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Comparative Study on the Potency of Four Multivariate Analysis of Variance (Manova) Test Statistics

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Abstract

Multivariate Analysis of Variance (MANOVA) is a very vital aspect of multivariate statistical Analysis. In order to estimate the mean vectors of k independent random samples of size n from p-variate standard population, it is essential to guarantee that the best test statistic(s) are engaged so as to achieve a positive result. The four MANOVA test statistics: Roy's largest root test statistic (θ), Pillai Bartlett trace test statistic (V), WilKs' lambda (Λ) and Lawley-Hotelling (U) test statistics are compared in terms of their powers. It was establish that the powers of these test statistics are function of the various configurations of the mean vectors. Roy's largest root test is the most potent when the mean vectors arrangement is one- dimensional, while pillai test takes the lead in terms of power when there are several dimensional configurations.

1.0 Introduction

Analysis of variance (ANOVA) is a technique for partitioning the total deviation of a set of data into several components in order to determine the fraction of the total difference attributable to each source of variation in the data set. It is used mostly in testing of hypotheses relating to population means [1].

In multivariate One-way Analysis of Variance (MANOVA), **k** independent random samples of size **n** are assumed to come from **p**-variate normal population with equal covariance matrices. It involves several dependent variables [2].

Multivariate analysis of variance is used first to investigate whether population means vectors differ, and if they do differ which mean components differ significantly [3].

Since the main point in Multivariate Analysis of variance (MANOVA) is to compare the mean vectors of the **K** samples for significant differences, it is important that suitable test statistic is used in order to have meaningful results.

1.10 The four Multivariate Analysis of Variance Test Statistics

The four test statistics of interest are:

1.1.1 Wilks' Test Statistics or Wilks' Lambda (*Λ*)

This is a likelihood ratio test of H_o: $\mu_1 = \mu_2 \ldots \mu_k$, where $\mu_1, \mu_2, \ldots, \mu_k$are the mean vectors of k independent random samples from a p – variate normal population with equal covariance matrices. It is given by

$$\Lambda = \frac{|\mathrm{E}|}{|\mathrm{E}+\mathrm{H}|} \qquad \textbf{(1.1)}$$

where E = sums of squares and products, not variances and covariances. To estimate Σ, we use $S_{P1} = \frac{\mathrm{E}}{(nk-k)}$,

so that $\mathrm{E}\left(\frac{\mathrm{E}}{nk-k}\right) = \Sigma$

H = between sum and squares on the diagonal for each of the P variables.

E+H =total sum of squares and products matrix.

This is known as Wilks' Λ .(It has also been called Wilks' U). We reject H_0 if $\Lambda \le \Lambda_{\alpha,\mathrm{P},\nu\mathrm{H},\nu\mathrm{E},}$. Note that rejection is for small values of Λ .

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P = number of variables (dimension),
ν_H = degrees of freedom for hypothesis,
ν_E = degrees of freedom for Error,
K = number of groups,
n = sample sizes.
Wilks' Λ can be expressed in terms of eigenvalues $\lambda_1, \lambda_2, ..., \lambda_s$ of $E^{-1}H$, as follows:

$$\Lambda = \prod_{i=1}^{s} \frac{1}{1+\lambda_i} \qquad (1.2)$$

Wilks' Λ transforms to an exact **F**-statistic when ν_H and P take on the values 1, 2. That is when ν_H = 1, 2 or when p = 1, 2. When the transformed values of Λ exceed the upper α-level percentage point of the exact F, H_o is rejected.

$$F = \frac{1 - \Lambda^{\frac{1}{t}}}{\Lambda^{\frac{1}{t}}} \frac{df_2}{df_1} \qquad (1.3)$$

With df_1 and df_2 denoting degrees of freedom,

where $df_1 = P\nu_H$, $df_2 = wt - \frac{1}{2}(P\nu_H - 2)$

$$w = \nu_E + \nu_H - \frac{1}{2}(P + \nu_E + 1)$$

$$t = \sqrt{\frac{P^2 \nu^2_H - 4}{P^2 + \nu_H^2 - 5}} \qquad (1.4)$$

When $P\nu_H = 2, t$ is equal to 1.

