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# The Variational Iteration Method for Solving Linear and Nonlinear Problems That Arise in Mathematical Physics

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## *Abstract*

***In this paper, Variational Iteration Method (VIM) is presented to further study and obtain the numerical solutions for the initial value problems of Klein-Gordon and sine-Gordon equations. Both linear and nonlinear cases are considered. Six examples are solved based on the method to illustrate its efficiency and applicability. The Method is flexible and has the ability to solve linear and nonlinear equations accurately and conveniently.***

**Keywords:**Variational iteration method, initial value problem, Klein-Gordon equation, sine-Gordon equation

## 1.0 Introduction

Differential equations play important role in the modeling and simulation of physical problems and the need to find the numerical solution to most of these equations became imperative since it is difficult to obtain the exact solution by the existing analytical methods. Many researchers [1-19] have used different numerical approaches to different types of differential equations.

Consider the equation of the form;

$$u_{tt}(x,t)-(c_1+c_2{}^2)u(x,t)_{xx}+au(x,t)+\alpha\sin u(x,t)+F(u(x,t))=h(x,t) \quad \text{...... (1)}$$

with initial condition

$$u(x,0)=f(x),u(x,0)=g(x) \quad \text{.......(2)}$$

Where $c_1$, $c_2$, $\alpha$, are all constants.

If $\alpha=c_2=0, c_1=1$ ......(3)

Then the equation becomes

$$u_{tt}(x,t)-u_{xx}(x,t)+au(x,t)+Fu(x,t)=h(x,t) \quad \text{......(4)}$$

Equ. (4) is known as the nonlinear Klein-Gordon equation, $h(x,t)$ is a source term and $F(u(x,t))$ is a nonlinear function of $u(x,t)$. This equation arises in field theory and describes nonlinear wave interaction. The equation is used to describe dispersive wave phenomena. It also appears in nonlinear optics and plasma physics [1-2].

If $c_1=0, a=0, F=0, h(x,t)=0$ .......(5)

Equ. (1) becomes:

$$u_{tt}(x,t)-c_2{}^2u(x,t)_{xx}+\alpha\sin u(x,t)=0 \quad \text{.......(6)}$$

Equ. (6) is the sine-Gordon which appears in physical phenomena such as the propagation of magnetic fluid and stability of fluids motions. It plays a major role in nonlinear physics [3-5]. This equation also arises in the study of differential geometry of surfaces.

The Solution of Klein-Gordon and sine-Gordon are numerically challenging. Authors [1-10] have used different numerical methods to solve these equations.

These equations plays important role in the propagation of fluxions in Josphson junction between two superconductors, stability of fluid motion, motion of rigid pendular attached to a stretched wire, solid state physics and nonlinear optics.

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In this paper, VIM proposed by He [11-14] is applied to obtain the approximate analytical solutions of various forms of Klein-Gordon and sine-Gordon equations. These equations plays important role in the propagation of fluxions in Josphson junction between two superconductors, stability of fluid motion, motion of rigid pendular attached to a stretched wire, solid state physics and nonlinear optics. Comparison with the exact solution are also be performed.
The solutions are obtained in the form of convergent series by using Maple 18.

## 2.0 Variational Iteration Method

The basic idea of the He's Variational Iteration Method (VIM) [11-14], can be explained by considering the following nonlinear partial differential equations ;

$$Lu(x,t)+Ru(x,t)+Nu(x,t)=g(x,t) \quad \ldots\ldots (7)$$

where L is a linear time derivative operator, R is a linear operator which has partial derivative with respect to x, N is a nonlinear operator and g is an inhomogeneous term. According to VIM, we can construct a correction functional as follows:

$$U_{n+1}(x,t)=U(x,t)+\int_0^t \lambda\left[LU_n+R\tilde{U}_n+N\tilde{U}_n-g\right]d\tau \quad \ldots\ldots(8)$$

where $\lambda$ is a Lagrange multiplier which can be identified optimally by variational iteration method. The subscript $n$ denote the $nth$ approximation, $\bar{u}_n$ is considered as a restricted variation i.e $\delta\bar{u}_n=0$. The successive approximation $u_{n+1}, n\geq 0$ of the solution $u$ can be easily obtained by determine the Lagrange multiplier and the initial guess $u_0$, consequently, the solution is given by $u=\lim_{n\to\infty} u_n$ .

