

## On some inequalities for Godunova-Levin and MT-Convex functions

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### Abstract

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*In this paper, we obtained some new Hadamard type inequalities for Godunova-Levin and MT-convex functions, using a simple analytical technique. An example to show the applicability of our results to means of real numbers is included.*

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### 1.0 Introduction

Let  $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a convex function and let  $a, b \in I$  with  $a < b$ . Then the following inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2} \quad (1.1)$$

is known in the literature as Hadamard inequality [1]. Both inequalities hold in the reversed direction if  $f$  is concave. Since its discovery in 1893, Hadamard's inequality has been proven to be one of the most useful inequalities in mathematical analysis. This inequality has the update for a lot of different types of convex function in the mathematics literature (see, [2]).

**Definition 1.1** [3]. A function  $f: I \rightarrow \mathbb{R}$  is Godunova-Levin function or said to belong to the class  $Q(I)$  if  $f$  is nonnegative and for all  $x, y \in I$  and  $t \in (0,1)$  satisfies the inequality:

$$f(tx + (1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t} \quad (1.2)$$

Some further properties of (1.2) are given in [4].

**Definition 1.2**[4]. We say that  $f: I \rightarrow \mathbb{R}$  is a  $P$ -function or that  $f$  belongs to the class  $P(I)$  if  $f$  is nonnegative and for all  $x, y \in I$  and  $t \in (0,1)$ , we have

$$f(tx + (1-t)y) \leq f(x) + f(y) \quad (1.3)$$

**Remark 1.3.** From definition 1.2 and 1.3,  $Q(I) \supset P(I)$ .

**Definition 1.4**[5]. A function  $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is said to belong to the class  $MT(I)$  if it is nonnegative and  $\forall x, y \in I$  and  $t \in (0,1)$ , satisfies the inequality:

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{\sqrt[2]{1-t}} f(x) + \frac{\sqrt{1-t}}{\sqrt[2]{t}} f(y) \quad (1.4)$$

Clearly,  $Q(I) \supset MT(I)$

In [4], Dragomir et al proved two new inequalities of Hadamard type for classes of Godunova-Levin functions and  $P$ -functions.

**Theorem 1.5.** Let  $f \in Q(I)$ ,  $a, b \in I$ , with  $a < b$  and  $f \in L_1([a, b])$ , Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{4}{b-a} \int_a^b f(x) dx \quad (1.5)$$

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**Theorem 1.6.** Let  $f \in Q(I)$ ,  $a, b \in \mathbb{R}$ , with  $a < b$  and  $f \in L_1([a, b])$ , Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a} \int_a^b f(x) dx \leq 2[f(a) + f(b)] \tag{1.6}$$

In [6], the following theorems were proved

**Theorem 1.7.** Let  $a, b \in (0, \infty)$ ,  $a < b$ ,  $I = [a, b]$  and  $f, g: [a, b] \rightarrow \mathbb{R}$  be two functions and  $f, g \in L_1[a, b]$ , If  $f \in Q(I)$  and  $g \in P(I)$ , then,

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{8(g(a) + g(b))}{b-a} \int_a^b f(x) dx \tag{1.7}$$

**Theorem 1.8.** Let  $a, b \in (0, \infty)$ ,  $a < b$ ,  $I = [a, b]$  with  $f, g: [a, b] \rightarrow \mathbb{R}$  be two functions and  $f, g \in L_1[a, b]$ , If  $f$  is convex and  $g$  belongs to the class  $P(I)$ , then,

$$\frac{1}{b-a} \int_a^b f(x)g(x) dx \leq \frac{M(a, b) + N(a, b)}{2} \tag{1.8}$$

Where  $M(a, b) = f(a)g(a) + f(b)g(b)$ ,  $N(a, b) = f(a)g(b) + f(b)g(a)$

Recently, Tunc et al [5] proved the following theorems;

**Theorem 1.9.** Let  $f, g: [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be two nonnegative MT-convex functions and  $f, g \in L_1[a, b]$ . Then, the following inequality holds;

$$\begin{aligned} &g(a) \frac{(b-x)\sqrt{(b-x)(x-a)}}{(b-a)^3} \int_a^b f(x) dx + g(b) \frac{(x-a)\sqrt{(b-x)(x-a)}}{(b-a)^3} \int_a^b f(x) dx \\ &+ f(a) \frac{(b-x)\sqrt{(b-x)(x-a)}}{(b-a)^3} \int_a^b g(x) dx + f(b) \frac{(x-a)\sqrt{(b-x)(x-a)}}{(b-a)^3} \int_a^b g(x) dx \\ &\leq \frac{1}{2} \left\{ \frac{1}{3} f(a)g(a) + \frac{1}{3} f(b)g(b) + \frac{1}{6} [f(a)g(b) + f(b)g(a)] \right\} \\ &+ \int_a^b (b-x)(x-a) f(x)g(x) dx \end{aligned} \tag{1.9}$$

Where  $M$  and  $N$  are as given in theorem (1.8) above.

