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Review on Some Subclasses of Analytic Functions with Fixed Argument of Coefficients

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Abstract

In this paper, we reviewed the work of J. Dziok and H.M. Srivastava using a new subclass of analytic functions with fixed point and fixed argument, provided alternative proofs supporting some of their claims.We also resolved the quadratic inequality in their work using a property of Bilinear Transformation. We intend to restrict our discussions to the coefficients estimates.

Keywords and Phrases: Analytic functions,generalized hypergeometric function, Hadamard product, convex functions,starlike functions

1.0 Introduction

Let A denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disk $E = \{z : |z| < 1\}$ in the complex plane with the usual normalization $f(0) = 0$, $f'(0) = 1$.

Let $A(\omega) \subset A$ be the class of functions of the form

$$f(z) = (z - \omega) + \sum_{k=2}^{\infty} a_k z^k \quad (n \in \mathrm{N} = 2,3,...) \tag{2}$$

normalized with $f(\omega) = 0$ and $f'(\omega) = 1$, where ω is a fixed point in E .

The function in (2) was introduced by Kanan and Ronning ,for more details see [1,2]

Def. 1.0

Let $T(k, k-1, \omega, A, B)$ denote the class of functions $f \in A(\omega)$ satisfying the condition

$$(z-\omega)^{-1}H(\alpha_1,...,\alpha_k:\beta_1,...,\beta_{k-1})f(z)\prec\frac{1+Az}{1+Bz} \tag{3}$$

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Journal of the Nigerian Association of Mathematical Physics Volume 35, (May, 2016), 21 – 28

Def. (2.0)

let $T_\theta(k,k-1,\omega,A,B)$ denote the subclass of the class

$T(k,k-1,\omega,A,B)$ of functions $f(z)$ of the form (1) such that arg($a_k\neq 0$) ($k\in\mathbb{N}$)

written in the form

$$f(z)=(z-\omega)+e^{i\theta}\sum_{n=2}^{\infty}|a_n|(z-\omega)^n \tag{4}$$

The class of functions introduced in (4) have fixed point and fixed argument.

For complex parameters $\alpha_1,...,\alpha_q$ and $\beta_1,...,\beta_s(\beta_j\neq 0,-1,-2,...,:j=1,...,s).$

hypergeometric function is

$${}_qF_s(\alpha_1,...,\alpha_q:\beta_1,...,\beta_s:z)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n...(\alpha_q)_n z^n}{(\beta_1)_n...(\beta_s)_n n!} \tag{5}$$

$(q\leq s+1; s\in\mathbb{N}_0=(0,1,2,...):z\in\mu)$

where $(\lambda)_n$ is the Pochhammer symbol defined in terms of the Gamma function Γ, by

$$(\lambda_n) = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \lambda(\lambda+1)...(\lambda+n-1) \qquad (n \in \mathbb{N} = \mathbb{N}_0 / \{0\})$$

with $(\lambda)_0 = 1$ [3,4] (6)

The convolution of functions $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z - w + \sum_{n=2}^{\infty} a_1 a_2 (z-w)^n$$

where * is the usual Hadamard product of $f_1(z)$,$f_2(z)$ as expressed in (1) . See [2-5]

Let k∈N and suppose that the parameters $\alpha_1,...,\alpha_k$ and $\beta_1,...,\beta_{k-1}$ are positive real numbers. Also let

$0 \leq \theta \leq \pi$,$0 \leq B \leq 1$ and $-B \leq A < B$.