## 3.0 Analysis of the Klein-Gordon and sine-Gordon Equations

In this section, we present the solution of equation (1) by means of VIM. The correction functional becomes

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda\left[(u_n)_{\tau\tau}-(c_1+c_2)^2(\bar{u}_n)_{xx}+au_n+\alpha\sin(\bar{u}_n)+F(u_n)-h(x,\tau)\right]d\tau,\ n\geq 0 \quad \ldots\ldots (9)$$

To find $\lambda(\tau)$ using variational calculus, it is obtained as:

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\delta\int_0^t \lambda(\tau)(u_n)_{\tau\tau}\delta\tau \quad \ldots\ldots(10)$$

So, the stationary conditions are:

$$1-\lambda^{l}(\tau)=0|_{\tau=t}$$
$$\lambda(\tau)=0|_{\tau=t} \quad \ldots\ldots(11)$$
$$\lambda^{ll}(\tau)=0|_{\tau=0}$$

hence,

$$\lambda(\tau)=\tau-t \quad \ldots\ldots(12)$$

And the formula becomes:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t(\tau-t)\begin{bmatrix}(u_n)_{\tau\tau}-(c_1+c_2)^2(u_n)_{xx}+\\ au_n+\alpha\sin\left(\bar{u}_n\right)+F(u_n)-h(x,\tau)\end{bmatrix}d\tau \quad \ldots\ldots(13)$$

## 4.0 Numerical Applications

To illustrate the applicability of the method, several examples are considered.
**Example 1:** Consider the inhomogeneous Klein-Gordon equation.

$$u_{\tau t}-u_{xx}-u=2-t^2. \quad \ldots\ldots(14)$$

subject to initial conditions:

$$u(x,0) = x, u_t(x,0) = x \qquad \text{…….(15)}$$

To solve eq.(13), the following variational iteration formula is applied.

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\tau - t)\left[(u_n)_{\tau\tau} - (u_n)_{xx} - u - 2 + \tau^2\right]d\tau. \quad n \geq 0 \qquad \text{…….(16)}$$

With initial approximation

$$u_0 = x + tx \qquad \text{…….(17)}$$

The following variables are easily obtained from (14) and given by

$$u = xt + x - \frac{1}{12}t^4 + \frac{1}{6}t^3x + \frac{1}{2}t^2x + t^2 \qquad \text{…….(18)}$$

$$u_2 = xt + x + \frac{1}{6}t^3x + \frac{1}{2}t^2x + t^2 - \frac{1}{360}t^6 + \frac{1}{120}t^5x + \frac{1}{24}t^4x. \qquad \text{…….(19)}$$

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$$u_5 = xt + x + \frac{1}{6}t^3x + \frac{1}{2}t^2x + t^2 + \frac{1}{120}t^5x + \frac{1}{24}t^4x + \frac{1}{5040}t^7x + \frac{1}{720}t^6x + \frac{1}{362880}t^9x + \frac{1}{40320}t^8x - \frac{1}{239500800}t^{12} + \frac{1}{39916800}t^{11}x + \frac{1}{3628800}t^{10}x. \qquad \text{…….(20)}$$

The exact solution is

$$u(x,t) = xe^t + t^2 \qquad \text{…….(21)}$$

The graph of u(exact) with u(vim) against t when x=0.5 is shown in Fig.1a. The graph of u(exact) with u(vim) against x when t=0.5 is shown in Fig.1b. Also, Fig.1c and Fig.1d show the 3-D graphs of 5-iterate of VIM and exact solution respectively

