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An EOQ model for delayed deteriorating items with inventory level dependent demand rate and partial backlogging

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Abstract

This paper presents an Economic Order Quantity (EOQ) model for delayed deteriorating items having inventory level dependent demand rate and shortages. Demand at any instant depends linearly on the on-hand inventory level at that instant. In the initial phase, inventory depletes down to a certain level of the inventory due to market demand only. In the second phase, the inventory level gets depleted due to the effect of both market demand and deterioration but still dependent on stock until the inventory level falls to zero. In the final phase of the cycle, shortages are allowed and the unsatisfied demand is partially backlogged at a rate which is a fixed fraction of demand rate during the shortage period. We establish the theoretical results for the optimal replenishment policy of the inventory system in order to minimize the total average system cost per unit time. Furthermore, we present some numerical examples to illustrate the application of the model developed and use the examples to study the effect of various changes in some possible combinations of model parameters on the decision variables of the system.

Keywords: Stock-dependent demand, delayed deterioration, partial backlogging rate.

1.0 Introduction

The amount of inventory to be maintained will naturally depend upon the consumption or usage of the commodity involved, and in many cases, the amount of inventory has a motivational effect on how customers purchase the commodity. It is a common belief that large piles of goods displayed in a shelf can attract large number of customers to buy more.

Whitin [1] observed, without empirical evidence that, ‘an increase in inventories may bring about increased sales of some items.’ Wolfe [2] presented empirical evidence of this relationship, noting that the sales of style merchandise, such as women’s dresses or sport clothes, are proportional to the amount of inventory displayed. Levin *et al.* [3] and Silver and Peterson [4] also observed that sales at the retail level tend to be proportional to inventory displayed and that a large pile of goods displayed in a supermarket will lead customers to buy more. This fact attracted a number of researchers to derive Economic Order Quantity (EOQ) models concentrated on stock-dependent demand rate patterns. Gupta and Vrat [5] were the first to incorporate these observations into inventory models, developing an inventory model with stock-dependent consumption rate, which is a function of the initial stock level. Baker and Urban [6] investigated an inventory model in which the demand rate in polynomial functional form is a function of the instantaneous stock level. Mandal and Phaudjar [7] assumed that the demand rate depends linearly on the on-hand inventory at any instant time. Datta and Pal [8] have discussed the model of inventory- level- dependent demand rate in which the demand rate depends upon inventory level down to a certain stock level and then it becomes constant for the rest of the cycle, until the inventory level is zero. Later on, Alfares [9] considered the inventory policy for an item with a stock-level-dependent demand rate and a storage-time-dependent holding cost.

In most inventory models a general assumption is that products have indefinitely long lives. However, in many cases, items deteriorate over time. Often the rate of deterioration is low and there is little need to consider the deterioration in the determination of inventory model. Nevertheless, there are many products in the real world that are subject to a significant rate of deterioration. Some examples of commodities which deteriorate over the time include blood, fish, meat, fruits, vegetables, gasoline, radioactive chemicals, potatoes, yams, products that have expiry date and so on.

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The inventory problem of deteriorating items was first studied by Whitin [1] who studied fashion items deteriorating at the end of the storage period. Then Ghare and Schrader [10] concluded in their study that the consumption of deteriorating items was closely relative to a negative exponential function of time. Mandal and Phaujdar [11] presented an Economic Production Quantity (EPQ) model concerning a single item with a variable rate of deterioration and the demand rate is dependent on instantaneous inventory level. Baraya and Sani [12] proposed an EPQ model for delayed deteriorating items with stock-dependent demand rate and linear time dependent holding cost. Padmanabhan and Vrat [13] developed an inventory model for initial stock dependent consumption rate and exponential decay. Datta and Pal [14] extended Mandal and Phaujdar's [11] work for deteriorating items with the assumption that the demand rate is a linear function of the on-hand inventory by allowing shortages, which are completely backlogged for both finite and infinite time horizons. Pal *et al*. [15] developed a deterministic inventory model by assuming that the demand rate is stock-dependent and the items deteriorate at constant rate. Later on, many researchers have done a lot of significant work on inventory models with inventory-level-dependent demand, such as considering, non-linear holding cost [16], perishable items [17,18,19], perishable items and non – linear holding cost [20]. Teng *et al*. [21] extended Datta and Pal's [14] EOQ model to allow not only deteriorating items but also non-zero ending inventory. Sana and Chaudhuri [22] analyzed a kind of EOQ model with current-stock-dependent demand where a supplier gives a retailer both a credit period and a price discount on the purchase of merchandise. Chang *et al.* [19] investigated inventory models with stock-and price-dependent demand rate for deteriorating items based on limited shelf space.

In the literature of inventory systems, inventory models for deteriorating items mostly assume that deterioration starts as soon as the retailer receives the commodities. However, in real life, many goods would have a span of maintaining quality or the original condition for some period. That is during that period there is no deterioration occurring, and that phenomenon is termed as "delayed deterioration" in Musa and Sani [23] or "non - instantaneous deterioration" in Wu *et al.* [24].

Wu *et al*. [24] developed an inventory model for non-instantaneous deteriorating items with stock-dependent demand and time-proportional partial backlogging rate. Jain *et al*. [25] suggested an inventory model with inventory level-dependent demand rate, non-instantaneous deterioration, partial backlogging and decrease in demand. Uthayakumar and Geetha [26] developed a finite planning horizon inventory model for non-instantaneous deteriorating items with stock dependent consumption rate. Here, shortages are allowed and partially backlogged. Mahata and Goswami [27] proposed Fussy EOQ Models for deteriorating items with stock dependent demand and non-linear holding costs. Of recent, Ruxian *et al.* [28] reviews the trends in deteriorating inventory studies including deteriorating items for inventory level-dependent demand inventory policies.