Lemma 1.0

Let $f(z) = z - \omega) + \sum_{n=2}^{\infty} |a_n| (z-\omega)^n$ and

$$H(\alpha_1,\dots,\alpha_k : \beta_1,...\beta_{k-1}) f(z) = h(\alpha_1,...,\alpha_k : \beta_1,...,\beta_{k-1} : z-\omega) * f(z) \qquad (7)$$

then $H(\alpha_1,...,\alpha_k : \beta_1,...,\beta_{k-1}) f(z) = (z-\omega) + \sum_{n=2}^{\infty} \Gamma_n^{-1} a_n (z-\omega)^n$ (8)

where

$$\Gamma_n = \frac{(\beta_1)_{n-1} \cdots (\beta_{k-1})_{n-1}}{(\alpha_1)_{n-1} \cdots (\alpha_k)_{n-1}} (n-1)! \qquad (9)$$

Proof: Given that

$$h\left(\alpha_1,...,\alpha_k:\beta_1,...,\beta_{k-1}:(z-\omega)\right)$$

$$=(z-\omega)\left\{\left(\frac{(\alpha_1)_0\cdots(\alpha_k)_0}{(\beta_1)_0\cdots(\beta_{k-1})_0}\right)+\left(\frac{(\alpha_2)_1\cdots(\alpha_k)_1}{(\beta_1)_1\cdots(\beta_{k-1})_1}\right)(z-\omega)+...\right\}.$$

Since satisfaction of equation (7) implies equation (8)
and by Hadamard product satisfying the condition

$$H(\alpha_1,...,\alpha_k:\beta_1,...,\beta_{k-1})f(z)=\left((z-\omega)+\sum_{n=2}^{\infty}\frac{(\alpha_1)_{n-1}\cdots(\alpha_k)_{n-1}}{(\beta_1)_{n-1}\cdots(\beta_{k-1})_{n-1}}\right)\frac{a_n(z-\omega)^n}{(n-1)!}. \quad (10)$$

On comparing equation (8) and equation (10) evidently yields

$$\Gamma_n^{-1}=\frac{(\alpha_1)_{n-1}\cdots(\alpha_k)_{n-1}}{(\beta_1)_{n-1}\cdots(\beta_{k-1})_{n-1}(n-1)!}$$

which implies that

$$\Gamma_n=\frac{(\beta_1)_{n-1}\cdots(\beta_{k-1})_{n-1}}{(\alpha_1)_{n-1}\cdots(\alpha_k)_{n-1}}.$$

Lemma 2.0

Let f be a function of the form (1) and $g(z)=(z-\omega)+\sum_{k=n}^{\infty}b_n(z-\omega)^k$ analytic in ,

where $|z|=r<1$ and $|\omega|=d.$

If $f\prec g$ and $g\in S^c(\omega)$ then $|a_n|(r+d)^{n-1}\leq 1$

Proof:

Since $f\prec g$ implies $f(z)=g(w(z))$ and $|w(z)|\leq 1$ satisfy that $f(z)\leq g(z)$.

Now $g(z)\in S^c(\omega)$ if and only if $f(z)=(z-\omega)g'(z)$ belongs to family of starlike functions . Consequently ,application of (Nevanlinna's theorem ,see pg. 111 of [6]) on the starlike functionsand following the implication of subordination in the first statement of the proof we get

$$\left|nb_n\right|(r+d)^{n-1} = \left|na_n\right|(r+d)^{n-1} \le n$$

which will yield $\left|a_n\right|(r+d) \le 1$.

Theorem 1.0

If a function of the form (4) belongs to the class $T_\theta(k, k-1, \omega, A, B)$ then

$$\sum_{k=n}^{\infty} \Gamma_n^{-1} \left|\overline{a}\right|(r+d)^{n-1} \le \gamma(\theta, \omega, A, B) \tag{11}$$

where $|z| = r < 1$ and $|w| = d$ and Γ_n is defined by (9) and

$$\gamma(\theta, \omega, A, B) = \frac{A-B}{B\cos\theta - \sqrt{1 - B^2 \sin^2\theta}} \tag{12}$$

Proof :

Since $(z-\omega)^{-1} H(\alpha_1, \ldots, \alpha_k : \beta_1, \ldots, \beta_{k-1}) f(z) = \dfrac{1 + A\phi(z)}{1 + B\phi(z)} = P(z)$

where $\phi(0) = 0$ and $|\phi(z)| \le 1$ for $z \in E$.