Table 1.1 is the table for the Wilks' Λ transformation to an exact upper Tail F-statistic when $\nu_H = 1 or 2$ and whenp=1 0r 2. The hypothesis is rejected when the transformed value of Λ exceeds the upper α - level percentage point of F-distribution.

Table 1.1 Transformations of Wilks' Λ to Exact Upper Tail F- Test

Parameters P,V_H	Statistics Having F-Distribution	Degrees of Freedom
P, $\nu_H = 1$	$\frac{1-\Lambda}{\Lambda} \frac{\nu_E - P + 1}{P}$	P, $\nu_E - P + 1$
P, $\nu_H = 2$	$\frac{1-\sqrt{\Lambda}}{\Lambda} \frac{\nu_E - P + 1}{P}$	$2_P, 2(\nu_E - P + 1)$
P = 1 any ν_H	$\frac{1-\Lambda}{\Lambda} \frac{\nu_E}{\nu_H}$	ν_H, ν_E
$P = 2, any \nu_H$	$\frac{1-\sqrt{\Lambda}}{\Lambda} \frac{\nu_E - 1}{\nu_H}$	$2\nu_H, 2(\nu_E - 1)$

1.1.2 Pillai –Bartlett Test Statistics

This is a multivariate analysis of variance (MANOVA) test statistic for H_o: $\mu_1 = \mu_2 = ... \mu_K$ if $\theta_{\alpha,s,m,N}$. The parameters s, m, and N are defined as s=min(v_H,p), m=$\frac{1}{2}(|\nu_H - p| - 1)$, N=$\frac{1}{2}(\nu_E - P - 1)$. It is based on the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_s$ of $E^{-1}H$.

The Pillai-Bartlett test statistic is given by

$$V^{(s)} = tr[(E+H)^{-1}H] = \sum_{i=1}^{s} \frac{\lambda_i}{1+\lambda_i} \; ... \qquad (1.5)$$

We reject H_o for $V^{(s)} \leq V_\alpha^{(s)}$.

1.1.3 Roy's Test

The Roy's test, known as the Roy's largest root test, is a multivariate statistical test for testing the significant difference of mean vectors based on the largest eigenvalue of $E^{-1}H$
. It is given by

$$\theta = \frac{\lambda_1}{1+\lambda_1} \qquad (1.6)$$

Where λ_1= largest eigenvalue of $E^{-1}H$

H_o: $\mu_1 = \mu_2 = \ldots \mu_k$ is rejected of $\boldsymbol{R} \geq \boldsymbol{R}_{\alpha,\ S,M,N}$

Where s = min (ν_H,P)

m = ½ ($|\nu_H - P|$-1)

N = ½ (V_E – P – 1)

1.1.4 Lawley – Hotelling Statistic

This a MANOVA test statistic for testing
H_o: $\mu_1 = \mu_2 = \ldots = \mu_k$. It is given by

$$U^{(s)} = tr\left(E^{-1}H\right) = \sum_{i-1}^{s} \lambda_i \qquad (1.7)$$

Ho is rejected for large values of $\frac{\nu_E}{\nu_H} U^{(s)}$

in two groups case where ν_H = $n_1 + n_{2-}2$ and ν_E = 1,
This is why Lawley-Hotelling test statistic is also known as the generalized T-test.

2.0 Methodology

To compare the mean vectors of k-samples for significant differences,
the hypothesis,

$H_o : \mu_1 = \mu_2 = \ . \ . \ . = \mu_{2,}$

becomes

$$H_o: \begin{bmatrix} \mu_{11} \\ \mu_{12} \\ \cdot \\ \cdot \\ \mu_{1p} \end{bmatrix} = \begin{bmatrix} \mu_{21} \\ \mu_{22} \\ \cdot \\ \cdot \\ \mu_{2p} \end{bmatrix} = \ . \ . \ . = \begin{bmatrix} \mu_{K1} \\ \mu_{K2} \\ \cdot \\ \cdot \\ \mu_{kp} \end{bmatrix}$$

Thus H_o implies P set of equalities:

$$\mu_{11} = \mu_{21} = \ldots = \mu_{k1}$$

$$\mu_{12} = \mu_{22} = \ldots = \mu_{K2}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\mu_{1P} = \mu_{2P} \ldots = \mu_{KP}$$

All $P(K-1)$ equalities must hold for H_o to be true; failure of only one equality will lead to a rejection of the hypothesis.