In practice, when shortages occur, some customers are willing to wait for backorder and others are impatient to wait and therefore would turn to buy from other competitors. For inventory models with stock dependent consumption rate, many inventory researchers assumed that shortages are completely backlogged such as [29], [30], [31], amongst others. On the other hand, authors such as Park [32]), Hollier and Mak [33], Wee [34], Urban [35] and Uthayakumar and Geetha [26] considered constant partial backlogging rates during the shortage period in their inventory models. Padmanabhan and Vrat [17] considered an EOQ model for perishable items with stock-dependent selling rate for sales environment. Under instantaneous replenishment with zero lead time, they present three models: without backlogging, with complete backlogging and partial backlogging for which they assumed that the amount of demand backlogged may depend on the amount of orders already backlogged. Chang and Dye [36] were among the first to introduce partial backlogging rate in Economic Order Quantity models. They assumed that, the length of the waiting time for the next replenishment becomes main factor for determining the backlogging rate. That is to say, the longer the waiting time is, the smaller the backlogging rate would be.

Many researchers considered time-proportional partial backlogging rate in their inventory models. Amongst them are Abad [37], Papachristos and Skouri [29, 38], Chang and Dye [42], Wang [39], Chern *et al*. [40], Min and Zhou [41], Dye *et al*. [42], Hou *et al*. [43], etc. Abad [44, 37] discussed a pricing and lot-sizing problem for a product with a variable rate of deterioration, allowing shortages and partial backlogging. Hou [31] afterwards extended the model of Padmanabhan and Vrat [17] by incorporating the effects of deterioration rate, inflation and time value of money to develop an inventory model with stock-dependent consumption rate and with complete backlogging for non-sales environment. Dye and Ouyang [45] developed an inventory model in which the proportion of customers, who would like to accept backlogging, is the reciprocal of a linear function of the waiting time. Wu *et al.* [24] discussed the problem of determining the optimal replenishment policy for non-instantaneous deteriorating items with stock-dependent demand. In the model, shortages are allowed; the backlogging rate is variable and dependent on the waiting time for the next replenishment.

In this paper, we present an EOQ model for delayed deteriorating items with inventory level dependent demand rate and partial backlogging rate. We assume a fixed fraction of demand rate to be the backlogging rate during the shortage period as in Wee [34]. In the initial stage, inventory depletes down to a certain level of the inventory due to market demand only. In the second stage the inventory level gets depleted due to the effect of both market demand and deterioration but still dependent on stock until the inventory level falls to zero. In the final phase of the cycle, shortages are allowed and the unsatisfied demand is partially backlogged at a rate which is a fixed fraction of demand rate during the shortage period. Several items such as cassava, yams, potatoes, some fruits and vegetables, and others exhibit this property. We establish the theoretical results for the optimal replenishment policy of the inventory system in order to minimize the total average system cost per unit time. Furthermore, we present some numerical examples to illustrate the application of the model developed and use the examples to study the effects of various changes in some model parameters on the decision variables of the system.

2. Notation and Modeling Assumptions

The inventory system is developed on the basis of the following notation and model assumptions:

2.1 Notation:

Q The Order Quantity which enters into inventory at time t = 0

S_d The inventory level up to which market demand rate is stock-dependent and depletion is based only on demand

T The length of replenishment cycle time

t_d The length of time in which the product has no deterioration and after this period of time, a constant fraction $\theta\ (0 < \theta < 1)$, of the on-hand inventory deteriorates

t_1 The length of time after which the shortage time begins

$I_1(t)$ The level of positive inventory at time t $(0 \le t \le t_d)$ in which the product has no deterioration

$I_2(t)$ The level of positive inventory at time t $(t_d \le t \le t_1)$ in which the product has deterioration

$B(t)$ The level of backorder at time t $(t_1 \le t \le T)$ in which the product has shortage

C_0 The ordering cost per order for each cycle

μ_1 The unit inventory holding cost per unit time of item, Naira/unit/unit time

μ_2 The unit deterioration cost per unit time of item, Naira/unit/unit time

μ_3 The unit shortage cost for backlogged items per unit time of item, Naira/unit/unit time

μ_4 The unit opportunity cost due to lost sales per unit time of item, Naira/unit/unit time

2.2 Assumptions:

(i) The unit cost of item and the traditional ordering cost are known and constant

(ii) The inventory system involves only one item and one stocking point in each cycle

(iii) The time horizon of the inventory system is infinite. Only a typical planning schedule of length is considered, all remaining cycles are identical

(iv) The demand rate D (t) at time t is assumed to be $D\left[I(t)\right] = \alpha + \beta I(t)$, where α is a positive constant, β is the stock dependent demand rate parameter, $0 < \beta < 1$, and $I(t)$ is the inventory level at time *t*.

(v) The deterioration of the on-hand inventory $I(t)$ begins after a time t_d from the instant of arrival in inventory.

(vi) Shortages are allowed and partially backlogged. We adopt the concept used in Wee (1995), where the unsatisfied demand is backlogged, and the fraction of shortages backordered is $\delta\ (0 < \delta < 1)$. The extreme cases $\delta = 0$ and $\delta = 1$ represent the scenarios of no shortage allowed and complete backlogging, respectively.

3.0 Mathematical Model and Analysis

Using the above assumptions, a typical cycle for the variation of inventory level with time where the shortages are allowed is shown in figure 1.

