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Semi-relativistic treatment of Hellmann potential using Supersymmetric Quantum Mechanics

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Abstract

We present an analytical solution of the spinless Salpeter equation for a Hellman potential using the supersymmetric quantum mechanics formalism. We obtain the energy eigenvalues and the corresponding eigenfunctions expressed in terms of hypergeometric functions.

Keywords: Hellmann potential, supersymmetric quantum mechanics, Bethe-Salpeter equation.

1.0 Introduction

The Bethe-Salpeter BSE equation is a semi-relativistic equation that describes the bound state of two body quantum field system in a relativistic covariant formalism [1-5]. This equation (BSE), after neglecting the spin degree of freedom with useful approximation gives what is known as spinless Salpeter equation (SSE)[1]. The SSE, because of its non-local nature cannot easily be solved analytically using common mathematical techniques. In the quantum relativistic regime, SSE is a generalization of the Schrodinger equation [6-7].In this work, we used supersymmetric quantum mechanics (SUSYQM) to solve SSE with Hellmann potential. The Hellmann potential is a superposition of an attractive Coulomb potential and a Yukawa potential, which is often used to compute bound-state normalizations and energy levels of neutral atoms. A lot of work has been done using BSE[8-14,16-17] The Bethe-Salpeter equation was first given at the American Physical Society meeting at the beginning of 1951[8]. Sameer Ikhdair and Ramazan Sever studied a perturbative treatment for the bound state of Hellmann potential[9]. In this paper, our aim is to solve the SSE for a Hellmann potential for any l-state using the SUSYQM. Hellmann suggested a superposition of the Coulomb plus Yukawa potential as follows[10],

$$V(r) = -\frac{a}{r} + b\frac{e^{-\delta r}}{r} \tag{1}$$

Where a and b are the strength of the Coulomb and the Yukawa potentials respectively and δ is the screening parameter [11]. The electron-core and electron-ions interaction have been investigated using the Hellmann potential [11]. The bound state energy spectra of Hellmann potential for different values of b and δ has been investigated by Dutt et al [12] using variational approach.

2.0 Spinless Salpeter equations in two body system

The SSE for body particle interacting in a spherically symmetric potential in the centre of mass system appear as [1-3, 13],

$$\left\{\sum_{i=1,2}\left(\sqrt{-\Delta + m_i^2} - m_i\right) + V(r) - E_{n,l}\right\}\chi(r) = 0, \Delta = \nabla^2, \tag{2}$$

Where $\chi(r) = R_{n,l}(r)Y_{l,m}(\theta,\phi)$. For heavy interacting particles and using appropriate transformation[14], we recast SSE of Eq.(1) as

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$$\left[-\frac{1}{2\mu}\frac{d^2}{dr^2}+\frac{l(l+1)}{2\mu r^2}+W_{nl}(r)-\frac{W_{nl}^2(r)}{2\tilde{m}}\right]\psi_{nl}(r)=0 \tag{3}$$

Where

$$W_{nl}(r)=V(r)-E_{nl}\ , \tag{5}$$

$$\mu=\frac{m_1m_2}{m_1m_2}\ , \tag{6}$$

$$\eta=\mu\left(\frac{m_1m_2}{m_1m_2-3\mu^2}\right)^{\frac{1}{3}}\ , \tag{7}$$

$$\tilde{m}=\frac{\eta^3}{\mu^2}=\frac{(m_1m_2\mu)}{(m_1m_2-3\mu^2)}\ , \tag{8}$$

In this work, we consider $V(r)$ as Hellmann potential. Substituting Eq.(1) and using Eqs.(5-8) in Eq.(3) yields

$$-\frac{d^2\psi_{nl}}{dr^2}+\left[\frac{l(l+1)}{r^2}-\frac{2\mu a}{\hbar^2 r}+\frac{2\mu be^{-\alpha r}}{\hbar^2 r}-\frac{2\mu E_{nl}}{\hbar^2}-\frac{\mu a^2}{\tilde{m}\hbar^2 r^2}\right]\psi_{nl}(r)$$
$$+\left[\frac{2\mu abe^{-\alpha r}}{\tilde{m}\hbar^2 r^2}-\frac{2\mu aE_{nl}}{\tilde{m}\hbar^2 r}+\frac{2\mu bE_{nl}e^{-\alpha r}}{\tilde{m}\hbar^2 r}-\frac{\mu b^2e^{-2\alpha r}}{\tilde{m}\hbar^2 r^2}-\frac{\mu E_{nl}^2}{\tilde{m}\hbar^2}\right]\psi_{nl}=0 \tag{9}$$