2.1 Tests Of Significance Using The Four Manova Test Statistics.

(1) Wilks Test Statistic

The Λ is calculated using $\Lambda = \frac{|E|}{|E - H|}$

$$\Lambda = \prod_{i=1}^{s} \frac{1}{1+\lambda_i}$$

Where $\lambda_1, \lambda_2, \ldots, \lambda_s$ are the eigenvalues of $E^{-1}H$ and s=min (ν_H ,P)

ν_H = between sum of squares and products matrix

$$H = n\sum_{i-1}^{k}\left(\bar{y}_{i.} - \bar{y}_{..}\right)\left(\bar{y}_{i.} - \bar{y}_{..}\right)'$$

$$= \sum_{i=1}^{k}\frac{1}{n}y_{i.}y_{i.}' - \frac{1}{kn}y_{..}y_{..}', \quad (1.9)$$

$$E = \sum_{i-1}^{k}\sum_{j-1}^{n}\left(y_{ij} - \bar{yi.}\right)\left(y_{ij} - \bar{y}_{i.}\right)' \quad = \sum_{ij} y_{ij}y_{ij}' - \sum_{i}\frac{1}{n}y_{i.}y' \quad (1.10)$$

$$\begin{bmatrix} SSH_{11} & SPH_{12} & \ldots & SPH_{1P} \\ SPH_{12} & SSH_{22} & \ldots & SPH_{2P} \\ \vdots & \vdots & \vdots & \vdots \\ SPH_{1P} & SPH_{2P} & \cdots & SSH_{PP} \end{bmatrix} \quad (2.1)$$

Where for example

$$SSH_{22} = n\sum\left(\bar{y}_{i.2} - \bar{y}_{..2}\right)^2 = \sum_{i}\frac{y_{i.2}^2}{n} - \frac{y_{..2}^2}{kn} \quad (2.2)$$

$$SPH_{12} = n\sum_{i=1}^{k}\left(\bar{y}_{i.1} - \bar{y}_{..1}\right)\left(\bar{y}_{i.2} - \bar{y}_{..2}\right) = \sum_{i}\frac{y_{i.1}y_{i.2}}{n} - \frac{y_{..1}y_{..2}}{kn}. \quad (2.3)$$

In these expressions, the subscript 1 or 2 indicate the first or second variables.
The matrix E can be expressed in the form:

$$E = \begin{bmatrix} SSE_{11} & SPE_{12} & \cdots & SPE_{1P} \\ SPE_{12} & SSE_{22} & \cdots & SPE_{2P} \\ \vdots & \vdots & \cdots & \vdots \\ SPE_{1P} & SPE_{2P} & \cdots & SSE_{PP} \end{bmatrix} \quad (2.4)$$

Where

$$SSE_{22} = \sum_{i=1}^{k}\sum_{j=1}^{n}\left(y_{ij2} - \bar{y}_{i.2}\right)^2 = \sum_{ij} y_{ij2}{}^2 - \sum_{i}\frac{y_{i.2}{}^2}{n},$$

$$SPE_{12} = \sum_{i=1}^{k}\sum_{j=1}^{n}\left(y_{ij1} - \bar{y}_{i.1}\right)\left(y_{ij2} - \bar{y}_{i.1}\right) = \sum y_{ij1}y_{ij2} - \sum_{i}\frac{y_{i.1}y_{i.2}}{n}$$