Observe that $\phi(z) = \dfrac{1 - P(z)}{BP(z) - A} = \dfrac{P(z) - 1}{A - BP(z)} = \overline{\phi(z)}$

from definition (1),

$$\phi(z) = \frac{1-(z-\omega)^{-1} H(\alpha_1,...,\alpha_k : \beta_1,...,\beta_{k-1}) f(z)}{B(z-\omega) + B\sum_{n=2}^{\infty} \Gamma_n^{-1} a_n (z-w)^n - A}$$

while

$$\overline{\phi(z)} = \frac{(z-\omega)^{-1} H(\alpha_1,...,\alpha_k : \beta_1,...,\beta_{k-1}) f(z) - 1}{A - B(z-\omega) + B\sum_{n=2}^{\infty} \Gamma_n^{-1} a_n (z-w)^n}$$

thus

$$|\phi(z)| = \frac{(z-\omega)^{-1} H(\alpha_1,...,\alpha_k : \beta_1,...,\beta_{k-1}) f(z) - 1}{B(z-\omega)^{-1} + B\sum_{n=2}^{\infty} \Gamma_n^{-1} a_n (z-w)^n} = \frac{\frac{H(\bullet) f(z)}{z-\omega} - (z-\omega)}{\frac{BH(\bullet) f(z)}{z-\omega} - (z-\omega)}$$

where $H(\bullet) = H(\alpha_1, \ldots, \alpha_k : \beta_1 ..., \beta_{k-1})$. Using the result of Lemma (1)

$$(z-\omega)^{-1} H(\bullet) f(z) = 1 + \sum_{n=2}^{\infty} \Gamma_n^{-1} a_n (z-\omega)^{n-1}$$ and

$$B(z-\omega)^{-1} H(\bullet) f(z) - A = (B-A) + Be^{i\theta} \sum_{n=2}^{\infty} \Gamma_n^{-1} a_n (z-\omega)^{n-1}$$

hence ,

$$|\phi(z)| = \left| \frac{\sum_{n=2}^{\infty} \Gamma_n^{-1} a_n (z-\omega)^{n-1}}{(B-A) + Be^{i\theta} \sum_{n=2}^{\infty} \Gamma_n^{-1} a_n (z-\omega)^{n-1}} \right| \le 1$$

.

Further simplification of the inequality yields

$$\left| \sum_{n=2}^{\infty} \Gamma_n^{-1} a_n (z-w)^{n-1} \right| < \left| (B-A) + e^{i\theta} \sum_{n=2}^{\infty} \Gamma_n^{-1} a_n (z-w)^{n-1} \right|$$

Consequently , $\sum_{n=2}^{\infty} \Gamma_n^{-1} |a_n| (r+d)^{n-1} < (B-A) + Be^{i\theta} \sum_{n=2}^{\infty} \Gamma_n^{-1} |a_n| (r+d)^{n-1}$

where $\Gamma^{-1}{}_n$ is same as in Lemma (1).

Let $K = \sum_{n=2}^{\infty} \Gamma_n^{-1} |a_n| (r+d)^{n-1}$ then one gets

$$|K| < |(B-A) + Be^{i\theta} K| > |(B-A) + BK\cos\theta|$$

(13)

neglecting the imaginary part since K is real . Squaring both sides of (13) and use the fact that $|\cos^2\theta| = 1$,will give

$$(1-B^2)K^2 - 2B[(B-A)\cos\theta]K - (B-A)^2 < 0$$

(14)

The method adopted in this thesis to solve the inequality in (14) is the use of inverse fixed-point property of bilinear transformation.