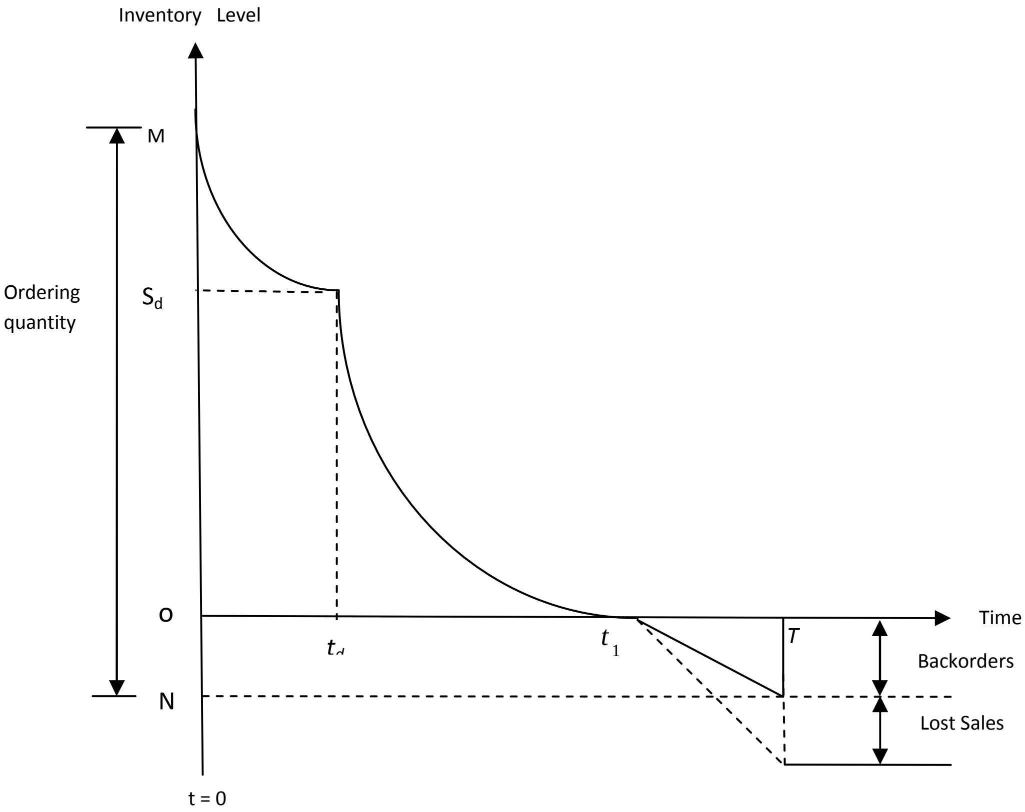

Figure 1: The graphical representation for the inventory system.

The inventory level dependent demand rate $D\left[I(t)\right]$ of the product at any instant of time t is a linear function of $I(t)$ and is given by $D\left[I(t)\right] = \alpha + \beta I(t)$. Maximum inventory M units of item arrive at the inventory system at the beginning of each cycle. During the time interval $\left[0, t_d\right]$, the on-hand inventory level gradually falls from M to the level S_d at time $t = t_d$, due to demand only. The inventory level is dropping to zero due to demand and deterioration during the time interval $\left[t_d, t_1\right]$. Then during the time interval $\left[t_1, T\right]$ shortages set in and unsatisfied demand is backlogged at a rate δ of the demand rate. The whole process is then repeated.

The inventory level at any time t in the interval $\left[0, t_d\right]$ can be represented by the following differential equation:

$$\frac{dI_1(t)}{dt} = -\left(\alpha + \beta I_1(t)\right), \quad 0 \le t \le t_d, \tag{1}$$

with the initial condition $I_1(0) = \mathrm{M}$.

Solution of equation (1) is obtained as follows:

$\frac{dI_1(t)}{dt} + \beta I_1(t) = -\alpha$ is a linear differential equation in $I_1(t)$ whose integrating factor is $e^{\beta t}$ so that

$$I_1(t)e^{\beta t} = -\frac{\alpha}{\beta}e^{\beta t} + k$$ where, k is a constant.

With the boundary condition $I_1(0) = M$, we have $k = M + \frac{\alpha}{\beta}$

and

$$I_1(t) = Me^{-\beta t} - \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) \tag{2}$$

During the time interval $[t_d, t_1]$ the product depletes owing to the effect of both market demand and deterioration. Given that θ is a constant fraction of the on-hand inventory that gets deteriorated per unit time, the differential equation governing the instantaneous state of $I_2(t)$ over the time period in $[t_d, t_1]$ is given by

$$\frac{dI_2(t)}{dt} + \theta I_2(t) = -\left(\alpha + \beta I_2(t)\right), \quad t_d \le t \le t_1 \tag{3}$$

with boundary condition $I_2(t_1) = 0$.

Solution of equation (3) is obtained as follows:

$\frac{dI_2(t)}{dt} + (\beta + \theta) I_2(t) = -\alpha$ is also a linear differential equation in $I_2(t)$ whose integrating factor is $e^{(\beta+\theta)t}$, which gives

$$I_2(t)e^{(\beta+\theta)t} = -\frac{\alpha}{\beta+\theta}e^{(\beta+\theta)t} + k_2$$ where, k$_2$ is a constant of integration.

With the boundary condition $I_2(t_1) = 0$, we have $k_2 = \frac{\alpha}{\beta+\theta}e^{(\beta+\theta)t_1}$ and we have

$$I_2(t) = \frac{\alpha}{\beta+\theta}\left(e^{(\beta+\theta)(t_1-t)} - 1\right) \tag{4}$$

During the shortage interval $[t_1, T]$, the demand at time t is partially backlogged at a fixed fraction $\delta\ (0 < \delta < 1)$ of the demand rate. Thus, the backorder level is governed by the differential equation:

$$\frac{dB(t)}{dt} = -\delta\left(\alpha + \beta\, B(0)\right),$$ since there is no stock after t$_1$

and so

$$\frac{dB(t)}{dt} = -\alpha\delta, \ t_1 \le t \le T \tag{5}$$

Solution of equation (5) is obtained as follows:

$B(t) = -\alpha\delta t + k_3$, where k_3 is a constant.