To estimate Σ, we use $S_{P1} = \mathrm{E}/(nk-k)$, so that

$$\mathrm{E}\left(\frac{\mathrm{E}}{nk-k}\right) = \Sigma.$$

0nce the values of W and B are evaluated, the ratio of the determinants of E and E +H are found in order to obtain Λ; that is

$$\Lambda = \frac{|\mathrm{E}|}{|\mathrm{E}+\mathrm{H}|}$$

$$\Lambda - \prod_{i=1}^{s} \frac{1}{1+\lambda_i}$$

Or the matrix $E^{-1}H$ is evaluate and its eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_s$ where s = min (ν_H,P) establish in order to assess. This means that the number of nonzero eigenvalues of $E^{-1}H$ is s = min (P,K-1) which is also the rank of H.

2.2 Obtaining the Eigen values of $E^{-1}H$

$E^{-1}H$ is a non symmetric matrix and so, many workstation programs do not provide its eigenvalues.
Below is the Cholesky approach and it can be used to get the eigenvalues of $(U')^{-1}HU^{-1}$.

$$(E^{-1}H - \lambda I)\,a = 0$$

$$(H - \lambda E)\,a = 0 \text{ (multiplying both sides by W)} \qquad (2.5)$$

Substitute $E = U'U$ into (2.5), multiplying by $(U')^{-1}$, and inserting $U^{-1}U = I$, we have

$$(H - \lambda U'U)a = 0,$$

$$(U')^{-1}(H - \lambda U'U)a = (U')^{-1}0 = 0,$$

$$[(U')^{-1}H - \lambda U]U^{-1}Ua = 0,$$

$$[(U')\text{-}1HU\text{-}1 - \lambda I]Ua = 0. \qquad (2.6)$$

Thus $(U')^{-1}HU^{-1}$ has the same eigenvalues as $E^{-1}H$ and has eigenvectors of the form **Ua**, where **a** is an eigenvector of $E^{-1}H$, with eigenvector Ua has the same eigenvalues as $W^{-1}B$ with eigenvector a.
It is important to note that if sample mean vectors are equal, B= O and $\Lambda = 1$ but becomes much "larger" than W, while Λ tends to zero as the sample mean vectors become more widely spread.

(a) Test of significance using Wilks' Lambda.

Having obtained the eigenvalues of $E^{-1}H$, or simply E and E + H as the case may be, the Λ statistics is evaluated. H_o is rejected if $\Lambda \leq$

$\Lambda_{\alpha, P}, \nu_H, \nu_E$, otherwise accepted.

(b) Test of Significance of differences using Lawley-Hotelling Statistic: With the eigenvalues of $W^{-1}B$ obtained, the Lawley-Hotelling Statistics is obtained as;

$$U^{(s)} = \sum_{i=1}^{s} \lambda_i$$

Then the test statistic $\frac{\nu_E}{\nu_E} U^{(s)}$ is calculated and its critical values are obtained from table. If $\nu_H = 1$ and P > 1, the relationship $U^{(1)} = T^2/V_E$ is used. H_o zero is rejected for large values of $\nu_E/\nu_H U^{(s)}$, otherwise accepted.

(c) Roy's Largest Root Test

The eigenvalues of $E^{-1}H$ obtained are used to evaluate $\theta = \frac{\lambda_1}{1+\lambda_1}$ where λ_1 is the largest eigenvalue of $E^{-1}H$.
If $\theta \geq \theta\alpha$, s, m, N, the null hypothesis H_o: $\mu_1 = \mu_2 = \ldots = \mu_K$ is rejected otherwise it is accepted.
Several software computer programs provide higher bound F approximation of θwhich is given by

$$F = \frac{(\nu_E - d - 1)}{d} \qquad (2.7)$$

With d and V_E-d-1 degrees of freedom.
It is better for Ho to be accepted instead of rejected. The decision of rejection of Ho might not be correct because $F > F_{\alpha}, \nu_E$-d-1 where d = max (P, ν_H).