**Theorem 1.10.** Let  $f, g \in [a, b] \rightarrow \mathbb{R}$  be two nonnegative MT-convex functions and  $c$ . Then,

$$\frac{8}{3} fg\left(\frac{a+b}{2}\right) \leq M(a, b) + N(a, b) \tag{1.10}$$

**Definition 1.11** [7] If  $f, g: X \rightarrow \mathbb{R}$  are two functions, for  $x, y \in X$ . Then, the following inequality;

$$(f(x) + f(y))(g(x) + g(y)) \geq 0 \tag{1.11}$$

Is said to be similarly ordered (briefly s.o.) for the two functions.

In this paper, we established some new inequalities involving two different kinds of convex functions; Godunova-Levin functions and MT-convex functions, using simple techniques in analysis. Our results are independent of the recent result of Tunc et al [5].

**2.0 Main Results**

**Theorem 2.1.** Let  $a, b \in [0, \infty)$ ,  $a < b$ ,  $I = [a, b]$  with  $f, g: [a, b] \rightarrow \mathbb{R}$  be two functions and  $f, g \in L_1([a, b])$ . If  $f \in Q(I)$  and  $g \in MT(I)$ . Then, the following inequality holds;

$$\frac{1}{(b-a)} \int_a^b \mu(x) f(x) g(x) dx \leq \frac{1}{2} \left\{ \frac{3\pi}{128} [f(a)g(a) + f(b)g(b)] + \frac{5\pi}{128} [f(a)g(b) + f(b)g(a)] \right\}$$

$$\text{Where } \mu(x) = \frac{(b-x)^2(b-x)^2}{(b-a)^4}, x \in [a, b] \tag{2.1}$$

**Proof:** Since  $f \in Q(I)$  and  $g$  is MT-convex, we have;

$$f(ta + (1-t)b) \leq \frac{1}{t} f(a) + \frac{1}{1-t} f(b) \tag{2.2}$$

$$g(ta + (1-t)b) \leq \frac{\sqrt{t}}{\sqrt[2]{1-t}} g(a) + \frac{\sqrt{1-t}}{\sqrt[2]{t}} g(b) \tag{2.3}$$

Multiplying (2.2) and (2.3), we get

$$\begin{aligned} & f(ta + (1-t)b)g(ta + (1-t)b) \\ & \leq \left[ \frac{1}{t} f(a) + \frac{1}{1-t} f(b) \right] \left[ \frac{\sqrt{t}}{\sqrt[2]{1-t}} g(a) + \frac{\sqrt{1-t}}{\sqrt[2]{t}} g(b) \right] \\ & \leq \frac{1}{2} \left\{ \frac{\sqrt{t}\sqrt{1-t}}{t(1-t)} f(a)g(a) + \frac{\sqrt{t}\sqrt{1-t}}{t^2} f(a)g(b) + \frac{\sqrt{t}\sqrt{1-t}}{(1-t)^2} f(b)g(a) + \frac{\sqrt{t}\sqrt{1-t}}{t(1-t)} f(b)g(b) \right\} \end{aligned} \tag{2.4}$$

If both sides of inequality (2.4) are multiplied by  $t^2(1-t)^2$ , equation (2.4) becomes;

$$\begin{aligned} & t^2(1-t)^2 [f(ta + (1-t)b)(g(ta + (1-t)b))] \\ & \leq \frac{1}{2} \left\{ f(a)g(a)t(1-t)\sqrt{t}\sqrt{1-t} + f(a)g(b)(1-t)^2\sqrt{t}\sqrt{1-t} + f(b)g(a)t^2\sqrt{t}\sqrt{1-t} \right. \\ & \quad \left. + f(b)g(b)t(1-t)\sqrt{t}\sqrt{1-t} \right\} \end{aligned} \tag{2.5}$$

By integrating both sides of (2.5), we obtain;

$$\begin{aligned} & \int_0^1 t^2(1-t)^2 [f(ta + (1-t)b)(g(ta + (1-t)b))] dt \\ & \leq \frac{1}{2} \left\{ f(a)g(a) \int_0^1 t(1-t)\sqrt{t}\sqrt{1-t} dt + f(a)g(b) \int_0^1 (1-t)^2\sqrt{t}\sqrt{1-t} dt + f(b)g(a) \int_0^1 t(1-t)\sqrt{t}\sqrt{1-t} dt \right\} \end{aligned} \tag{2.6}$$

By substituting,  $x = ta + (1-t)b$ , equation (2.6) gives;

$$\begin{aligned} & \frac{(b-x)^2(x-a)^2}{(b-a)^3} \int_a^b f(x)g(x) dx \leq \frac{1}{2} \left\{ \frac{3\pi}{128} f(a)g(a) + \frac{5\pi}{128} f(a)g(b) + \frac{5\pi}{128} f(b)g(a) + \frac{8\pi}{128} f(b)g(b) \right\} \\ & = \frac{1}{2} \left\{ \frac{3\pi}{128} [f(a)g(a) + f(b)g(b)] + \frac{3\pi}{128} [f(a)g(b) + f(b)g(a)] \right\} \end{aligned}$$

The proof is completed.

