

Short Cut to the Multiple Binomial Price Model for Option Valuation

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Abstract

This paper develops a method by which a fair price value of options could be obtained without going through the cumbersome nature of multiple binomial price model especially when asset transaction period are extended up to three four, five etc, periods. As we continue to equal partition the time interval, it become very necessary for us to use a more convenient method in order to avoid errors of computations. The method in this paper is referred to as a short cut to the multiple binomial price model. This is achieved by applying coefficient of expansion obtained from the Pascal triangle and fixing coefficients into the binomial tree.

1.0 Introduction

In finance, an option is a derivative financial instrument that specifies a contract between two parties for future transaction on an asset at a reference price which is usually called the strike price. An option which convey the right and not the obligation to buy an asset at a specifies price is called a call – option while the option which convey the right and not the obligation to sell an asset at a specifies price is called a put - option .The process of activating an option and trading the underlying asset at a said price is called exercising. The option not exercised at the expiration date becomes void and worthless.[1]

The standard derivative or plain vanilla options are the European and American options, a European option allows exercise only at maturity while an American option allows for exercise at anytime during the life span of the option. Other options like Asian, Barrier, Bermudan etc are called exotic or non standard derivatives.[2]

A European option gives a financial agent the right, but not the obligation to buy (call option) or to sell (put option) on the expiry or maturity date at a specifies price called the strike price or exercise price [3]. The strike price shall be denoted as K and maturity date as T. We shall also denote the price or payoff of a European call(put) by P^{EC} . The payoff P^{EC} to an investor who hold a long position on a European call with strike price K and expiration date T is

$$P_{long}^{EC} = \text{Max} \{ S(T) - K, 0 \} = \{ S(T) - K \}^+ \tag{1.1}$$

If $S(T) > K$, the option is in money, meaning the buyer will buy.

If $S(T) < K$, the option is out of money, meaning the buyer will not buy.

If $S(T) = K$, the option is at the money, meaning the buyer can decide to buy or not.

The payoff to an investor who holds a short position on a European call is

$$P_{short}^{EC} = - \text{Max} \{ S(T) - K, 0 \} = \text{Min} \{ K - S(T), 0 \}^+ \tag{1.2}$$

For a put option,

$$P_{long}^{EC} = \text{Max} \{ K - S(T), 0 \} = \{ K - S(T), 0 \}^+ \tag{1.3}$$

$$P_{short}^{EC} = - \text{Max} \{ K - S(T), 0 \} = \text{Min} \{ S(T) - K, 0 \}^+ \tag{1.4}$$

On the other hand, American option gives a financial agent the right, but not obligation to buy (call) or sell (put) an option on or prior to expiry date at a specifies price K.

Actually there are basically four possible positions;

A long position on a call – option , a short position on call – option , a long position on put – option , and a short position on a put – option . Each of these positions are usually characterized in terms of the payoff at maturity to the agent [4].

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Determining the price of an option has been a major challenge in the financial world, making Mathematical modeling simulation an inevitable tool in the financial industry .[4]

Banks, Brokerage houses, Investors and other consulting services spent countless hours to simulate and predict price movement for financial instrument like option. Much of these Mathematics employed in this field, particularly in academic research is highly sophisticated [1] which spans the field of Analysis, Probability and stochastic, Statistics, Differential equations and Numerical analysis. Obviously, numerical analysis are becoming familiar with financial application and terminology associated with them .[5] Although in recent development, models and algorithm has been proposed which reach the state of the art in numerical analysis. [2]

The binomial model is one of the various numerical methods for determining the price of an option, it is very useful especially when the holder of an option has early exercise decision to make prior to the maturity or when exact formulas are not available. It can accommodate complex option pricing problems [1].