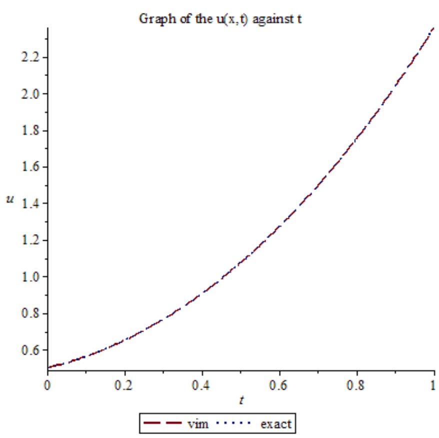


Figure 1a: u(exact) with u(vim.) when x=0.5

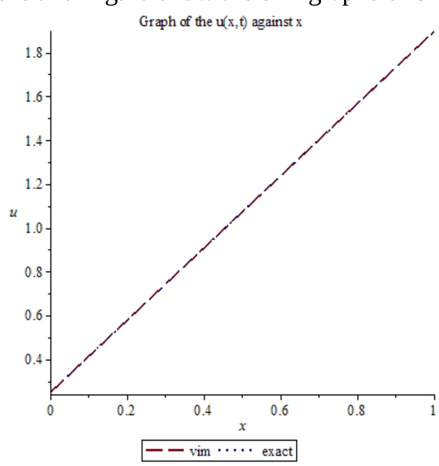


Figure 1b: u(exact) with u(vim.) when t=0.5

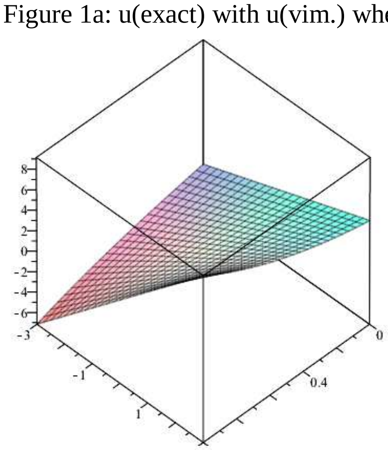

Figure 1c- 3-D graph of 5-iterate of VIM

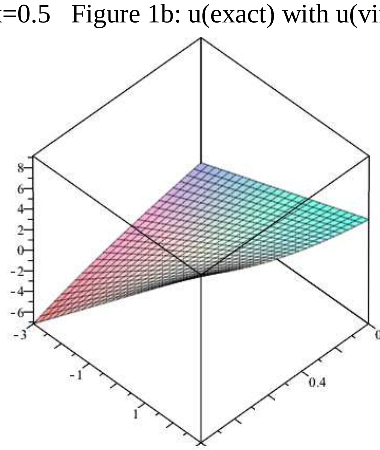

Figure 1d- 3-D graph of exact solution

**Example 2**. Consider the equation

$$u_{tt} - u_{xx} - 2u = 0 \qquad \text{…….(22)}$$

$$u(x,0)=\cos x, u_t(x,0)=\cos x. \qquad \text{…….(23)}$$

To solve equation (20), the following VIM formula is applied

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\tau - t)\left[(u_n)_{\tau\tau} - (u_n)_{xx} - 2u_n\right]d\tau, \;\; n \geq 0 \qquad \text{…….(24)}$$

$$u_0(x,t)=\cos x + t\cos x. \qquad \text{…….(25)}$$

The following iterates are easily obtained:

$$u_1 = \frac{1}{6}\cos x\left(t^3+3t^2+6t+6\right). \qquad \text{…….(26)}$$

$$u_2 = \frac{1}{120}\cos x\left(t^5+5t^4+20t^3+60t^2+120t+120\right). \qquad \text{…….(27)}$$

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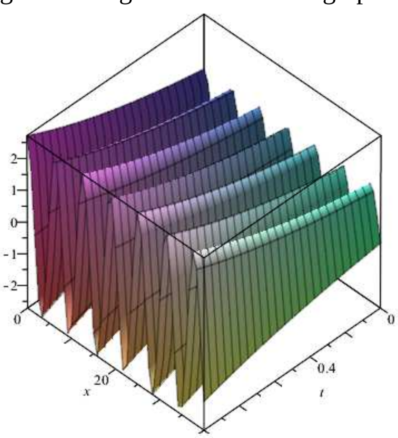
$$u_5 = \frac{1}{39916800}\cos x\begin{pmatrix} t^{11}+11t^{10}+110t^9+990t^8+7920t^7+55440t^6+332640t^5+1663200t^4 \\ +6652800t^3+19958400t^2+39916800t+39916800\end{pmatrix} \qquad \text{…….(28)}$$

The exact solution is

$$u(x,\tau) = e^t \cos x. \qquad \text{…….(29)}$$

Fig.2a and Fig.2b show the 3-D graphs of 5-iterate of VIM and exact solution of equ. (22) respectively.