**Theorem 2.2.** Let  $f, g: [a, b] \rightarrow R$  be two functions with  $a, b, \in [0, \infty], a < b, I = [a, b]$  and  $f, g \in L_1([a, b])$ . If  $f \in Q(I)$  and  $g \in MT(I)$ . Then, the following inequality holds;

$$\begin{aligned} & \frac{1}{2} \left\{ g(a) \frac{\sqrt{(b-a)^3} \sqrt{(x-a)^3}}{(b-a)^3} \int_a^b f(x) dx + g(b) \frac{\sqrt{(b-a)^3} \sqrt{(x-a)^3}}{(b-a)^3} \int_a^b f(x) dx \right\} \\ & + f(a) \frac{(b-x)(x-a)^2}{(b-a)^4} \int_a^b g(x) dx + f(b) \frac{(b-x)(x-a)^2}{(b-a)^4} \int_a^b g(x) dx \quad (2.7) \\ & \leq \frac{1}{2} \left\{ \frac{3\pi}{128} M(a,b) + \frac{5\pi}{128} N(a,b) \right\} + \frac{(x-a)^2(b-x)^2}{(b-a)^5} \int_a^b f(x)g(x) dx \end{aligned}$$

Where  $M(a,b) = f(a)g(a) + f(b)g(b)$ ,  $N(a,b) = f(a)g(b) + f(b)g(a)$

**Proof:** Since  $f \in Q(I)$  and  $g \in MT(I)$ , then

$$\begin{aligned} f(ta + (1-t)b) & \leq \frac{1}{t}f(a) + \frac{1}{1-t}f(b) \\ g(ta + (1-t)b) & \leq \frac{\sqrt{t}}{\sqrt[2]{1-t}}g(a) + \frac{\sqrt{1-t}}{\sqrt[2]{t}}g(b) \end{aligned} \quad (2.8)$$

Recall that, for  $p, q, r, s \in R^+$ , then  $pq + rs \leq ps + qr$ . Thus, (2.8) gives;

$$\begin{aligned} & f(ta + (1-t)b) \left( \frac{\sqrt{t}}{\sqrt[2]{1-t}}g(a) + \frac{\sqrt{1-t}}{\sqrt[2]{t}}g(b) \right) + g(ta + (1-t)b) \left( \frac{1}{t}f(a) + \frac{1}{1-t}f(b) \right) + g(ta \\ & + (1-t)b) \left( \frac{1}{t}f(a) + \frac{1}{1-t}f(b) \right) \\ & \leq \left( \frac{1}{t}f(a) + \frac{1}{1-t}f(b) \right) \left( \frac{\sqrt{t}}{\sqrt[2]{1-t}}g(a) + \frac{\sqrt{1-t}}{\sqrt[2]{t}}g(b) \right) + f(ta + (1-t)b)g(ta + (1-t)b) \end{aligned} \quad (2.9)$$

Simplifying the inequality (2.9), we have

$$\begin{aligned} & g(a) \frac{\sqrt{t}}{\sqrt[2]{1-t}}f(ta + (1-t)b) + g(b) \frac{\sqrt{1-t}}{\sqrt[2]{t}}f(ta + (1-t)b) + f(a) \frac{1}{t}g(ta + (1-t)b) + f(b) \frac{1}{1-t}g(ta + (1-t)b) \\ & \leq \frac{\sqrt{t}}{\sqrt[2]{1-t}}f(a)g(a) + \frac{\sqrt{1-t}}{\sqrt[2]{t}}f(a)g(b) + \frac{\sqrt{t}}{\sqrt[2]{1-t}}f(b)g(a) \\ & + \frac{\sqrt{1-t}}{\sqrt[2]{t}}f(b)g(b) + f(ta + (1-t)b)g(ta + (1-t)b) + g(a) \frac{\sqrt{t}\sqrt{1-t}}{2(1-t)}f(ta + (1-t)b) \\ & + g(b) \frac{\sqrt{t}\sqrt{1-t}}{2t}f(ta + (1-t)b) \\ & + f(a) \frac{1}{t}g(ta + (1-t)b) + f(b) \frac{1}{1-t}g(ta + (1-t)b) \end{aligned}$$

Further simplification of equation (2.10) gives

$$\begin{aligned} & \frac{1}{2} \{ g(a)t^2(1-t)\sqrt{t}\sqrt{1-t}f(ta + (1-t)b) + g(b)t(1-t)^2\sqrt{t}\sqrt{1-t}f(ta + (1-t)b) \} \\ & + f(a)t(1-t)^2g(ta + (1-t)b) + f(b)t^2(1-t)g(ta + (1-t)b) \end{aligned}$$