(d)Pillai-Bartlett Statistic

The Statistic

$$V^{(s)} = \sum_{i=1}^{s} \frac{\lambda_i}{1+\lambda_i}$$

is evaluated using the eigenvalues of $E^{-1}H$ and Hoi: $\mu_1 = \mu_2 = \ldots = \mu_s$ is rejected if $V^{(s)} > V\alpha^{(s)}$, otherwise it is accepted.

2.3 Relationship between the Four MANOVA Tests Statistics and T^2.

(a) Lawley-Hotelling and T^2

We already know that in case of two groups, that is where ν_H = K-1 = 1 and s=1,

$$H = \sum n\left(\bar{y}_{i.} - \bar{y}_{..}\right)\left(\bar{y}_{i.} - \bar{y}_{..}\right)' \qquad (2.8)$$

$$= \frac{n_1 n_2}{n_1 + n_2}\left(\bar{y}_{1.} - \bar{y}_{2.}\right)\left(\bar{y}_{1.} - \bar{y}_{2.}\right)'$$

$$= C\left(\bar{y}_{1.} - \bar{y}_{2.}\right)\left(\bar{y}_{1.} - \bar{y}_{2.}\right)'$$

Where $C = \frac{n_1 n_2}{n_1+n_2}$ (2.9)

$$But U^{(S)} = \sum_{i=1}^{s} \lambda_i$$

$$= tr\,(E^{-1}H)$$

$$= \frac{T^2}{n_1+n_2-2}$$

$U^{(1)} = \frac{T^2}{n_1+n_2-2}$ or $T^2 = (n_1 + n_2 - 2)U^{(1)}$ (2.10)

It is however, very important to see that when $V_H = 1$ and s = 1 all the four test statistics are functions of each other and thus yield equivalent results since there is only one non zero eigenvector. They can be expressed in terms of θ as;

(b) **Roy's largest root**, $\theta = \frac{\lambda_1}{1+\lambda_1}$

(C) **Pillai Bartlett test,** $V^{(s)} = \sum_{i=1}^{s} \frac{\lambda_1}{1+\lambda_1} = \frac{\lambda_1}{1+\lambda_1} = \theta$

(d) **Wilks' Lambda**Λ, $\Lambda = \prod_{i-1}^{s} \frac{1}{1+\lambda_1} - \frac{1}{1+\lambda_1} - 1 - \frac{1}{1+\lambda_1} = 1-\theta$

2.4 The relationship between T^2 and Wilks' Λ, pillai and Roy's largest root is by simply involving;

(a)Lawley -Hotelling , $U^{(1)} = \sum_{i=1}^{s} \lambda_1 = \lambda_1 = \frac{\theta}{1-\theta}$

(b) Wilks Λ and T^2 are related as

$T^2 = (n_1 + n_2 - 2)\frac{1-\Lambda}{\Lambda}$

(c) Pillai Bartlett and T^2

$$T^2 = (n_1 + n_2 - 2)\frac{V^{(s)}}{1-V^{(s)}}$$

(d) Roy's Largest root and T^2;

$$T^2 = (n_1 + n_2 - 2)\frac{\theta}{1-\theta}$$

Appropriate to the multidimensional landscape of the space that contain the denote vectors $\mu_1, \mu_2, \ldots, \mu_K$, the four singular tests which are accurate tests in that each test has chance α of rejecting H_o when H_o is true, may lead to singular conclusions, even when H_o is true in a given sample. H_o may be accepted by some, while it is rejected by others.

In any case where the conclusion of these MANOVA tests differ, F-test (on the same α level) should be carried out on each individual variable if it leads to the rejection of H_o in order to see which variables that in fact contribute to the rejection.

EXAMPLE

The four MANOVA test statistics can be compared, using three drugs A,B,C,with known contaminated effects which are apparent in several measurable features of an organism ingesting them, are used on trial male rats of essentially equal ages and weights drawn from a common laboratory strain and randomly divided among the drugs. The measured features are ; Y_1= loss of weight and Y_2=rate increase.