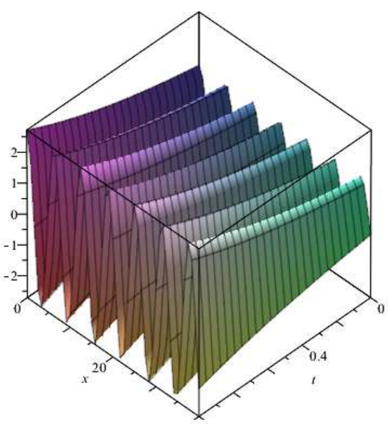




Figure 2a- 3-D graph of 5-iterate of VIM Figure 2b- 3-D graph of exact solution

**Example 3:** Consider the Equation

$u_{tt} - u_{xx} - u = 0$ …….(30)

With initial conditions

$$u(x,0) = 0, u_t(x,0) = \sin x. \qquad \text{…….(31)}$$

The following VIM formula is applied

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^{\tau} (\tau - t)\left[(u_n)_{\tau\tau} - (u_n)_{xx} - u_n\right]d\tau, \quad n \geq 0 \qquad \text{…….(32)}$$

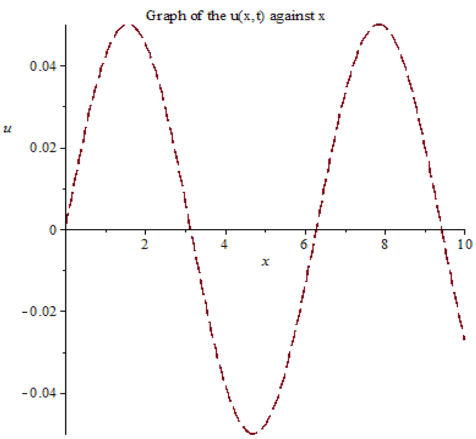
With $u_0(x,t) = t\sin x$ …….(33)

After 1-iterate,

$$u_1 = t\sin x. \qquad \text{…….(34)}$$

Equ.(34) is the exact solution to equ.(30).

The graph of u(vim) against x when t=0.05 is shown in Fig.3a. Also, Fig.3b shows the 3-D graphs of 1-iterate of VIM which is equivalent to the exact solution.



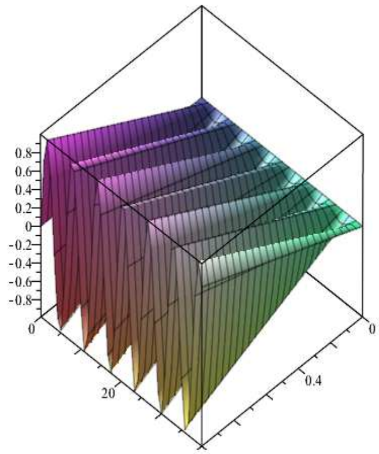



Figure 3a: u(vim.) when t=0.05 Figure 3b- 3-D graph of exact solution

**Example 4:** Consider the nonlinear Klein-Gordon equation.

$$u_{tt} - u_{xx} + u^2 = \left(t^2 + x^2\right)^2 \quad \text{.......(35)}$$

$$u(x,0) = x^2, u_t(x,0) = 0 \quad \text{.......(36)}$$

The VIM formula is

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\tau - t)\left[(u_n)_{\tau\tau} - (u_n)_{xx} + (u_n)^2 - \left(\tau^2 + x^2\right)^2\right] d\tau. \quad n \geq 0 \quad \text{.......(37)}$$

The following iterates are easily obtained:

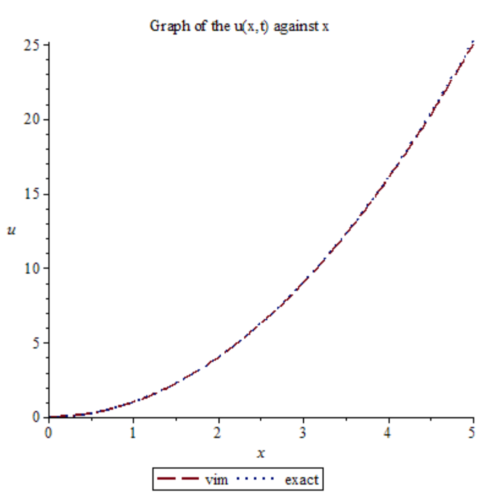
$$u_1 = x^2 + \frac{1}{12}t^4 - \frac{1}{2}t^2x^4 + \frac{1}{2}t^2x^2 + t^2. \quad \text{.......(38)}$$

