Journal of the Nigerian Association of Mathematical Physics
***Volume 25, No. 2* (November, 2013), pp 355 – 362**
© J. of NAMP

Fractional Derivatives and Decay-Growth Problem

[1]Modebei M. I., [2]Olaiya O. O, [3]Otaide I.

[1,2]Department of Mathematics Programme,
National Mathematical Centre, Abuja, Nigeria.
[3]Department of Mathematics,
Delta State University, Abraka

Abstract

Radioactive substances decay and as a result they create other substances which also have the affinity to decay. An example is Radium which decays to Radon and thus further decays to Polonium. In real sense, these decays take place in fractions. Suppose that initially a sample of pure radium decays. It is of interest to determine how much radium and radon the sample contain at time t. In this work we make a close comparison of the integer order and fractional order derivatives using the decay-growth model equation of a radioactive substance thus giving the amount of sample it contains after disintegration at time t.

Keywords: Fractional derivatives, Fractional Differential Equation, Matrix exponential function, Mittag-Laffler function.

1.0 Introduction

Fractional calculus is as old as normal conventional integer-order calculus. In a letter correspondence, Marquis de L'Hospital asked Leibniz "What if the order of the derivative is $1/2$?´´. To this end, Leibniz replied in a prophetical way, "Thus it follows that will be equal to $x^{2}\sqrt[2]{2dx:x)}$, an apparent paradox, from which one day useful consequences will be drawn". This letter of Leibniz was dated 30th September, 1695 [1].

In spite of its long history, fractional calculus was not considered eligible for any applications. This was due to its high complexity and lack of physical and geometric interpretation. Application of fractional calculus to real-world problems is only four decades old. Applications can be broadly categorized into: Modeling of Systems and Fractional-order Control [2].

Rigorous mathematical theory has been developed. Geometrical interpretation or physical meaning exists. But it is not as straight forward as for the integer-order derivatives [3].

A typical example of differential equations involving fractional derivatives. is the Bagley-Torvik equation of oscillatory processes with fractional damping [4]:

$$\frac{d^2}{dx^2}f(x)+\alpha D_t^{\frac{3}{2}}f(x)+\beta f(x)=g(x) \qquad (1)$$

There are likewise Ordinary Differential Eqauations (ODEs) and Partial Differential Equatioms (PDEs). which have fractional derivatives that are Linear and nonlinear [4]. Existence and uniqueness of solutions of fractional differential equations have been established, and analytical solutions are difficult to evaluate. Dedicated, elegant numerical methods exist as well [5].

Some of the merits of fractional derivatives are [2]:

1. Calculating time-fractional derivative of a function $f(t)$ at some time $t=t_1$ which requires all the past history, i.e. all $f(t)$ from $t=0$ to $t=t_1$.
2. Employing fractional derivatives for modeling systems with memory.
3. Calculating space-fractional derivative of a function $f(x)$ at $x=x_1$ which requires all non-local $f(x)$ values.
4. Employing fractional derivatives for modeling distributed parameter systems.

Corresponding author: ***Modebei M. I.,*** E-mail: gmarc.ify@gmail.com, Tel.: +2348051749774

In this paper we illustrate the rate at which a radioactive substance disintegrates with time fractionally as compared to the disintegration when modeled using the conventional differential equation with integer order

2.0 Preliminaries

2.1 Gamma Function

This is otherwise refered to as the generalization of the factorial for all real numbers, defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad x \in R \tag{2}$$

with the property that

$$\Gamma(x+1) = x\Gamma(x)$$

$$\Gamma(x) = (x-1)!$$

2.2 Mittag-Leffler Function

This function is a direct generalization of the conventional exponential function e^x, and it play a very important role in fractional calculus. The Mittag-Leffler function is defined in terms of power series as follows

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \ \alpha > 0 \tag{3}$$

For some particular values of α and β, the Mittage-Leffler function reduces to some farmiliar functions. For example

$$E_{1,1}(x) = e^x$$

$$E_{1,2}(x) = \frac{e^x - 1}{x}$$

2.3 Riemann-Liouville Fractional Integral

Definition 2.1 [6, 8] *Let* v *be a non negative number,* f *be piecewise continuous on* $I = (0, \infty)$ *and integrable on any finite subinterval of* $J = [0, \infty]$. *Then for* $t > 0$ *we define the Riemann-Liouville fractional integral of* f *of order* v *as*

$${}_cD_x^{-v} f(x) = \frac{1}{\Gamma(v)} \int_c^x (x-t)^{v-1} f(t) dt, \quad v > 0 \tag{4}$$

This definition can be obtained in several ways but we shall consider the linear differential equation.We first state a lemma.