2.0 The binomial model

Suppose that S_0 is the stock price at the beginning of a given time period, then the binomial price model assumed that at the end of each time period ,the price will either go up to S_0u with probability p or down to S_0d with probability $(1 - p)$, where u and d are up and down factors with $d < 1 < u$ [6]

The parameter u, d and p satisfy the condition for risk –neutral valuation (a situation where all expected returns must be zero).Thus we have

$$S_0 e^{r\Delta T} = S_0 p u + S_0 (1-p) d = e^{-r\Delta T} = P u + (1 - P) \tag{2.1}$$

Since S follows a lognormal distribution, its variance is given as

$$\text{var}(S_0) = S_0^2 e^{-2r\Delta T} (e^{\sigma^2 \Delta T} - 1) \tag{2.2}$$

But $\text{var}(S) = E(S^2) - (E(S))^2$

Thus

$$S_0^2 e^{-2r\Delta T} (e^{\sigma^2 \Delta T} - 1) = P u^2 S_0^2 + (1 - P) d^2 S_0^2 - (P u S + (1 - P) d S)^2 \tag{2.3}$$

Which can be simplified to

$$e^{-2r\Delta T + \sigma^2 \Delta T} = P u^2 + (1 - P) d^2 \tag{2.4}$$

Suppose $u = \frac{1}{d}$, then from (2.1) and (2.4)

$$P = \frac{e^{r\Delta T} - d}{u - d}, \quad d = e^{-\sigma\sqrt{\Delta T}} \tag{2.5}$$

The P in (2.5) is called the risk neutral probability which represent an upward movement of stock price.

2.1 One period binomial price model

This involve pricing an option or an underlying asset whose price today is S_0 and where the price of the option today is f . Suppose that the option’s life span is T , we must find f which is the fair price or value, in this regard, we assume that during the option’s life, the price of the asset can move up from S_0 to S_0u or down from S_0 to S_0d where $u > 1$ and $0 < d < 1$ [6]

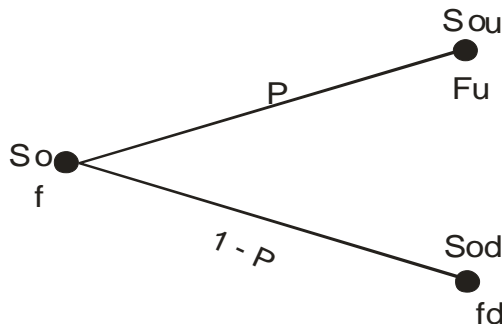


Fig (2.1)

$$f = e^{-rT} [P f u + (1 - P) f d] \tag{2.6}$$

For American option, the derivative may not be alive till the expiration date, hence there is need for some modification if using the binomial model. This is to allow for the possibility of early exercise of an American option. The option price today is then the maximum of the exercise value and the value of the option is kept alive[4]. Thus the price f today of an American option is

$f = \text{Max}[\text{value of exercised now}; e^{-rT}(Pfu + (1 - P))fd]$ with P as in (2.5)

for example , if the American option is a put option, then

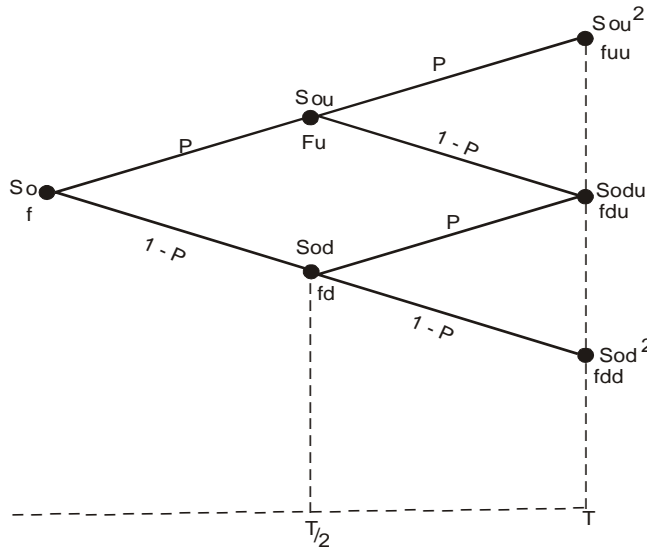
$$f = \text{Max} (K - S_0)^+, e^{-rT}(Pfu + (1 - P)fd) \tag{2.7}$$

$$\text{with } fu = (K - S_0u)^+ \text{ and } fd = (K - S_0d)^+ \tag{2.8}$$

Where K is the strike price and S_0 is the asset price today

3.0 Multiple – period binomial price model

This is generated by equi-partitioning the life span of the derivative security into small intervals and generating a binomial tree.