Equation(9) is complicated and cannot be solved analytically because of the presence of centrifugal term. We employed the following approximations[15]:

$$\frac{1}{r^2}=\frac{\alpha^2}{(1-e^{-\alpha r})^2}\ \text{and}\ \frac{1}{r}=\frac{\alpha}{(1-e^{-\alpha r})} \tag{10}$$

Substituting equation (10) into (9) we have

$$-\frac{d^2\psi_{nl}(r)}{dr^2}+\frac{1}{(1-e^{-\alpha r})^2}\left[\left(-\frac{2\mu b\alpha}{\hbar^2}-\frac{2\mu b\alpha E_{nl}}{\tilde{m}\hbar^2}-\frac{\mu^2 b\alpha^2}{\tilde{m}\hbar^2}\right)e^{-2\alpha r}\right]\psi_{nl}(r)$$
$$+\frac{1}{(1-e^{-\alpha r})^2}\left[\left(\frac{2\mu b\alpha}{\hbar^2}+\frac{2\mu a\alpha}{\hbar^2}+\frac{2\mu ab\alpha^2}{\tilde{m}\hbar^2}+\frac{2\mu a\alpha E_{nl}}{\tilde{m}\hbar^2}+\frac{2\mu b\alpha E_{nl}}{\tilde{m}\hbar^2}\right)e^{-\alpha r}\right]\psi_{nl}(r)$$
$$+\frac{1}{(1-e^{-\alpha r})^2}\left[\left(\alpha^2 l(l+1)-\frac{2\mu a\alpha}{\hbar^2}-\frac{\mu a^2\alpha^2}{\tilde{m}\hbar^2}-\frac{2\mu a\alpha E_{nl}}{\tilde{m}\hbar^2}\right)\right]\psi_{nl}(r)=\left[\frac{2\mu E_{nl}}{\hbar^2}+\frac{\mu E_{nl}^2}{\tilde{m}\hbar^2}\right]\psi_{nl}(r) \tag{11}$$

Simplifying more explicitly, yields

$$-\frac{d^2\psi_{nl}(r)}{dr^2}+V_{eff}(r)\psi_{nl}(r)=\tilde{E}\psi_{nl}(r) \tag{12}$$

Where,

$$V_{eff\,(r)}=\frac{Ae^{-2\alpha r}+Be^{-\alpha r}+C}{(1-e^{-\alpha r})^2}\ , \tag{13}$$

$$\tilde{E}_{nl}=\frac{2\mu E_{nl}}{\hbar^2}+\frac{\mu E_{nl}^2}{\tilde{m}\hbar^2} \tag{14}$$

$$A=-\frac{2\mu b\alpha}{\hbar^2}-\frac{2\mu b\alpha E_{nl}}{\tilde{m}\hbar^2}-\frac{\mu^2 b\alpha^2}{\tilde{m}\hbar^2} \tag{15}$$

$$B=\frac{2\mu b\alpha}{\hbar^2}+\frac{2\mu a\alpha}{\hbar^2}+\frac{2\mu ab\alpha^2}{\tilde{m}\hbar^2}+\frac{2\mu a\alpha E_{nl}}{\tilde{m}\hbar^2}+\frac{2\mu b\alpha E_{nl}}{\tilde{m}\hbar^2} \tag{16}$$

$$C=\alpha^2 l(l+1)-\frac{2\mu a\alpha}{\hbar^2}-\frac{\mu a^2\alpha^2}{\tilde{m}\hbar^2}-\frac{2\mu a\alpha E_{nl}}{\tilde{m}\hbar^2} \tag{17}$$