Lemma 2.2 (Replacement Lemma) [6, 8] *Suppose that* $f : [a,b] \to R$ *is continuous, then*

$$\int_a^x \int_a^{x'} f(t) dt dx' = \int_a^x (x-t) f(t) dt, \quad x \in [a,b]$$

Proof. Now consider the nth order linear differential equation with the initial conditions

$$\begin{cases} y^{(n)} = f(x); \\ y(c) = y'(c) = y''(c) = \cdots = y^{(n-1)}(c) = 0 \end{cases}$$

Integrating n times from c to x and from the second integration to the nth integration, we apply Lemma 2.2, we obtain

$$y(x) = \int_c^x \frac{(x-t)^{n-1}}{(n-1)!} f(t) dt + \frac{(x-c)^{n-1}}{(n-1)!} y^{(n-2)}(c) + \cdots + (x-c) y'(c) + y(c)$$

since $y(c) = y'(c) = y''(c) = \cdots = y^{(n-1)}(c) = 0$ we get

$$y(x) = \int_c^x \frac{(x-t)^{n-1}}{(n-1)!} f(t) dt$$

We may rightly say that $y(t)$ is the nth integral of $f(x)$, thus

$${}_cD_x^{-v} f(x) = \frac{1}{(n-1)!} \int_c^x (x-t)^{n-1} f(t) dt,$$

Replacing n with $v > 0$ and recalling that $\Gamma(n) = (n-1)!$, we therefore have

$$ {}_cD_x^{-v} f(x) = \frac{1}{\Gamma(v)} \int_c^x (x-t)^{v-1} f(t)dt, \quad v > 0 $$

2.4 Riemann-Liouville Fractional Derivative

The Riemann-Liouville fractional derivative can be defined using the definition of the fractional integral.

Definition 2.3 [7, 8] *let* $v = n-u$ *where* $0 < v < 1$ *and* n *is the smallest integer greater than* u *. The fractional derivative of* $f(x)$ *of order* u *is*

$$ D^u f(x) = D^n[D^{-v} f(x)] \tag{5} $$

If we wish to find the fractional derivative of $f(x) = x^{\lambda}$ of order v, where $\lambda \geq 0$, in order to use the definition above, we interchange u and v and let $u = n - v$ where $0 < u < 1$. Then we have that $n = 1$ and $u = 1 - v$, so that

$$ \begin{aligned} D^v f(x) &= D^1[D^{1-v} f(x)] \\ &= D^1[D^{1-v} x^{\lambda}] \\ &= D^1\left[\frac{\Gamma(\lambda+1)}{\Gamma((\lambda-v+1)+1)} x^{\lambda-v+1}\right] \\ &= \frac{(\lambda-v+1)\Gamma(\lambda+1)}{(\lambda-v+1)\Gamma(\lambda-v+1)} x^{\lambda-v} \end{aligned} $$

$$ D^v f(x) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-v+1)} x^{\lambda-v} $$

3.0 Fractional Differential Equation and Laplace Transform.

Here we shall apply Laplace transform to solve some fractional differential equation. This procedure ia analogous to solving the conventional differential equation with integer order using Laplace transform.

Definition 3.1 (Laplace Transform.) [7, 8]. *Let* $f(x)$ *be given for* $t \geq 0$ *and suppose that* $f(x)$ *is defined on some interval* I *. The Laplace transform of* $f(t)$ *denoted by* $\mathcal{L}\{f(t)\}$ *or equivalently* $F(s)$ *is defined by the equation*

$$ \mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)dt \tag{6} $$

Theorem 3.2 *Suppose that*

(I) f is piecewise continuous on the interval $0 \leq t \leq A$ for any positive *A*,

(II) $|f(t)| \leq ke^{st}$ where $t \geq M$, where k, a, M are real constants and $k, M > 0$.

Then, $\mathcal{L}\{f(t)\} = F(s)$ defined in (6) above exist for $s > a$.

Such functions are described as piecewise continuous, and of exponential order as $t \longrightarrow \infty$.

$\mathcal{L}^{-1}\{f(t)\}$ is the inverse Laplace transform of $f(t)$ and is unique.