With the boundary condition $B(t_1) = 0$, we have $k_3 = \alpha\delta t_1$

$$\therefore \quad B(t) = -\alpha\delta(t - t_1) \tag{6}$$

Since $I(t)$ must be continuous at $t = t_d$, then from equations (2) and (4), we get

$$Me^{-\beta t_d} - \frac{\alpha}{\beta}\left(1 - e^{-\beta t_d}\right) = \frac{\alpha}{\beta+\theta}\left(e^{(\beta+\theta)(t_1-t_d)} - 1\right)$$

or

$$M = \frac{\alpha}{\beta+\theta}\left(e^{(\beta+\theta)(t_1-t_d)} - 1\right)e^{\beta t_d} + \frac{\alpha}{\beta}\left(e^{\beta t_d} - 1\right) \tag{7}$$

Substituting equation (7) into equation (2), we obtain

$$I_1(t)=\left[\frac{\alpha}{\beta+\theta}\left(e^{(\beta+\theta)(t_1-t_d)}-1\right)e^{\beta t_d}+\frac{\alpha}{\beta}\left(e^{\beta t_d}-1\right)\right]e^{-\beta t}-\frac{\alpha}{\beta}\left(1-e^{-\beta t}\right)$$

$$=\left[\frac{\alpha}{\beta+\theta}\left(e^{(\beta+\theta)(t_1-t_d)}-1\right)e^{\beta(t_d-t)}+\frac{\alpha}{\beta}\left(e^{\beta(t_d-t)}-e^{-\beta t}\right)\right]-\frac{\alpha}{\beta}\left(1-e^{-\beta t}\right)$$

$$=\left[\frac{\alpha}{\beta+\theta}\left(e^{(\beta+\theta)(t_1-t_d)}-1\right)e^{-\beta(t-t_d)}+\frac{\alpha}{\beta}\left(e^{-\beta(t-t_d)}-e^{-\beta t}\right)\right]-\frac{\alpha}{\beta}\left(1-e^{-\beta t}\right) \qquad (8)$$

Substituting $t=T$ in equation (6), we obtain the maximum amount of demand backlogged per cycle which is given by

$$N=-B(T)$$

$$=\alpha\delta(T-t_1) \qquad (9)$$

Combining equations (7) and (9), we obtain the order quantity, Q, as

$$Q=M+N$$

$$=\frac{\alpha}{\beta+\theta}\left(e^{(\beta+\theta)(t_1-t_d)}-1\right)e^{\beta t_d}+\frac{\alpha}{\beta}\left(e^{\beta t_d}-1\right)+\alpha\delta(T-t_1) \qquad (10)$$

3.1 Total inventory during the complete cycle time T

This is given by

$$C_h=\int_0^{t_d}I_1(t)\,dt+\int_{t_d}^{t_1}I_2(t)\,dt$$

$$=\int_0^{t_d}\left(\frac{\alpha}{\beta+\theta}\left(e^{(\beta+\theta)(t_1-t_d)}-1\right)e^{-\beta(t-t_d)}+\frac{\alpha}{\beta}\left(e^{-\beta(t-t_d)}-e^{-\beta t}\right)-\frac{\alpha}{\beta}\left(1-e^{-\beta t}\right)\right)dt$$

$$+\frac{\alpha}{\beta+\theta}\int_{t_d}^{t_1}\left(e^{(\beta+\theta)(t_1-t)}-1\right)dt$$

$$=\left[\frac{\alpha}{\beta(\beta+\theta)}\left(1-e^{(\beta+\theta)(t_1-t_d)}\right)-\frac{\alpha}{\beta^2}-\frac{\alpha t_d}{\beta}\right]-\left[\frac{\alpha e^{\beta t_d}}{\beta(\beta+\theta)}\left(1-e^{(\beta+\theta)(t_1-t_d)}\right)-\frac{\alpha e^{\beta t_d}}{\beta^2}\right]$$

$$-\frac{\alpha}{\beta+\theta}\left[\left(t_1+\frac{1}{\beta+\theta}\right)-\left(t_d+\frac{e^{(\beta+\theta)(t_1-t_d)}}{\beta+\theta}\right)\right]$$

$$=\frac{\alpha}{\beta(\beta+\theta)}\left[1-e^{(\beta+\theta)(t_1-t_d)}-e^{\beta t_d}\left(1-e^{(\beta+\theta)(t_1-t_d)}\right)\right]-\left[\frac{\alpha}{\beta^2}+\frac{\alpha t_d}{\beta}-\frac{\alpha e^{\beta t_d}}{\beta^2}\right]$$

$$-\frac{\alpha}{\beta+\theta}\left[\left(t_1+\frac{1}{\beta+\theta}\right)-\left(t_d+\frac{e^{(\beta+\theta)(t_1-t_d)}}{\beta+\theta}\right)\right]$$

$$= \frac{\alpha}{\beta(\beta+\theta)}\left(1-e^{(\beta+\theta)(t_1-t_d)}\right)\left(1-e^{\beta t_d}\right)+\frac{\alpha}{\beta^2}\left(e^{\beta t_d}-\beta t_d-1\right)$$

$$+\frac{\alpha}{(\beta+\theta)^2}\left(e^{(\beta+\theta)(t_1-t_d)}-(\beta+\theta)(t_1-t_d)-1\right) \tag{11}$$

