Journal of the Nigerian Association of Mathematical Physics
Volume 35, **(***May, 2016***), pp1 – 4**
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On Higher Order Curvature Invariants of Petrov Type-III

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Abstract

In this paper, space-time curvature invariants are generalized for vacuum type-III solutions with a non-vanishing cosmological constant Λ. It is shown that all curvature invariants containing derivatives of the Weyl tensor vanish if a type-III space-time admits a non-expanding and non-twisting null geodesic congruence. A non-vanishing curvature invariant containing first derivatives of the Weyltensor is found in the case of type-III space-time with expansion or twist.

1.0 Introduction

It is by now well known (see for example [1]) that for Petrov type-Nvacuum space times with a non-expanding and non-twisting null geodesic congruence all curvature invariants constructed from the Weyl tensor and their derivatives of arbitrary order vanish. The same result was established for non-expanding and non-twisting Petrov type-III vacuum space times in [2]. In this paper, therefore we are considering only expanding space-times $\rho \neq 0$where ρ is the complex expansion since it has been shown that the differential invariants of any order vanish for type III and type N space-times with zero expansion. Aside classification of gravitational fields, curvature invariants also provide a measure of the amplitude of the gravitational field. In the case of type-III vacuum space-time with expansion or twist we find a nonzerocurvature invariant of the first and second order using a procedure similar to that for type-Nvacuumspace-times given in [1]. For a proper understanding and rigorous reconstruction of the proofs in this paper, the work done in [1] as well as [2] and the book [3] are invaluable tools.

2.0 Type III Space-Time Invariants

The local properties of the gravitational field can be described by the curvature tensor and its covariant derivatives of order one and higher. These properties will show up in scalars formed from them by contracting their tensor products. In particular, the appearance of singularities in such scalars is an indication of a local singularity in the field. The converse is not true: the mere absence of singularities in these scalars is no proof of regularity of the field. For example, the C-metric describes a space-time which is singularal though all the known invariants are regular. In this paper, we are interested in studying type-III space-times from the point of view of invariants. One invariant has already been given in [1] and it has been used to classify algebraically special gravitational fields in [4]. However, this invariant is neither the simplest nor the only one that exists. In this section, we show the construction of an invariant of order one (i.e. using only first order derivatives of the Weyl tensor) and its expressions for the known solutions available. It will immediately be apparent that they contain singularities. In the next section another invariant of order two, is obtained and some comments are made about the expression of it for a Robinson-Trautman type III metrics. First, let us recall some basic relations from spinor calculus and Newman-Penrose formalism which we would find useful in deriving the first order curvature invariant.We can use basis o_A and l_A satisfying the relations

$$O_A l^A = 1, \quad O_A O^A = 0, \ \ l_A l^A = 0 \tag{2.1}$$

to decompose the Weyl spinor

$$\Psi_{ABCD} = \Psi_0 l_A l_B l_C l_D - 4\Psi_1 O_{(A} l_B l_C l_{D)} + 6\Psi_2 O_{(A} O_B l_C l_{D)}$$
$$-4\Psi_3 O_{(A} O_B O_C l_{D)} + \Psi_4 O_{(A} O_B O_C O_{D)} \tag{2.2}$$

where

$$\Psi_0 = \Psi_{ABCD} O^A O^B O^C O^D$$
$$\Psi_1 = \Psi_{ABCD} O^A O^B O^C l^D$$
$$\Psi_2 = \Psi_{ABCD} O^A O^B l^C l^D \tag{2.4}$$

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$\Psi_3 = \Psi_{ABCD} \quad O^A l^B l^C l^D$
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There exist four principal spinors $\alpha_A,\ \beta_A, \gamma_A, \delta_A$, such that

$$\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)} \tag{2.5}$$

Since for in type-IIIspace-times, three principal spinors of Ψ_{ABCD} coincide,it is therefore convenient to choose this repeated principal spinor as a basis spinor O_A.Then

$$\Psi_{ABCD} = O_{(A}O_B O_C O_{D)} \tag{2.6}$$

and

$$\Psi_0 = \Psi_1 = \Psi_2 = 0 \tag{2.7}$$

For the Weyl spinor we get

$$\Psi_{ABCD} = -4\Psi_3\ O_{(A}O_B O_C l_{D)} + \Psi_4\ O_{(A}O_B O_C O_{D)} \tag{2.8}$$

We choose the second basis spinor l_A to satisfy

$$Dl_A = 0 \tag{2.9}$$

which implies that a complex null tetrad induced by O_A and l_A is parallel propagated along the geodesic null congruence. This implies

$$\sigma = \kappa = \varepsilon = \pi = 0 \tag{2.10}$$

Equations (2.1) and (2.7) ensure that all invariant quantities constructed from Ψ_{ABCD} without derivatives vanish, that is the zeroth order curvature invariants vanish. To obtain a nonzero quantity we have to use the first covariant derivative.