The Laplace transform is a Linear operator. that is closed under addition and scalar multiplication. The laplace transform of t^p where $p > -1$ is given by

$$ \mathcal{L}\{t^p\} = \int_0^{\infty} e^{-st} t^p dt = \frac{1}{s^{p+1}} \int_0^{\infty} e^{-x} x^p dx = \frac{\Gamma(p+1)}{s^{p+1}} $$

and

$$ \mathcal{L}\{e^{\alpha t}\} = \int_0^{\infty} e^{(\alpha-s)t} dt = \frac{1}{s-\alpha} $$

3.1 Laplace Transform of Fractional Integral

The Laplace transform of $f(t)$ of differential order v is given as

$$\mathrm{L}\{D^{-v} f(t)\} = \frac{1}{\Gamma(v)}\mathrm{L}\{t^{v-1}\}\mathrm{L}\{f(t)\} = s^{-v}F(s) \qquad (7)$$

Which is the Laplace transform of (4).

3.2 Laplace Transform of Fractional Derivative

The Laplace transform of integer order operator of $f^{(n)}$ is given as

$$\begin{aligned}\mathrm{L}\{f^{(n)}(t)\} &= s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)\\ &= s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0)\end{aligned}$$

we thus recall from (5) and noting that $u = n - v$, then

$$D^v f(t) = D^n[D^{-(n-v)} f(t)]$$

Assume that $\mathrm{L}\{f(t)\}$ exist, then

$$\begin{aligned}\mathrm{L}\{D^v f(t)\} &= \mathrm{L}\{D^n[D^{-(n-v)} f(t)]\}\\ &= s^n\mathrm{L}\{D^n[D^{-(n-v)} f(t)]\} - \sum_{k=0}^{n-1} s^{n-k-1} D^k[D^{-(n-v)} f(t)]|_{t=0}\\ &= s^n[s^{-(n-v)}F(s)] - \sum_{k=0}^{n-1} s^{n-k-1} D^{k-(n-v)} f(0)\\ &= s^v F(s) - \sum_{k=0}^{n-1} s^{n-k-1} D^{k-(n-v)} f(0)\end{aligned}$$

More precisely. if we take $n = 2$ we obtain

$$\mathrm{L}\{D^v f(t)\} = s^v F(s) - sD^{-(2-v)} f(0) - D^{-(1-v)} f(0), \quad 0 < v \le 2$$

To illustrate this we thus solve the fractional differential equation $D^{1/2} f(t) = \alpha f(t)$.
Finding the Laplace transform of both sides we obtain

$$s^{1/2}F(s) - D^{-1/2} f(0) = \alpha F(s)$$

It is quite easy to see that $D^{-1/2} f(0)$ is a constant say λ, so that we have

$$s^{1/2}F(s) - \lambda = \alpha F(s)$$

$$F(s) = \frac{\lambda}{s^{1/2} - \alpha}$$

Finding the inverse Laplace transform we finally obtain

$$f(t) = \mathrm{L}^{-1}\left\{\frac{\lambda}{s^{1/2} - \alpha}\right\} = \lambda t^{-1/2} E_{1/2,1/2}(\alpha t^{1/2})$$

4.0 Main Result

We consider the case where the decay of a sample A is the growth of another sample B and the dacay of a sample B is the growth of another sample C as follows. Suppose that x_1 at time $t = 0$ denoted by $x_1(0)$ is the initial mass of a sample A i.e $x_0(0) = m_0$

$$A(= x_1(t)) \underbrace{\rightarrow}_{\alpha\ decay\ constant} B(= x_2(t)) \underbrace{\rightarrow}_{\beta\ decay\ constant} C(= x_3(t)) \underbrace{\rightarrow}_{\gamma\ decay\ constant} \cdots\cdots$$

with the following dynamics

$$\frac{dx_1(t)}{dt} = -\alpha x_1(t) \tag{8}$$

$$\frac{dx_2(t)}{dt} = \alpha x_1(t) - \beta x_2(t) \tag{9}$$

$$\frac{dx_3(t)}{dt} = \beta x_2(t) - \gamma x_3(t) \tag{10}$$

$$x_1(0) = m_0, \quad x_2(0) = x_3(0) = 0 \tag{11}$$

Notice that (8) is a decay process while (9) and (10) are both decay and growth processes. For the purpose of this work, we shall consider only equation (8) and (9) for simplicity sake. Thus, considering equation (8) and (9) we have the equivalent form as

$$x'(t) = Ax(t) \tag{12}$$

where

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \qquad A = \begin{pmatrix} -\alpha & 0 \\ \alpha & -\beta \end{pmatrix}, \qquad x(0) = \begin{pmatrix} m_0 \\ 0 \end{pmatrix}$$