fig(3.1)

Figure (3.1) is the case of a two – period binomial model, we get

$$fu = e^{-rT/2} [Pfu + (1 - p)]fdu \tag{3.1.1}$$

$$fd = e^{-rT/2} [Pfd + (1 - p)]fdd \tag{3.1.2}$$

$$f = e^{-rT/2} [Pfu + (1 - P)]fd \tag{3.1.3}$$

Substituting (3.1.1) and (3.1.2) into (3.1.3), we have

$$f = e^{-rT} [P^2 fu + 2p(1 - p)fdu + (1 - P)^2 fdd] \tag{3.1.4}$$

The number P^2 , $2P(1 - P)$ and $(1 - P)^2$ are the risk neutral probabilities that the underlying asset price S_{0u^2} , S_{0du} and S_{0d^2} respectively are attained.

It is glaring from the above that determining the current ($t = 0$) price f of an option by multi- period binomial tree , we begin from the end of the tree, with each nodes giving rise to a one period binomial model which is easily priced.

For a contingent claim whose payoff only depends on the asset price $S(T)$ is of the form $g(S(T))$, then the current value of the option is given by

$$f = e^{-rT} \sum_{j=1}^N \binom{N}{j} g(S_0 u^j d^{N-j}) P^j (1 - P)^{N-j} \tag{3.5}$$

The coefficient $\binom{N}{j} P^j (1 - P)^{N-j}$ is the probability of the asset price being $S_0 u^j d^{N-j}$ at time T.

Since transaction is a continuous process, we can extend to a three period binomial model as;

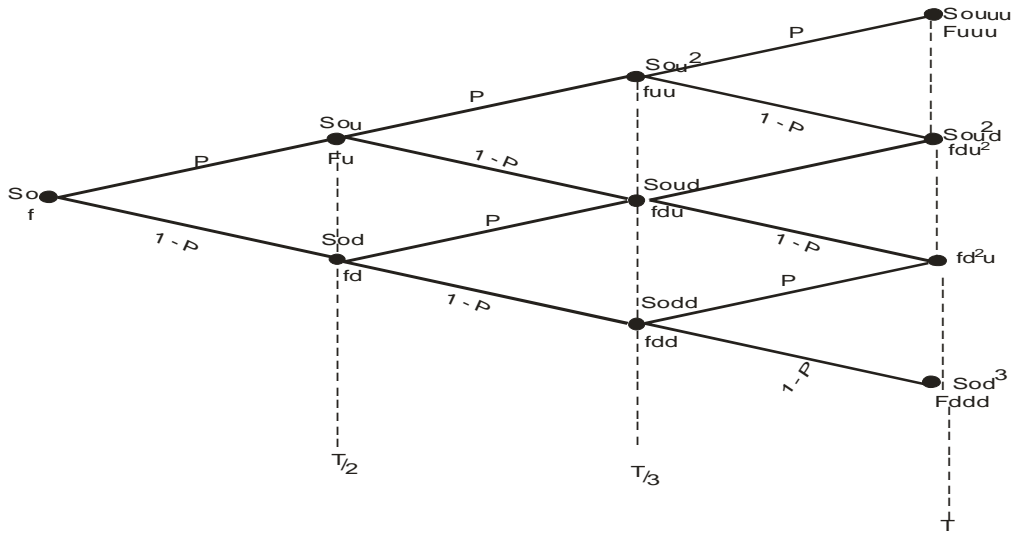


Fig (3.2)