3.0 Expanding or Twisting Solutions

The Bianchi identity using (2.7) and (2.10) gives

$$D\Psi_3 = 2\rho\Psi_3 \tag{3.1}$$

Using (2.1) one can show that all first order invariants of the Weyl tensor vanish if they contain only squares or cubes in $C_{\alpha\beta\gamma\delta;\varepsilon}$. However, there is a non-vanishing curvature invariant which is found to be

$$I = C^{\alpha\beta\gamma;\varepsilon}C_{\alpha\mu\gamma\nu;\varepsilon}C^{\lambda\mu\rho\nu;\sigma}C_{\lambda\beta\rho\delta;\sigma} \tag{3.2}$$

In terms of the Newman-Penrose quantities (3.2) can be written as

$$I = (48\rho\bar{\rho}\Psi_3\bar{\Psi}_3)^2 \tag{3.3}$$

which is a non-vanishing invariant of the first order. This invariant can be used in analyzing singularities in type III space-times with expansion or twist. For example, the general type III solution admitting a geodesic, shear free, twist free and diverging null congruence is the Robinson-Trautman metric of Petrov type III. This metric is known to satisfy the equation

$$ds^2 = \frac{2r^2}{P^2}d\xi d\bar{\xi} - 2dudr - \left(\Delta \ln P - 2r(\ln P)_{,u}\right)du^2 \tag{3.4}$$

where $P(u,\xi,\bar{\xi})$ satisfies

$$\Delta\Delta P = 0, \qquad (\Delta P)_{,\xi} \neq 0, \qquad \Delta \equiv 2P^2\partial_\xi\partial_\xi \tag{3.5}$$

and $\partial/\partial r$ is the repeated null eigenvector [5]. In an appropriately chosen null vector we have

$$\sigma = \kappa = \varepsilon = \pi = \Psi_0 = \Psi_1 = \Psi_2 = 0, \qquad \rho = -\ \frac{1}{r} \tag{3.6}$$

$$\Psi_3 = -\frac{P}{r^2}(\Delta \ln P)_{,\bar{\xi}} \tag{3.7}$$

$$\Psi_4 = \frac{1}{r^3}\left(P^2\left(\frac{1}{2}\Delta \ln P - r(\ln P)_{,u}\right)_{,\bar{\xi}}\right)_{,\bar{\xi}} \tag{3.8}$$

Substituting (3.6), (3.7) and (3.8) into (3.3) we get

$$I = \left(\left(\frac{48}{r^6}\right)P\bar{P}(\Delta \ln P)_{,\bar{\xi}}(\Delta \ln \bar{P})_{,\bar{\xi}}\right)^2 \tag{3.9}$$

This invariant, which is non-zero in general, can be used for analyzing singularities in Robinson-Trautman solutions.

4.0 Non-Expanding and Non-Twisting Solutions

Non-expanding and non-twisting solutions satisfying (2.10) as well as the condition $\rho = 0$ belong to Kundt's class and are completely known. It has been shown in [2] that type-Nvacuum space-times, without expansion and without twist, have all curvature invariants of all orders vanishing. The same result with a similar proof, with slightmodifications, has been shown to be also valid for type-IIIvacuum space-times without expansion andwithout twist. Thus we give here only the basic ideas of the proof. The following Newman-Penrose equations with the operator D are needed $D\tau = 0, D\alpha = 0,\ \ D\beta = 0, D\gamma = \tau\alpha + \bar{\tau}\beta - R/24\,, D\lambda = 0, D\mu = R/12,$
(4.1) $D\nu = \bar{\tau}\mu + \lambda\tau + \Psi_3$
The operator D also satisfy the commutation relations

$$(\Delta D - D\Delta) = (\gamma + \bar{\gamma})D - \tau\bar{\delta} - \bar{\tau}\delta \tag{4.2}$$