In fractional order we have

$$D^v x(t) = Ax(t), \quad D^{-(1-v)}x(0) = \xi, \quad v \in (0,1) \tag{13}$$

If $v \equiv 1$, the the system (13) becomes (12). In this work we make a very close comparison between the systems (12) and (13) as $v \to 1$.

Theorem 4.1 *If matrix* $A \in R^{n\times n}$ $(or\ C^{n\times n})$ *has a distinct eigenvalues* $(\lambda_i \neq \lambda_j, i, j = 1,2,...,m)$, *then it has (at least)* m *linearly independent eigenvectors.*

Corollary 4.2 *If all eigenvalues of* A *are distinct then* A *is diagonalizable, that is there exist an invertible matrix* P *and a diagonal matrix* $D = \begin{pmatrix} \lambda_1 & 0 & \cdots \\ 0 & \ddots & 0 \\ \vdots & 0 & \lambda_n \end{pmatrix}$ *such that* $A = PDP^{-1}$.

Now to determine the exponential of a matrix A we may employ Corollary 4.2 that is

$$e^{At} = Pe^{Dt}P^{-1}$$

Recall that in this case $A = \begin{pmatrix} -\alpha & 0 \\ \alpha & -\beta \end{pmatrix}$, the eigenvalues are $\lambda_1 = -\alpha$ and $\lambda_2 = -\beta$, so that

$$D = \begin{pmatrix} -\alpha & 0 \\ 0 & -\beta \end{pmatrix}, \qquad e^{Dt} = \begin{pmatrix} e^{-\alpha t} & 0 \\ 0 & e^{-\beta t} \end{pmatrix}$$

Corresponding to the eigenvalues are the eigenvectors $\begin{pmatrix} \frac{\beta-\alpha}{\alpha} \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively. Hence

$$P = \begin{pmatrix} \frac{\beta-\alpha}{\alpha} & 0 \\ 1 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} \frac{\alpha}{\beta-\alpha} & 0 \\ \frac{\alpha}{\alpha-\beta} & 1 \end{pmatrix}$$

So that by computation we have that

$$e^{At} = Pe^{Dt}P^{-1} = \begin{pmatrix} e^{-\alpha t} & 0 \\ \frac{\alpha}{\beta-\alpha}(e^{-\alpha t} - e^{-\beta t}) & e^{-\beta t} \end{pmatrix}$$

Hence the solution of system (12) becomes

$$\vec{x} = e^{At}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad where \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} m_0 \\ 0 \end{pmatrix}$$

then obtaining

$$\vec{x} = \begin{pmatrix} e^{-\alpha t} & 0 \\ \frac{\alpha}{\beta-\alpha}(e^{-\alpha t} - e^{-\beta t}) & e^{-\beta t} \end{pmatrix}\begin{pmatrix} m_0 \\ 0 \end{pmatrix}$$

then

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} m_0 e^{-\alpha t} \\ \frac{m_0\alpha}{\beta-\alpha}(e^{-\alpha t} - e^{-\beta t}) \end{pmatrix} \tag{14}$$

We need to find the exponential matrix in other to solve system (13). We shall need the following results.

Theorem 4.3 *If A is a square matrix, then the solution of system (13) is given by*

$$_v\vec{x}(t) = K_{v,v}(At^v)C \tag{15}$$

where $K_{v,v}(At^v) = t^{v-1}E_{v,v}(At^v)$ and $E_{v,v}(At^v)$ is the Mittag-Laffler function and C in an $n \times 1$ matrix.

Proof. For proof see [7].