To solve equation (12), we have to first solve the associated Riccati equation[16-20]

$$W^2(r)\mp W'(r)=V_{eff}(r)-\tilde{E}_{0,l}, \tag{18}$$

in which we proposed a superpotential of the form

$$W(r)=\frac{fe^{-\alpha r}}{\left(1-e^{-\alpha r}\right)}+q \tag{19}$$

Therefore, substituting Eq.(19) into Eq.(18), we obtain the Ricaati equation for our study as,

$$\frac{f^2e^{-2\alpha r}}{\left(1-e^{-\alpha r}\right)^2}+\frac{2fq\ e^{-\alpha r}}{\left(1-e^{-\alpha r}\right)}+q^2+\frac{\alpha fe^{-\alpha r}}{\left(1-e^{-\alpha r}\right)^2}=\frac{Ae^{-2\alpha r}+Be^{-\alpha r}+C}{\left(1-e^{-\alpha r}\right)^2}-\tilde{E}_{0,l} \tag{20}$$

Solving equation (20) completely, we obtain the following parameters:

$$\tilde{E}_{0,l}=C-q^2 \tag{21}$$

$$p=\frac{-\alpha\pm\sqrt{\alpha^2+4(A+B+C)}}{2} \tag{22}$$

$$q=\frac{f^2-A+C}{2f} \tag{23}$$

Thus, we construct the supersymmetric partner potentials as follows

$$V_+(r)=\frac{f\left(f+\alpha\right)e^{-2\alpha r}}{\left(1-e^{-\alpha r}\right)^2}+\frac{2fqe^{-\alpha r}}{\left(1-e^{-\alpha r}\right)}+q^2$$

$$V_-(r)=\frac{f\left(f-\alpha\right)e^{-\alpha r}}{\left(1-e^{-\alpha r}\right)^2}+\frac{2fqe^{-\alpha r}}{\left(1-e^{-\alpha r}\right)}+q^2 \tag{24}$$

Thus, it is shown that $V_+(r)$ and $V_-(r)$ are shape invariant, satisfying the shape-invariant condition

$$V_+(r,a_0)=V_-(r,a_1)+R(a_1), \tag{25}$$

with $a_0=f$ and a_1 is a function of a_0, i.e $a_1=f(a_0)=a_0+\alpha$. Therefore $a_n=a_0+\alpha n$, Thus, we can see that the shape invariance holds via a mapping of the form $a_0\to a_0+\alpha$. Therefore from Eq.(25) ,we have

$$R(a_1)=\left(\frac{(a_0)^2-A+C}{2a_0}\right)^2-\left(\frac{(a_1)^2-A+C}{2a_1}\right)^2,$$

$$R(a_2)=\left(\frac{(a_1)^2-A+C}{2a_1}\right)^2-\left(\frac{(a_2)^2-A+C}{2a_2}\right)^2,$$

$$-$$
$$-$$
$$- \qquad (26)$$

$$R(a_n)=\left(\frac{(a_{n-1})^2-A+C}{2a_{n-1}}\right)^2-\left(\frac{(a_n)^2-A+C}{2a_n}\right)^2,$$

The energy eigenvalues can be obtained as follows

$$\tilde{E}_{nl}=\tilde{E}_{nl}^{-}+\tilde{E}_{0,l} \quad , \qquad (27)$$

Where,

$$\tilde{E}_{nl}^{-}=\sum_{k=1}^{\infty}R(a_k)=\left(\frac{(a_0)^2-A+C}{2a_0}\right)^2-\left(\frac{(a_n)^2-A+C}{2a_n}\right)^2, \qquad (28)$$

By substituting Eqs.(21) and (28) into Eq.(27), We get the energy spectrum for Hellman potential in the semi-relativistic theory as,

$$\frac{2\mu E_{nl}}{\hbar^2}+\frac{\mu E_{nl}^2}{\tilde{m}\hbar^2}+\frac{1}{4}\left[\frac{\alpha^2 l(l+1)-\frac{2\mu a\alpha}{\hbar^2}-\frac{\mu a^2\alpha^2}{\tilde{m}\hbar^2}-\frac{2\mu a\alpha E_{nl}}{\tilde{m}\hbar^2}+\frac{2\mu b\alpha}{\hbar^2}+\frac{2\mu b\alpha E_{nl}}{\tilde{m}\hbar^2}+\frac{\mu^2 b\alpha^2}{\tilde{m}\hbar^2}}{(n+\sigma)}+(n+\sigma)\right]^2$$
$$+\alpha^2 l(l+1)-\frac{2\mu a\alpha}{\hbar^2}-\frac{\mu a^2\alpha^2}{\tilde{m}\hbar^2}-\frac{2\mu a\alpha E_{nl}}{\tilde{m}\hbar^2}=0 \qquad (21)$$