3.2 Total amount of items which deteriorate during the cycle time

This is given by

$$C_{\det} = I_2(t_d) - \text{demand during the time interval } [t_d, t_1]$$

$$= \frac{\alpha}{\beta+\theta}\left(e^{(\beta+\theta)(t_1-t_d)}-1\right)-\left(\int_{t_d}^{t_1}\alpha\,dt+\frac{\alpha\beta}{\beta+\theta}\int_{t_d}^{t_1}\left(e^{(\beta+\theta)(t_1-t)}-1\right)dt\right)$$

$$= \frac{\alpha}{\beta+\theta}\left(e^{(\beta+\theta)(t_1-t_d)}-1\right)-\alpha(t_1-t_d)-\frac{\alpha\beta}{(\beta+\theta)^2}\left(e^{(\beta+\theta)(t_1-t_d)}-(\beta+\theta)(t_1-t_d)-1\right)$$

$$= \frac{\alpha}{(\beta+\theta)^2}\left\{(\beta+\theta)\left(e^{(\beta+\theta)(t_1-t_d)}-1\right)-(\beta+\theta)^2(t_1-t_d)-\beta\left(e^{(\beta+\theta)(t_1-t_d)}-(\beta+\theta)(t_1-t_d)-1\right)\right\}$$

$$= \frac{\alpha}{(\beta+\theta)^2}\left\{\theta\, e^{(\beta+\theta)(t_1-t_d)}-(\beta+\theta)-(\beta+\theta)^2(t_1-t_d)+\beta(\beta+\theta)(t_1-t_d)+\beta\right\}$$

$$= \frac{\alpha}{(\beta+\theta)^2}\left\{\theta\, e^{(\beta+\theta)(t_1-t_d)}-\theta-(\beta+\theta)(t_1-t_d)\left((\beta+\theta)-\beta\right)\right\}$$

$$= \frac{\alpha}{(\beta+\theta)^2}\left\{\theta\, e^{(\beta+\theta)(t_1-t_d)}-\theta-\theta(\beta+\theta)(t_1-t_d)\right\}$$

$$= \frac{\alpha\theta}{(\beta+\theta)^2}\left\{e^{(\beta+\theta)(t_1-t_d)}-(\beta+\theta)(t_1-t_d)-1\right\} \tag{12}$$

3.3 Total amount of shortage (backordered) items due to backlog during the cycle time

This is given as:

$$C_s = \int_{t_1}^{T}\left(-B(t)\right)dt$$

$$= \alpha\delta\int_{t_1}^{T}(t-t_1)\,dt$$

$$= \alpha\delta\left\{\left(\frac{T^2}{2}-Tt_1\right)-\left(\frac{t_1^2}{2}-t_1^2\right)\right\}$$

$$= \frac{\alpha\delta}{2}(T-t_1)^2 \tag{13}$$

3.4 Total amount of demand items unsatisfied during the cycle time.

This is given by

$$C_{lost} = \int_{t_1}^{T}\alpha(1-\delta)\,dt$$

$$= \alpha(1-\delta)(T-t_1) \tag{14}$$

3.5 Total relevant inventory cost per unit time

This is obtained as:

$$\text{TVC}\left(\text{t}_1, T\right) = \frac{1}{\text{T}}\left\{\begin{array}{l}\text{ordering cost + inventory holding cost per cycle + the deterioration cost per cycle}\\ \text{+ shortage cost per cycle due to backlog + opportunity cost per cycle due to lost sales}\end{array}\right\}$$

$$= \frac{1}{T}\left\{C_0 + \mu_1 C_h + \mu_2 C_{\text{det}} + \mu_3 C_s + \mu_4 C_{lost}\right\}$$

$$= \frac{1}{T}C_0 + \frac{1}{T}\mu_1 C_h + \frac{1}{T}\mu_2 C_{\text{det}} + \frac{1}{T}\mu_3 C_s + \frac{1}{T}\mu_4 C_{lost}$$

$$= \frac{C_0}{T} + \frac{\mu_1}{T}\left\{\frac{\alpha}{\beta(\beta+\theta)}\left(1 - e^{(\beta+\theta)(t_1 - t_d)}\right)\left(1 - e^{\beta t_d}\right) + \frac{\alpha}{\beta^2}\left(e^{\beta t_d} - \beta t_d - 1\right)\right.$$
$$\left. + \frac{\alpha}{(\beta+\theta)^2}\left(e^{(\beta+\theta)(t_1 - t_d)} - (\beta+\theta)(t_1 - t_d) - 1\right)\right\}$$
$$+ \frac{\mu_2}{T}\left\{\frac{\alpha\,\theta}{(\beta+\theta)^2}\left(e^{(\beta+\theta)(t_1 - t_d)} - (\beta+\theta)(t_1 - t_d) - 1\right)\right\}$$
$$+ \frac{\mu_3}{T}\left\{\frac{\alpha\delta}{2}(T - t_1)^2\right\} + \frac{\mu_4}{T}\left\{\alpha(1-\delta)(T - t_1)\right\}$$

$$= \frac{C_0}{T} + \frac{\mu_1}{T}\left\{\frac{\alpha}{\beta(\beta+\theta)}\left(1 - e^{(\beta+\theta)(t_1 - t_d)}\right)\left(1 - e^{\beta t_d}\right) + \frac{\alpha}{\beta^2}\left(e^{\beta t_d} - \beta t_d - 1\right)\right\}$$
$$+ \frac{\alpha(\mu_1 + \theta\mu_2)}{(\beta+\theta)^2 T}\left\{e^{(\beta+\theta)(t_1 - t_d)} - (\beta+\theta)(t_1 - t_d) - 1\right\}$$
$$+ \frac{\mu_3}{T}\left\{\frac{\alpha\delta}{2}(T - t_1)^2\right\} + \frac{\mu_4}{T}\left\{\alpha(1-\delta)(T - t_1)\right\} \qquad (15)$$

