}$$

2.8 Inversion Formula for Fractional Laplace Transform.

This section we derive the important inverse fractional Laplace Transform of a type of Mittag-Leffler function given **[23]** as

$$\boldsymbol{L^{-1}\left[\frac{m!s^{\alpha-\beta}}{(s^\alpha - a)^{m+1}}\right] = t^{\alpha m+\beta-1}E_{\alpha,\beta}^{(m)}} \tag{14}$$

We approach it through a combination of integration and differentiation under the integral. We begin by computing the Laplace transform of the function

$$t^{\beta-1}E_{\alpha,\beta}(at^\alpha)$$

By definition $L\left[t^{\beta-1}E_{\alpha,\beta}(at^\alpha)\right] = \int_0^\infty e^{-st}t^{\beta-1}\sum_{n=0}^\infty \frac{a^n t^{\alpha n}}{\Gamma(\alpha n+\beta)}dt$

$$= \sum_{n=0}^\infty \frac{a^n}{\Gamma(\alpha n+\beta)}\int_0^\infty e^{-st}t^{\alpha n+\beta-1}dt$$

$let\ st = r \Rightarrow dt = dr/s$ And the integral transforms to