$(\delta D - D\delta) = (\bar{\alpha} + \beta)D$ (4.3)

The Bianchi identity (3.1) in this case then takes the form

$D\Psi_3 = 0$ (4.4)

Consider the transformation

$O'^A = aO^A,\ l'^A = a^{-1}l^A$ (4.5)

A quantity Ω, which transforms in the form

$\Omega' = a^q\Omega$ (4.6)

has the boost-weight$b(\Omega) = q$. For Ψ_3, this we have $b(\Psi_3) = -2$. Also we have

$b(K) = n_1 + n_2 - m_1 - m_2$

where

$m_1 = O^{A_{1,}} \dots O^{A_{m_1}}$

$m_2 = \bar{O}^{\dot{X}_{1,}} \dots \bar{O}^{\acute{X}_{m_2}}$

$n_1 = l^{B_{1,}} \dots l^{B_{n_1}}$ (4.7)

$n_2 = \bar{l}^{\dot{Y}_{1,}} \dots \bar{l}^{\acute{Y}_{n_2}}$

It is then easy to show that the Newman-Penrose equations imply $K = 0$, if $b(K) \geq 0$ and as a result of this all invariants vanish.

5.0 An Invariant of the Second Order

Besides the invariant of order one presented in the previous section,there exist invariants of order two. Invariants of higher order are somewhat toocomplicated to be applicable. However, in this section we present another invariant of ordertwo andits expression can be calculated for the simple case of a Robinson-Trautman solution using the symbolic computational software Maple. A question that comes readily in mind is that why should one bother with a higher order invariant when we have already established the singularity of the Robinson-Trautman solutions? For the simple reason that it gives a different type of information about the solutions of Type III metrics. Now it is obvious that all curvature invariants of order zero vanish for Type-IIIand Type-Nvacuum space-times as a consequence of the relation $O_AO^A = 0$. The question arises, whether there exist some non-vanishing curvature invariants of higher order. Specifically, we investigate this question in this paper for second order invariants. To achieve this, we work in the null tetrad in which the Weyl tensor is given by

$C_{abcd} = \Psi_3(N_{ab}M_{cd} + M_{ab}N_{cd}) + \Psi_4N_{ab}N_{cd}$ (5.1)

where

$N_{ab} = n_am_b - m_an_b$

$M_{ab} = n_al_b - l_an_b + \bar{m}_am_b - m_a\bar{m}_b$

are bivectors and $n, m, \bar{m}, l$are basis vectors which are chosen to be orthogonal null vector fields, two of which are real and the other two complex conjugated such that the scalar product of the two real one's equals 1 and of the two complex one's equals -1. That is $l_an^a = 1, m_a\bar{m}^a = -1$ and all others scalar products vanish.

Let

$D_{rst} = C_{abcd;r}C^{abcd}_{;st}$ (5.2)

Substituting the expressions for $C_{abcd;r}$and $C^{abcd}_{;st}$ we get

$D_{rst} = -96\Psi_3\vartheta_s\vartheta_t\left(\Psi_{3,r} + \Psi_3U_r + \Psi_4\vartheta_r\right) +$

$96\Psi_3\vartheta_r\left[(\Psi_3\vartheta_s)_{;,t} + \vartheta_t\left(\Psi_{3,s} + \Psi_3U_s + 2\Psi_4\vartheta_s\right)\right]$ (5.3)

To get an invariant one has to contract the expression for$D_{[rs]t}$by its complex conjugate thus

$J_2 = D_{[rs]t}\bar{D}^{[rs]t}$ (5.4)

The general expression for J_2 is often very messy, however for simple space-times it can be computed with the aid of the symbolic computation software Maple.

6.0 Conclusion

In this paper, we have shown that for type-IIIvacuum space-times without twist and expansion all Weyl's invariants of all orders vanish. However, in the case of type-III space-times with expansion or twist, only invariants of the zeroth order vanish while the invariants of first and second order do not vanish. The general expression for the non-vanishing invariant of the second order is often very messy, however for simple space-times it can be computed with the aid of the symbolic computation software Maple.

7.0 References

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