Table 3.1 Illustrative Example

A		B		C	
Y1	Y2	Y1	Y2	Y1	Y2
5	6	7	6	15	21
5	4	8	7	14	11
9	8	9	12	17	12
		6	8		

The result of the variables obtained is stated below,

SPSS OUT PUT: SAMPLE SIZES, n.
Table 3.2 Grouping of Variables

		Value Label	n
Group	1.00	Group A	**3**
	2.00	Group B	**4**
	3.00	Group C	**3**

Table 3.3 Multivariate Tests[c]

Effect		Value	F	Hypothesis df	Error df	Sig.
Group	Pillai's-B Trace	1.10	4.36	4.0	14	.017
	Wilks' Lambda	.10	6.51[a]	4.0	12	.005
	Hotelling's Trace	6.94	8.67	4.0	10	.003
	Roy's Largest Root	6.62	23.176[b]	2.0	7	.001

Table 3.4 Box's Test of Equality of Covariance Matrices[a]

Box's M	**8.3**
F	**.78**
df1	**6**
df2	**661.6**
Sig.	**.59**

Table 3.4 for Box's M test of Equality of covariance matrices, accepts the null hypothesis at α=0.01 indicate equal covariance matrices.

SPSS OUTPUT: EQUAL SAMPLE SIZES, n.
Table 3.5 Between-subjects factors of group Variables

		Value Label	n
Group	**1**	**A**	**3**
	2	**B**	**4**
	3	**C**	**3**

Table 3.6 Eigen value and % Variance

Function n	Eigenvalue	% of Variance
1	**6.62[a]**	**95**
2	**.32[a]**	**5**

Table 3.6 show that there is one eigenvalue (6.62) and a very small eigenvalue (0.32) using one dimensional of the mean vectors.

Table 3.7 Box's test of Equality of Covariance Matrices[a]

Box's M	13.6
F	1.2
Df1	6
Df2	897.2
sig	.33

The null hypothesis is rejected at α=0.01, indicating unequal covariance matrices.

Table 3.8 Multivariate tests[c]

Effect	Value	F	Hypothesis df	Error df	Sig.
GROUP Pillai's-B Trace	**1.6**	**7.3**	**6**	**12**	**.002**
Wliks' Lambda	**.01**	**13.4[a]**	**6**	**10**	**.000**
Hotelling's Trace	**33.1**	**22.1**	**6**	**8**	**.000**
Roy's Largest Root	**31.6**	**63.2[b]**	**3**	**6**	**.000**

3.1 Effect of Eigenvalues on Mean Vectors.

Given that eigenvalues of E^{-1} are, λ_1 =1.9, λ_2 =0.13, λ_3 = 0.1, λ_4 =0.04, then

$$P_\lambda = \frac{1.9}{1.9 + 0.13 + 0.1 + .004} = 0.8675 = \text{Pr}\,oportion \quad of \quad the \quad 1st \quad eigenvalue.$$

This approximation show that the total parcentage is 86% and the mean vector is in one dimension MANOVA.

4.0 CONCLUSION

The potency of the four MANOVA tests depend on the design of the mean vectors which be available from the eigenvalues of $E^{-1}H$.When there is only one great eigenvalue, while others be minute, the mean vectors design is to one dimension or simply collinear. In this case,Pillai-Bartlett test appears most superior to the other MANOVA tests in this circulate mean vectors arrangement.

Roy's largest root test, which only uses the largest eigenvalue, is the most robust and potency test to be employed. This shows a line design of the mean vectors.

The reverse is the case when the mean vectors configuration is spread out in several dimensions as indicated by the presence of several large eigenvalues.

However, with respect to heterogeneity of covariance matrices in equal group sizes, all the four MANOVA tests are sufficiently robust.

All the mean vectors are at the same point when the null hypothesis, $H_o : \mu_1 = \mu_2 = \ldots = \mu_k$, is true, thus yielding the same Type 1error for all the tests.

The tests can be prearranged in terms of rule as; $\theta > U^{(s)} > \Lambda > V^{(S)}$, for one dimensional design and $V^{(s)} > \Lambda > U^{(S)} > R$, f or dim case.

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