Let K_1 and $\overline{K_1}$ be solutions of (14) such that

$$K_1 = \frac{(B-A)}{1-B^2} \left(B\cos\theta - \sqrt{1 - B^2 \sin^2\theta} \right)$$

(15)

and

$$\overline{K_1} = \left(\frac{B-A}{1-B^2}\right)\left(B\cos\theta + \sqrt{1-B^2\sin^2\theta}\right) \quad .$$

(16)

Clearly ,

$$K_1\overline{K_1} = \frac{-(B-A)^2}{(1-B^2)} \quad .$$

(17)

Since the transformation is bilinear we claim that

$$\overline{K_1} = \frac{-1}{K_1}$$

(18)

using (17) and (18) ,one observes that (17) can be written as

$(B-A)^2 = 1 - B^2.$

hence , (15) and (16) can be written respectively as

$$K_1 = \frac{1}{(B-A)}\left(B\cos\theta - \sqrt{1-B^2\sin^2\theta}\right)$$

$$\overline{K_1} = \frac{1}{(B-A)}\left(B\cos\theta + \sqrt{1-B^2\sin^2\theta}\right) \quad ,$$

substituting for K_1 and $\overline{K_1}$ in (18) will give

$$\overline{K_1} = \frac{1}{\frac{-1}{(B-A)}\left(B\cos\theta - \sqrt{1-B^2\sin^2\theta}\right)}$$

$$= \frac{(A-B)}{B\cos\theta - \sqrt{1-B^2\sin^2\theta}} = \gamma(\theta,\omega,A,B)$$

that is

$$\sum_{n=2}^{\infty} \Gamma_n^{-1} \left|\overline{a_n}\right| (r+d)^{n-1} \le \frac{A-B}{B\cos\theta - \sqrt{1-B^2\sin^2\theta}}$$

.

Corollary 1.0

A function f of the form (4) belongs to the class $T_\pi(k,k-1,\omega,A,B)$ if and only if

$$\sum_{n=2}^{\infty} \Gamma_n^{-1} \left|\overline{a_n}\right| \le \frac{B-A}{1+B}$$

(19)

Theorem 2.0

A function f of the form (4) belongs to the class $T_\pi(k,k-1,\omega,A,B)$ if and only if

$$\sum_{n=2}^{\infty} \Gamma_n^{-1} \left|\overline{a_n}\right| (r+d)^{n-1} = \sum_{n=2}^{\infty} \Gamma_n^{-1} \left|a_n\right| (r+d)^{n-1}$$

(20)

where Γ_n is defined by equation (9)

Proof

From theorem 2 of [3] , $\sum_{n=2}^{\infty} \Gamma_n^{-1} \left|a_n\right| (r+d)^{n-1} \le \frac{B-A}{1+B}$ and here, following corollary (1) then

(20) is satisfied.

2.0 Conclusion

The work showed that Generalized Hypergeometric function has the ability to transform a subclass of analytic function with fixed point and fixed argument to Schwarz function and provided their upper and lower bounds. Introduction of fixed point to the original function studied by the Authors [3] did not change the bounds. The reflexive, inverse property of bilinear transformation was used to provide solution of a quadratic inequality.

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4.0 References

[1]. S. Kanans and F. Ronning, Uniformly starlike and convex functions and related classes of univalent functions , Ann. Univ. Mariae Curie-Sklodowska Sect. A 53 (1999), 87-93.

[2]. Emelike Ukeje , Generalization of coefficient Inequalities and Convolution Properties of a class of Analytic Functions with Respect to a Differential Operator. Journal of the Nigerian Associationof Mathematical Physics Volume 34,(March,2016),pp 5-12

[3]. J. Dziok and H. M. Srivastava , Some Subclasses of Analytic Functions with Fixed Argument of Coefficients Associated with the Generalized Hypergeometric Function. AdvancedStudies in Contemporary Mathematics 5 (2002) No 2 pp. 115-125.

[4]. H. M. Srivastava and S. Owa (Editors) , Current Topic in Analytic Function

Theory, World Scientific Publishing Company , Singapore , New Jersey, London and Hong Kong 1992.

[5]. A. T. Oladipo. Coefficient Estimate For Some Families of Analytic Univalent Functions Associated With q-Analogue of Dziok-Srivastava Operator. Analele Universitatii Oradea Fasc Mathematica , Tom XXX (2015) , Issue No. 1, 141-153.

Journal of the Nigerian Association of Mathematical Physics Volume 35, (May, 2016), 21 – 28

[6]. A. W. Goodman, Univalent functions Vol. 1 Mariner Publishing Company 1983