Theorem 4.4 *(Putzer's algorithm) If $\lambda_i,\ i = 1,2,...,n$ are eigenvalues of an $n \times n$ matrix A, then*

$$K_{v,v}(At^v) = \sum_{k=0}^{n-1} p_{k+1}(t)M_k$$

where

$$M_0 = I, \quad M_k = \prod_{i=1}^{k}(A - \lambda_i I), \quad 1 \le k \le n-1$$

and

$$D^v p_1(t) = \lambda_1 p_1(t), \quad D^v p_k(t) = \lambda_k p_k(t) + p_{k-1}(t),$$

$$D^{-(1-v)}p_0 = 1,\ D^{-(1-v)}p_k(0) = 0,\ \ 2 \le k \le n$$

By Putzer's algorithm it can be easily seen that

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ & \end{pmatrix}, \qquad M_1 = \begin{pmatrix} 0 & 0 \\ \alpha & \alpha-\beta \\ & \end{pmatrix}$$

Solving

$$D^v p_1(t) = \lambda_1 p_1(t),\ and\ D^v p_2(t) = \lambda_2 p_2(t) + p_1(t),$$

we obtain

$$p_1(t) = t^{v-1}E_{v,v}(\lambda_1 t^v)\ and\ \ p_2(t) = \frac{t^{v-1}}{\lambda_1-\lambda_2}(E_{v,v}(\lambda_1 t^v) - E_{v,v}(\lambda_2 t^v))$$

But

$$K_{v,v}(At^v) = p_1(t)M_0 + p_2(t)M_1$$

substituting and simplifying, also taking to account that $\lambda_1 = -\alpha$ and $\lambda_2 = -\beta$, we obtain

$$K_{v,v}(At^v) = \begin{pmatrix} t^{v-1}E_{v,v}(-\alpha t^v) & 0 \\ \frac{\alpha t^{v-1}}{\beta-\alpha}(E_{v,v}(-\alpha t^v) - E_{v,v}(-\beta t^v)) & t^{v-1}E_{v,v}(-\beta t^v) \\ & \end{pmatrix}$$

$$K_{v,v}(At^v) = t^{v-1}\begin{pmatrix} E_{v,v}(-\alpha t^v) & 0 \\ \frac{\alpha}{\beta-\alpha}(E_{v,v}(-\alpha t^v) - E_{v,v}(-\beta t^v)) & E_{v,v}(-\beta t^v) \\ & \end{pmatrix}$$

By Theorem 4.3, the solution of system (13) where $\xi = \begin{pmatrix} m_0 \\ 0 \\ \end{pmatrix}$ is given as

$${}_v\vec{x}(t) = K_{v,v}(At^v)C$$

$${}_v\vec{x}(t) = t^{v-1}\begin{pmatrix} E_{v,v}(-\alpha t^v) & 0 \\ \frac{\alpha}{\beta-\alpha}(E_{v,v}(-\alpha t^v) - E_{v,v}(-\beta t^v)) & E_{v,v}(-\beta t^v) \\ & \end{pmatrix}\begin{pmatrix} m_0 \\ 0 \\ \end{pmatrix}$$

Hence

$$\begin{pmatrix} {}_v\vec{x}_1(t) \\ {}_v\vec{x}_2(t) \\ \end{pmatrix} = \begin{pmatrix} m_0 t^{v-1}E_{v,v}(-\alpha t^v) \\ m_0\frac{\alpha}{\beta-\alpha}(E_{v,v}(-\alpha t^v) - E_{v,v}(-\beta t^v)) \\ \end{pmatrix} \tag{16}$$

Using the following constants $m_0 = 1620$, $-\alpha = 4.25\times10^{-4}$, $-\beta = 2.1\times10^{-6}$ (See [9]).

Plotting x_1 and ${}_v\bar{x}_1$ in equations (14) and (16) respectively against time t yields Figure (1) which shows the amount of Radium decaying with time, respectively for v= $0.92,\ 0.94,\ 0.96,\ 0.98,\ 0.99$.

Plotting x_2 and ${}_v\bar{x}_2$ in equations (14) and (16) respectively against time t yields Figure (2) which shows the amount of Radon growing with time, respectively for v= $0.92,\ 0.94,\ 0.96,\ 0.98,\ 0.99$.