Similarly the following equations arise

$$fdd = e^{-rT/3}(Pfd^2u + (1 - P)fd^2) \tag{3.5.1}$$

$$fdu = e^{-rT/3}(Pfd^2u + (1 - P)fd^2) \tag{3.5.2}$$

$$fuu = e^{-rT/3}(Pfu^3 + (1 - P)fdu^2) \tag{3.5.3}$$

$$fu = e^{-rT/2}(Pfu^2 + (1 - P)fdu) \tag{3.5.4}$$

$$fd = e^{-rT/2}(Pfd^2u + (1 - P)fd^2) \tag{3.5.5}$$

$$f = e^{-rT/2}(Pfu + (1 - P)fd) \tag{3.5.6}$$

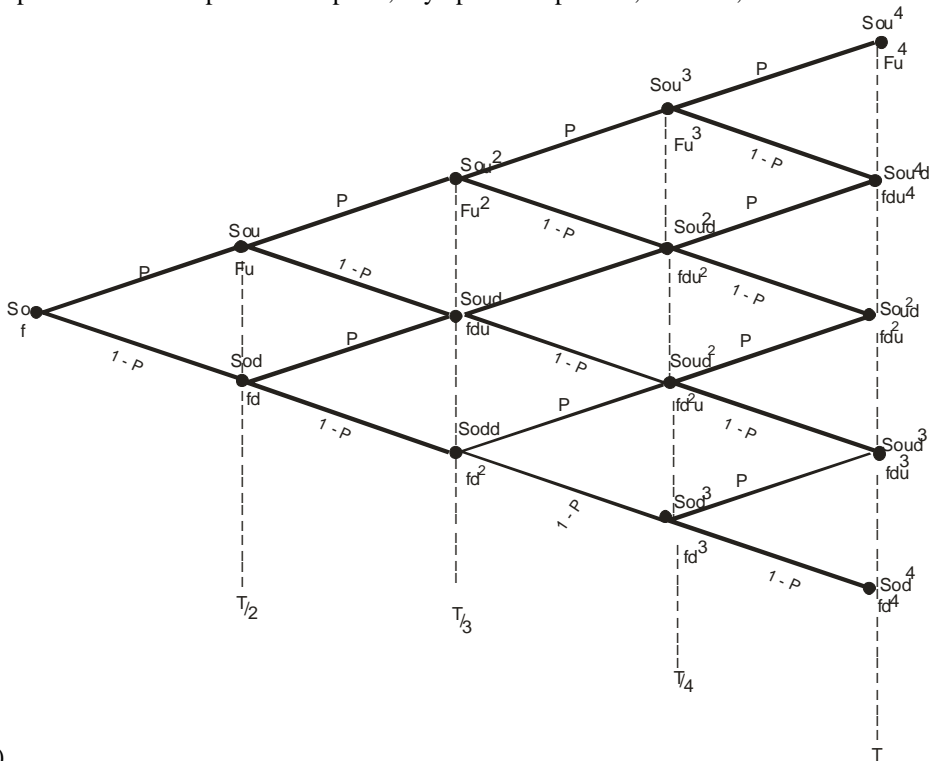
By substituting (3.5.4) and (3.5.5) into (3.5.6), we have

$$f = e^{-rT}(P^2fu^2 + 2P(1 - P)fdu + P(1 - P)^2fd^2) \text{ which is same as (3.1.3)}$$

Again if we substitute (3.5.3), (3.5.2) and (3.5.1) into (3.1.3) we obtain

$$f = e^{-rT}(P^3fu^3 + 3P^2(1 - P)fdu^2 + 3P(1 - P)^2fd^2u + (1 - P)fd^3) \tag{3.5.7}$$

Now if we continue to partition the life span of the option, say up to four periods, we have;



Fig(3.3)