Where,

$$\sigma=\frac{1}{2}\left(1\pm\sqrt{1+\frac{4}{\alpha^2}\left(\begin{array}{l}-\frac{2\mu b\alpha}{\hbar^2}-\frac{2\mu b\alpha E_{nl}}{\tilde{m}\hbar^2}-\frac{\mu^2 b\alpha^2}{\tilde{m}\hbar^2}+\frac{2\mu b\alpha}{\hbar^2}+\frac{2\mu a\alpha}{\hbar^2}+\frac{2\mu ab\alpha^2}{\tilde{m}\hbar^2}+\frac{2\mu a\alpha E_{nl}}{\tilde{m}\hbar^2}+\frac{2\mu b\alpha E_{nl}}{\tilde{m}\hbar^2}+\\ \alpha^2 l(l+1)-\frac{2\mu a\alpha}{\hbar^2}-\frac{\mu a^2\alpha^2}{\tilde{m}\hbar^2}-\frac{2\mu a\alpha E_{nl}}{\tilde{m}\hbar^2}\end{array}\right)}\right) \qquad (30)$$

The corresponding wave function is determined as

$$U_{nl}(r)=N_{nl}(e^{-\alpha r})^{\sqrt{w_3}}(1-e^{-\alpha r})^{1/2+\sqrt{w_1-w_2+w_3+1/4}}\,{}_2F_1(-n,n+2\sqrt{w_3^s}+2\sqrt{w_1-w_2+w_3+1/4}+1,2\sqrt{w_3}+1,e^{-\alpha r}), \qquad (31)$$

Where,

$$w_1 = \frac{1}{\alpha^2}\left(A + \frac{2\mu E_{nl}}{\hbar^2} + \frac{\mu E_{nl}^2}{\tilde{m}\hbar^2}\right),$$

$$w_2 = \frac{1}{\alpha^2}\left(-B + 2\left(\frac{2\mu E_{nl}}{\hbar^2} + \frac{\mu E_{nl}^2}{\tilde{m}\hbar^2}\right)\right),$$

$$w_3 = \frac{1}{\alpha^2}\left(C + \frac{2\mu E_{nl}}{\hbar^2} + \frac{\mu E_{nl}^2}{\tilde{m}\hbar^2}\right). \tag{31}$$

Also, $N_{n,J}$ is a normalization constant and the Jacobi polynomials are defined as [21-22]

$$P_n^{(c,d)}(z) = 2^{-n}\sum_{p=0}^{n}\binom{n+c}{p}\binom{n+d}{n-p}(1-z)^{n-p}(1+z)^p, \tag{32}$$

$$P_n^{(c,d)}(z) = \frac{\Gamma(n+c+1)}{n!\Gamma(n+c+d+1)}\sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(n+c+d+r1)}{\Gamma(n+c+1)}\left(\frac{z-1}{2}\right)^r, \tag{33}$$

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}. \tag{34}$$

and it is related to the hypergeometric function ${}_2F_1(-n, n+c+d+1; 1+c; x)$ as,

$$P_n^{(c,d)}(1-2x) = \frac{\Gamma(n+1+c)}{n!\Gamma(1+c)}\,{}_2F_1(-n, n+c+d+1; 1+c; x), \tag{35}$$

3.0 Conclusion

In this paper, we have obtained the approximate solutions of the spinless Sapelter equation for Hellman potential using the SUSYQM. We have obtained the energy eigenvalues and corresponding wave functions in terms of the hypergeometric functions. As special cases, when we set $a = 0, b = -V_0$, the Hellmann potential reduced to the Yukawa potential reported in ref.[13] and when $a = -2V_0, \delta \to 0, b = V_0$,the Hellman potential becomes a Coulomb potential. The results obtained in this work are in agreement with the previously released ones[13]

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