4.0 Solution Procedure

Based on the non-linear objective function (15), TVC (t_1, T) is a function of t_1 and T. The necessary conditions for the total system cost per unit time in equation (15) to be minimized are

$$\frac{\partial}{\partial t_1}\left(TVC(T, t_1)\right) = 0 \text{ and } \frac{\partial}{\partial T}\left(TVC(T, t_1)\right) = 0 \qquad (16)$$

The differential equation $\frac{\partial}{\partial t_1}\left(TVC(T, t_1)\right) = 0$ is equivalent to

$$\frac{\partial}{\partial t_1}\left(TVC(T, t_1)\right) = -\frac{\alpha\mu_1}{\beta T}\left(1 - e^{\beta t_d}\right)e^{(\beta+\theta)(t_1 - t_d)} + \frac{\alpha\mu_1}{(\beta+\theta)T}\left(e^{(\beta+\theta)(t_1 - t_d)} - 1\right)$$
$$+ \frac{\alpha\mu_2\theta}{(\beta+\theta)T}\left(e^{(\beta+\theta)(t_1 - t_d)} - 1\right) - \frac{\mu_3\alpha\delta}{T}(T - t_1) - \frac{\mu_4\alpha}{T}(1-\delta) = 0 \qquad (17)$$

Multiplying both sides of equation (17) by $T\beta(\beta+\theta)$, simplifying and rearranging the terms, yields,

$$-\alpha\mu_1(\beta+\theta)\left(1 - e^{\beta t_d}\right)e^{(\beta+\theta)(t_1 - t_d)} + \alpha\mu_1\beta\left(e^{(\beta+\theta)(t_1 - t_d)} - 1\right)$$
$$+ \alpha\mu_2\theta\beta\left(e^{(\beta+\theta)(t_1 - t_d)} - 1\right) - \mu_3\alpha\delta\beta(\beta+\theta)(T - t_1) - \mu_4\alpha\beta(\beta+\theta)(1-\delta) = 0 \qquad (18)$$

Grouping the terms in (18), we have

$$\left\{\alpha\beta(\mu_1+\theta\mu_2)-\alpha\mu_1(\beta+\theta)\left(1-e^{\beta t_d}\right)\right\}e^{(\beta+\theta)(t_1-t_d)}-\mu_3\alpha\delta\beta(\beta+\theta)T+\mu_3\alpha\delta\beta(\beta+\theta)t_1$$
$$-\ \alpha\beta\left\{(\mu_1+\mu_2\theta)\ +\mu_4(\beta+\theta)(1-\delta)\right\}\ =\ 0$$

which can be written as

$$\omega_1 e^{(\beta+\theta)(t_1-t_d)}+\omega_2 T+\omega_3 t_1+\omega_4\ =\ 0 \tag{19}$$

where

$$\omega_1\ =\ \alpha\beta(\mu_1+\theta\mu_2)-\alpha\mu_1(\beta+\theta)\left(1-e^{\beta t_d}\right),\ \omega_2=-\mu_3\alpha\delta\beta(\beta+\theta),$$
$$\omega_3=\ \mu_3\alpha\delta\beta(\beta+\theta),\ \omega_4=\ -\ \alpha\beta\left\{(\mu_1+\mu_2\theta)\ +\mu_4(\beta+\theta)(1-\delta)\right\}$$

Similarly, $\frac{\partial}{\partial T}\left(TVC(T,t_1)\right)=0$ is equivalent to

$$\frac{\partial}{\partial T}\left(TVC(T,t_1)\right)=-\frac{C_0}{T^2}-\frac{\mu_1}{T^2}\left\{\frac{\alpha}{\beta(\beta+\theta)}\left(1-e^{(\beta+\theta)(t_1-t_d)}\right)\left(1\ -\ e^{\beta t_d}\right)\ +\ \frac{\alpha}{\beta^2}\left(e^{\beta t_d}\ -\ \beta t_d\ -\ 1\right)\right\}$$
$$-\ \frac{\alpha(\mu_1+\theta\mu_2)}{(\beta+\theta)^2T^2}\left\{e^{(\beta+\theta)(t_1-t_d)}\ -\ (\beta+\theta)(t_1-t_d)-\ 1\right\}$$
$$+\frac{\mu_3}{T^2}\left\{\frac{\alpha\delta}{2}\left(T^2-t_1^2\right)\right\}+\frac{\mu_4}{T^2}\left\{\alpha(1-\delta)\,t_1\right\}\ =\ 0 \tag{20}$$

Multiplying both sides of equation (20) by $T^2(\beta+\theta)^2\beta^2$, we have,

$$-C_0(\beta+\theta)^2\beta^2-\mu_1\left\{\alpha\beta(\beta+\theta)\left(\ e^{(\beta+\theta)(t_1-t_d)}-1\right)\left(e^{\beta t_d}-1\right)\ +\ \alpha(\beta+\theta)^2\left(e^{\beta t_d}\ -\ \beta t_d\ -\ 1\right)\right\}$$
$$-\ \alpha\beta^2(\mu_1+\theta\mu_2)\left\{e^{(\beta+\theta)(t_1-t_d)}\ -\ (\beta+\theta)(t_1-t_d)-\ 1\right\}$$
$$+\mu_3\beta^2(\beta+\theta)^2\,\frac{\alpha\delta}{2}\left(T^2-t_1^2\right)+\mu_4\beta^2(\beta+\theta)^2\,\alpha(1-\delta)\,t_1\ =\ 0 \tag{21}$$