Fig(3.3) give us the following equations,

$$fd^3 = e^{-rT/4} (Pfd^3u + (1 - P)fd^4) \tag{3.6.1}$$

$$fd^2u = e^{-rT/4} (Pfd^2u^2 + (1 - P)fd^3u) \tag{3.6.2}$$

$$fdu^2 = e^{-rT/4} (Pfd^3u + (1 - P)fd^2u^2) \tag{3.6.3}$$

$$fu^3 = e^{-rT/4} (Pfu^4 + (1 - P)fdu^3) \tag{3.6.4}$$

$$fd^2 = e^{-rT/3} (Pfd^2u + (1 - P)fd^3) \tag{3.6.5}$$

$$fdu = e^{-rT/3} (Pfd^2u + (1 - P)fd^2u) \tag{3.6.6}$$

$$fu^2 = e^{-rT/3} (Pfu^2 + (1 - P)fdu^2) \tag{3.6.7}$$

$$fu = e^{-rT/2} (Pfu^2 + (1 - P)fdu) \tag{3.6.8}$$

$$fd = e^{-rT/2} (Pfd^2u + (1 - P)fd^2) \tag{3.6.9}$$

$$f = e^{-rT/2} (Pfu + (1 - P)fd) \tag{3.6.10}$$

similarly if we solve the above equations, we have

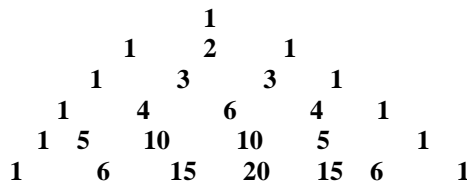
$$f = e^{-rT} (P^4fu^4 + 4P^3(1 - P)fdu^3 + 6P^2(1 - P)^2fd^2u^2 + 4P(1 - P)^3fd^3u + (1 - P)^4fd^4) \tag{3.6.11}$$

From the above, it is obvious that the more we increase the number of steps, the more difficult it becomes to obtain fs

4.0 The short cut method

Due to the cumbersome nature of the above method, it will be more convenient to introduced an alternative method whereby a lot of equations will not be generated. This method is refer to in this paper as the short cut method to the multi-binomial price model. This method will help researchers to obtain option price easily rather than generating series of equations

Consider the Pascal triangle



(fig4.1)

For one period,

$$f = e^{-rT} (Pfu + (1 - P)fd)$$

This imply that, the coefficient of P and (1 - P) are both one and the powers are one too.

For two periods, we picked out the coefficients of expansion from fig (4.1) as 1, 2, 3, next step is to draw the binomial tree . see Fig (3.1) and locate f as shown below,

From S_0 to S_0u^2 , we have two P's and fuu, i. e in the upward movement, also in the down ward movement, we have two (1 - P)'s and fd^2 , the product of which gives $(1-P)^2fd^2$. At the middle, we have one upward movement and one downward movement with fdu.

Combining the above and fixing the coefficient, we have (3.1.4).

Similarly for a three periods, the binomial tree, see fig (3.2) is in four section;

From S_0 to S_0u^3 , we have three P's and fu^3 giving P^3fu^3

From S_0 to S_0d^3 , we have three (1-P)'s and fd^3 giving $(1 - P)^3 fd^3$

From fd to S_0u^2d we have two P's, one (1 - P) and fdu^2 giving $P^2(1 - P)fd u^2$

From fd^2 to S_0ud^2 , we have one p, two (1 - P)'s and fd^2u giving $P(1 - P)^2fd^2u$

Now the coefficient of expansion from fig (4.1) are 1, 3, 3 and 1, by combining the above result and fixing the coefficients, we have (3.5.7)

Again the tree for four periods, is in five sections, see fig (3.3)

From S_0 to S_0u^4 we have P^4fu^4 , from S_0d to S_0u^3d we have $P^3(1 - P)fdu^3$

From S_0d^2 to $S_0d^2u^2$ we have $P^2(1 - P)^2fd^2u^2$, from S_0d^3 to S_0d^3u we have $P(1 - P)^3fd^3u$ and from S_0 to S_0d^4 we have $(1 - P)^4fd^4$. By fixing coefficients of expansion, we have (3.6.11).

We can continue in this manner as trading continues, however trading periods could be extended with ease up to 5, 6, 7,

5.0 Summary and conclusion

This paper have examined the multi-binomial price model as one of the numerical methods of option valuation. It is assumed to be the simplest model for stock price fluctuations, that is in small time interval Δt , the stock price will either go up to S_0u or go down to S_0d with probability P and $(1-P)$ respectively. This method is good for pricing options like the American option where a decision need to be made at each time step.

We also developed a method by which a fair price of option could be obtained without going through the cumbersome nature of the multi- binomial price model especially when transactions period are extended up to three , four , five etc. periods.

With this method, a financial agent can determine the fair price of an option within numerous trading periods.

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