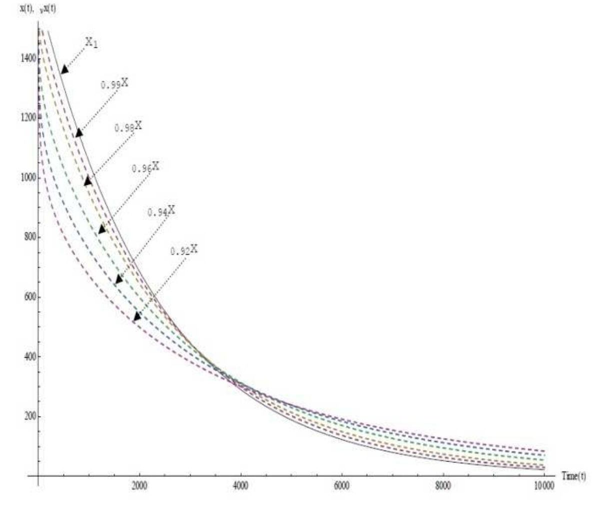

Figure 1. Comparison of the curves $x_1(t)$ and $_v x_1(t)$, i.e decay of Radium atom

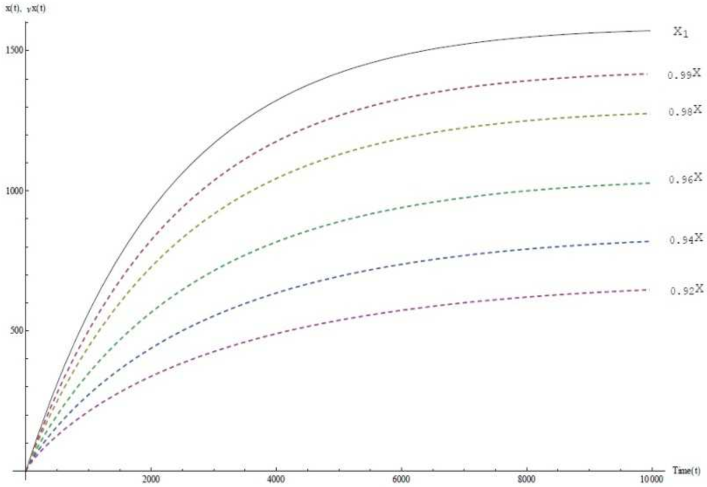

Figure 2. Comparison of the curves $x_2(t)$ and $_v x_2(t)$ i.e growth of Radon atom

It can be noticed from the graphs that as $v \to 1$, $_v x_{1,2}(t) \to x_{1,2}(t)$ that is the solutions of the fractional order approaches the solutions of the integer order. Notice also that it is close to impossible to distinguish the $x_{1,2}(t)$ and $_v x_{1,2}(t)$ in the case where $v = 0.99$.

4.1 Conclusion

From the analysis made so far we have been able to see that fractional orders tells precisely how the atom in this subject disintegrate fractionally and as well grow fractionally as this can be summed up to the approximated integer order case. As a matter of fact, fractional calculus has the potential of presenting intriguing and useful applications in the future. It is also important to note that a lot of researches is still on going in this direction [2,7,8].

References

[1] Ross, B. (Editor): Fractional Calculus and its Applications, Lecture Notes in Mathematics no. 457, Springer Verlag, Berlin 1975. [Proc. Int. Conf. held at Univ. of New Haven, USA, 1974]

[2] Samko S.G., Kilbas, A.A. and O.I. Marichev: Fractional Integrals and Derivatives, Theory and Applications, Gordon and Breach, Amsterdam 1993. [Engl. Transl. from Russian, Integrals and Derivatives of Fractional Order.

[3] Barret, J. H.: Differential equations of non-integer order, Canad. J. Math., 6 (1954), 529-541.

[4] Beyer, H. and S. Kempfle: Definition of physically consistent damping laws with fractional derivatives, ZAMM, 75 (1995), 623-635.

[5] Matignon D. : Stability results for fractional differential equations with applications to control processing. Proceedings Computational Engineering in Systems and Application multiconference, IMACS, IEEE-SMC, Lille, France, July 1966, pp. 963-968.

[6] Miller, K.S. and B. Ross: An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York 1993.

[7] Kimeu, Joseph M., "Fractional Calculus: Definitions and Applications" (2009). Masters Theses & Specialist Projects. Paper 115. http://digitalcommons.wku.edu/theses/115

[8] Modebei M. I.**,** Olaiya O. O. – Application of Fractional Differential Equation to Mortgage Problem. Inter Journ of App Math Resr IJAMR Vol 2 (4) (2013) pp505-511 ©*Science Publishing Corp (http://www.sciencepubco.com/index.php/ijamr/article/view/1306)*

[9] V.M. Chisté, B, C. Dulieu. Decay data evaluation of Radium – 226 and its daughters.) Laboratoire National Henri Becquerel Saclay, 2010