Simplifying and rearranging the terms in (21), we get

$$\left\{\alpha\mu_1\beta(\beta+\theta)\left(1-e^{\beta t_d}\right)-\ \alpha\beta^2(\mu_1+\theta\mu_2)\right\}e^{(\beta+\theta)(t_1-t_d)}$$
$$+\mu_3\beta^2(\beta+\theta)^2\,\frac{\alpha\delta}{2}T^2-\mu_3\beta^2(\beta+\theta)^2\,\frac{\alpha\delta}{2}t_1^2$$
$$+\left\{\alpha\beta^2(\beta+\theta)(\mu_1+\theta\mu_2)\ +\mu_4\beta^2(\beta+\theta)^2\,\alpha(1-\delta)\right\}t_1$$
$$-\mu_1\left\{\alpha\beta(\beta+\theta)\left(1\ -\ e^{\beta t_d}\right)\ +\ \alpha(\beta+\theta)^2\left(e^{\beta t_d}\ -\ \beta t_d\ -\ 1\right)\right\}\qquad-\ \alpha\beta^2(\mu_1+\theta\mu_2)\left\{\ (\beta+\theta)t_d-\ 1\right\}$$
$$-C_0(\beta+\theta)^2\beta^2\ =\ 0 \tag{22}$$

which also reduces to

$$\Upsilon_1 e^{(\beta+\theta)(t_1-t_d)}+\Upsilon_2T^2+\Upsilon_3 t_1^2+\Upsilon_4 t_1+\Upsilon_5\ =\ 0 \tag{23}$$

where

$$\Upsilon_1\ =\ \ \alpha\mu_1\beta(\beta+\theta)\left(1-e^{\beta t_d}\right)-\ \alpha\beta^2(\mu_1+\theta\mu_2),$$

$$\Upsilon_2 = \mu_3\beta^2(\beta+\theta)^2\frac{\alpha\delta}{2},\ \Upsilon_3 = -\mu_3\beta^2(\beta+\theta)^2\frac{\alpha\delta}{2}$$

$$\Upsilon_4 = \alpha\beta^2(\beta+\theta)(\mu_1+\theta\mu_2) + \mu_4\beta^2(\beta+\theta)^2\,\alpha(1-\delta),$$

$$\Upsilon_5 = -\mu_1\left\{\alpha\beta(\beta+\theta)\left(1 - e^{\beta t_d}\right) + \alpha(\beta+\theta)^2\left(e^{\beta t_d} - \beta t_d - 1\right)\right\}$$
$$- \alpha\beta^2(\mu_1+\theta\mu_2)\left\{(\beta+\theta)t_d - 1\right\} - C_0(\beta+\theta)^2\beta^2$$

Solving the two non-linear equations (19) and (23) simultaneously will give the optimum values t_1* of t and T* of T provided these values t_1* and T* satisfy the conditions

$$\frac{\partial^2 TVC(T,t_1)}{\partial T^2} > 0 \tag{24}$$

and

$$\det\left[H_m(T,t_1) > 0\right], \tag{25}$$

where, $\det\left[H_m(T,t_1)\right]$, is the determinant of the Hessian matrix given by

$$H_m(T,t_1) = \begin{bmatrix} \dfrac{\partial^2 TVC(T,t_1)}{\partial T^2} & \dfrac{\partial^2 TVC(T,t_1)}{\partial T\partial t_1} \\ \dfrac{\partial^2 TVC(T,t_1)}{\partial t_1\partial T} & \dfrac{\partial^2 TVC(T,t_1)}{\partial t_1^2} \end{bmatrix}$$

It is not however easy to get the solution of equations (19) and (23) by an analytic method. Hence, there is need for an approximation solution method. From (19), we have

$$T = -\frac{1}{\omega_2}\left(\omega_1 e^{(\beta+\theta)(t_1-t_d)} + \omega_3 t_1 + \omega_4\right) \tag{26}$$

Substituting this value into (23), we get an equation in a single variable t_1 as,

$$\Upsilon_1 e^{(\beta+\theta)(t_1-t_d)} + \frac{\Upsilon_2}{(\omega_2)^2}\left(\omega_1 e^{(\beta+\theta)(t_1-t_d)} + \omega_3 t_1 + \omega_4\right)^2 + \Upsilon_3 t_1^2 + \Upsilon_4 t_1 + \Upsilon_5 = 0 \tag{27}$$

We can now use equation (27) to solve for t_1 using Newton-Raphson method and then substitute the solution obtained for (27) into (26) to compute T. These solutions T* and t_1* will now jointly make the optimal solution of (19) and (23) provided equations (24) and (25) are satisfied. By using the optimal values of t_1* and T* in (15) and (10), optimal minimum system cost per unit time $TVC\left(t_1^*, T^*\right)$ and optimal order quantity Q* respectively are obtained.

5.0 Numerical examples

Four tables (Table 1, Table 2, Table 3 and Table 4) of examples are presented to illustrate the model developed and the effects of changes of some model parameters on the decision variables. From Table 1, an inventory system in the first example has the following values of parameters: $\alpha = 180$, $\beta = 0.7$, $\theta = 0.2$, $\delta = 0.2$, C_o = ₦ 80 per order, μ_1 = ₦ 0.5 per unit, μ_2 = ₦ 2 per unit, μ_3 = ₦ 2.5 per unit, μ_4 = ₦ 1.2 per unit, $t_d = 0.1$ time units. By using Newton-Raphson method, the optimal length of time with positive inventory t_1* is obtained using expression (27) as $t^* = 0.724$ units, the optimum cycle length of time T* is determined using expression (26) as $T^* = 1.028$ units and corresponding minimum total system cost per unit time is determined using expression (15) as TVC* = ₦153.04 while the optimal order quantity per cycle is determined using expression (10) as Q* = 224.12 units. For other values of the parameters as reflected in the table, the optimal length of time with positive inventory t_1*, the optimum cycle length of time T*, the corresponding minimum total system cost per unit time TVC*, the optimal order quantity per cycle Q* are similarly determined. The results are tabulated in Table 1. Also, Table 2, Table 3 and Table 4 are similarly drawn up, with changes as indicated in the heading of the table. Note that in all examples given, the sufficient conditions expressed by equations (24) and (25) are satisfied implying that t_1* and T* so determined are indeed the minimum values in the considered region.

Table 1: Effect of changing the constant demand rate α on the decision variables

S/No	α	β	δ	θ	μ_1	μ_2	μ_3	μ_4	C_o	t_d	t_1*	$T*$	Q*	TVC*
1	180	0.7	0.2	0.2	0.5	2	2.5	1.2	80	0.1	0.724	1.028	224.12	153.04
2	185	0.7	0.2	0.2	0.5	2	2.5	1.2	80	0.1	0.718	1.015	227.01	154.67
3	190	0.7	0.2	0.2	0.5	2	2.5	1.2	80	0.1	0.711	1.002	229.86	156.28

Table 2: Effect of changing the stock-dependent demand rate β on the decision variables

S/No	α	β	δ	θ	μ_1	μ_2	μ_3	μ_4	C_o	t_d	t_1*	$T*$	Q*	TVC*
1	150	0.3	0.7	0.15	0.5	2	2.5	1.2	120	0.08	1.083	1.501	251.73	182.94
2	150	0.4	0.7	0.15	0.5	2	2.5	1.2	120	0.06	1.047	1.472	255.40	184.65
3	150	0.5	0.7	0.15	0.5	2	2.5	1.2	120	0.06	1.013	1.446	258.96	186.30

Table 3: Effect of changing the deterioration rate θ on the decision variables

S/No	α	β	δ	θ	μ_1	μ_2	μ_3	μ_4	C_o	t_d	t_1*	$T*$	Q*	TVC*
1	160	0.8	0.9	0.15	0.7	2	2.5	1.2	80	0.08	0.680	1.040	203.67	150.34
2	160	0.8	0.9	0.20	0.7	2	2.5	1.2	80	0.08	0.643	1.013	196.16	153.26
3	160	0.8	0.9	0.25	0.7	2	2.5	1.2	80	0.08	0.609	0.990	189.69	155.81

Table 4: Effect of changing the backlogging rate δ on the decision variables

S/No	α	β	δ	θ	μ_1	μ_2	μ_3	μ_4	C_o	t_d	t_1*	$T*$	Q*	TVC*
1	170	0.8	0.5	0.1	0.5	1.5	2	1.7	80	0.08	0.913	0.958	242.92	152.06
2	180	0.8	0.55	0.1	0.5	1.5	2	1.7	80	0.08	0.909	1.021	247.89	151.02
3	180	0.8	0.6	0.1	0.5	1.5	2	1.7	80	0.08	0.902	1.066	251.49	149.57

6.0 Results and Discussion

A careful study of the computational results as shown in Tables 1 - 4, and within the range of values of the chosen parameters, reveals the following observations:

(1) From Table 1, a higher value of α results in higher values of Q* and TVC*, but lower values of t* and T*. This implies that increase in the demand rate will result in the increase of the optimal order quantity per cycle Q* and the minimum total system cost per unit time TVC*, but decrease in the length of time with positive inventory t_1* and the optimum cycle length of time T*. This is expected since if demand rate is higher, stock will take less time to finish and so t_1* and T* decrease. Increasing the total demand will, also in turn, increase the order quantity and the overall system cost per unit time.

(2) From Table 2, a higher value of β results in higher values of Q* and TVC*, but lower values of t* and T*. This implies that increase in the stock-dependent demand rate will result in the increase of the optimal order quantity per cycle and the minimum total system cost per unit time, but decrease in the length of time with positive inventory and the optimum cycle length of time. This is expected since if stock dependent demand rate is higher, stock will finish earlier and so T* will decrease. On the other hand, increasing the stock- dependent demand parameter β will effectively increase the total demand which, in turn, will increase the overall system cost per unit time.

(3) From Table 3, higher values of θ result in higher values of TVC*, but lower values of t_1*, T* and Q*. In this case, it implies that increase in deterioration rate will result in increase of the minimum total system cost per unit time, but decrease in the optimal length of time with positive inventory, optimum cycle length of time and optimal order quantity per cycle. The total variable cost per unit time, however, increases in this case, which is also expected since when deterioration cost increases, the total variable cost per unit time will also increase. With deterioration rate higher however, stock will finish earlier resulting in lower T*.

(4) From Table 4, higher values of δ result in higher values T* and Q*, but lower values of t_1* and TVC*. In this case, it implies that increase in backlogging rate will result in increase of the optimum cycle length of time and the optimal order quantity per cycle and , but decrease in the values of the optimal length of time with positive inventory and the optimal minimum total system cost per unit time. The total variable cost per unit time in this case is certainly expected to decrease due to zero stocking cost. It is also expected that increasing the backlogging rate δ increases the order quantity Q.

7.0 Conclusion and Suggestions

In this article, an economic order quantity (EOQ) model is presented for delayed deteriorating items with stock-dependent demand, allowing shortages and fixed backlogging rate. The impact of stock dependent demand rate, constant rate of deterioration and partial backlogging parameters on order quantity, maximum inventory level and total system cost per unit time are reported. We present some numerical examples to illustrate the application of the model developed and use the examples to study the effect of various changes in some possible combinations of model parameters on the decision variables of the system. We find from the above results that the effects of stock dependent demand rate, deterioration and backlogging rate on the optimal replenishment policy are significant, and hence should not be ignored in developing inventory models. The proposed model can be used in inventory control of certain delayed deteriorating items such as food items, electronic components, fashionable commodities, vegetables, fruits, yams, potatoes, and so on. The model we have presented in this study provides a basis for several possible extensions. For future research, the model can be enriched by incorporating inflation, time value of money, variable holding cost, and so on.

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