$$u_2 = x^2 + \frac{1}{6}t^4 - \frac{1}{2}t^2x^2 + t^2 - \frac{1}{12960}t^{10} + \frac{1}{672}t^8x^4 - \frac{1}{672}t^8x^2 - \frac{1}{336}t^8 - \frac{7}{180}t^6x^2 - \frac{1}{120}t^6x^8 + \frac{1}{60}t^6x^6$$
$$+ \frac{1}{40}t^6x^4 - \frac{1}{30}t^6 + \frac{1}{12}t^4x^6 - \frac{1}{12}t^4x^4 - \frac{2}{3}t^4x^2 \quad \text{.......(39)}$$

The exact solution is:

$$u(x,t) = t^2 + x^2 \quad \text{.......(40)}$$

The graph of u(exact) with u(vim) against x when t=0.25 is shown in Fig.4a. The graph of u(exact) with u(vim) against x when t=0.5 is shown in Fig.4b. Also, Fig.4c and Fig.4d show the 3-D graphs of 2-iterate of VIM and exact solution respectively.



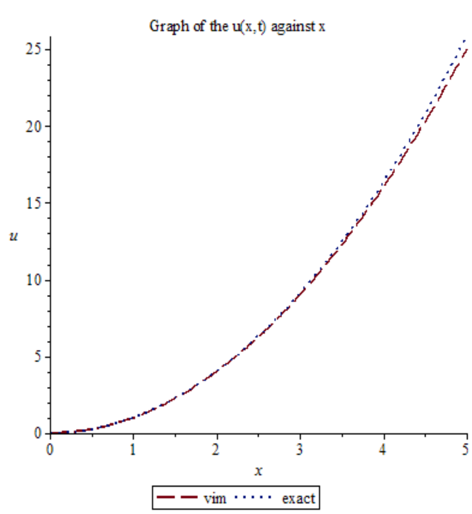
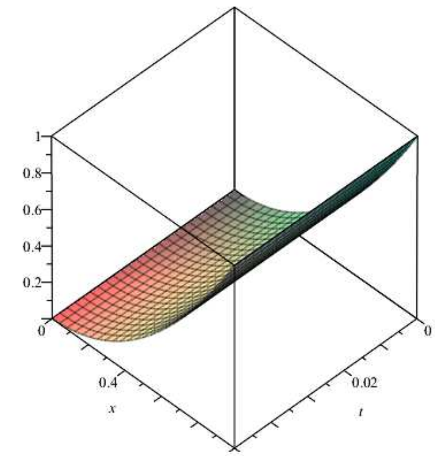




Figure 4a: u(exact) with u(vim.) when t=0.25 Figure 4b: u(exact) with u(vim.) when t=0.5



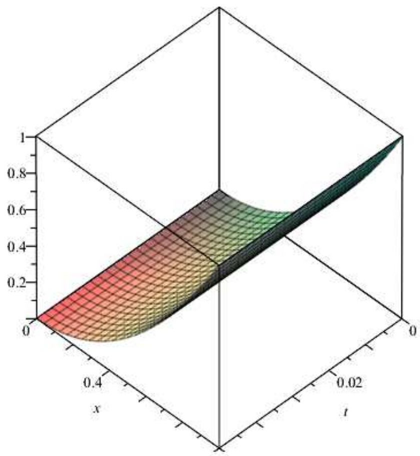




Figure 4c- 3-D graph of 2-iterate of VIM

Figure 4d- 3-D graph of exact solution

**Example 5:** Consider the equation

$$u_{tt} - u_{xx} - u = -\sin x \sin t \qquad \text{…….(41)}$$

Subject to initial conditions:

$$u(x,0) = 0, u_t(x,0) = \sin x \qquad \text{…….(42)}$$

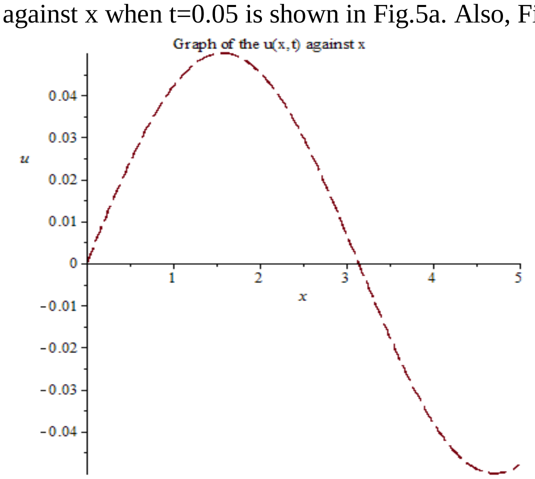
The VIM formula is given as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\tau - t)[(u_n)_{\tau\tau} - (u_n)_{xx} - u_n + \sin x \sin \tau] d\tau, \quad n \geq 0 \qquad \text{…….(43)}$$

which converges to

$$u_1 = \sin x \sin t \qquad \text{…….(44)}$$

After 1-iterate, the scheme produced Equ. (44) which is the exact solution of Equ.(41) The graph of u(exact) with u(vim) against x when t=0.05 is shown in Fig.5a. Also, Fig.5b shows the 3-D graphs of 1-iterate of VIM.



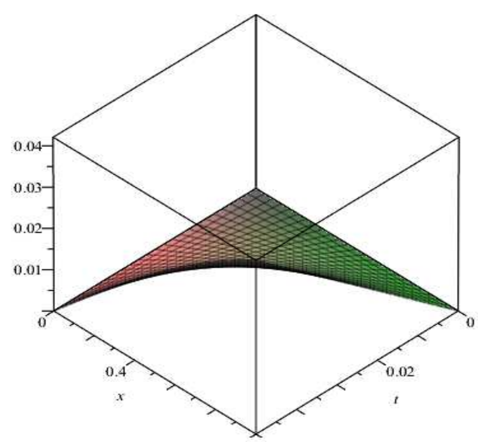




Figure 5a: u(vim.) when t=0.05

Figure 5b- 3-D graph of exact solution

**Example 6:** Consider Sine-Gordon equation

$$u_{tt} - u_{xx} - \sin u, x \in [a,b], \quad t \geq 0 \qquad \text{…….(45)}$$

Subject to:

$$u(x,0) = 4\tan^{-1}(\exp(\gamma x)), x \in [a,b]$$

$$u_t(x,0) = \frac{-4(y\exp(\gamma x))}{1+\exp(2\gamma x)}. \qquad \text{…….(46)}$$

The VIM formula becomes:

$$u_{n+1}(x,t) + \int_0^t (\tau - t)[(u_n)_{\tau\tau} - (u_n)_{xx} - (\sin(u_n))] d\tau \qquad \text{…….(47)}$$

Where

$$u_0(x,t) = 4\tan^{-1}(\exp(\lambda x)) - \frac{4cy\exp(\gamma x)}{1+\exp(2\lambda x)}t \qquad .......(48)$$

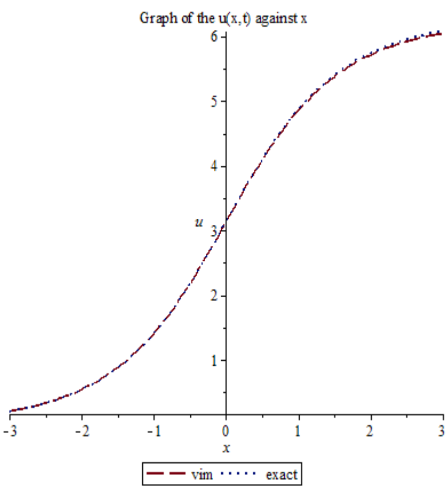
The exact solution of Equ.(45) is

$$u(x,t) = 4\tan^{-1}(\exp(\gamma(x-ct))) \qquad .......(49)$$

Where

$$\gamma = \frac{1}{\sqrt{1-c^2}} \qquad .......(50)$$

Figures 6(a)-(d) show the comparison of 1-iterate of VIM and the exact solution when. $c = 0.0005$.
The graph of u(exact) with u(vim) against x when t=0.25 is shown in Fig.6a. The graph of u(exact) with u(vim) against x when t=0.5 is shown in Fig.6b. Also, Fig.6c and Fig.6d show the 3-D graphs of 1-iterate of VIM and exact solution respectively.



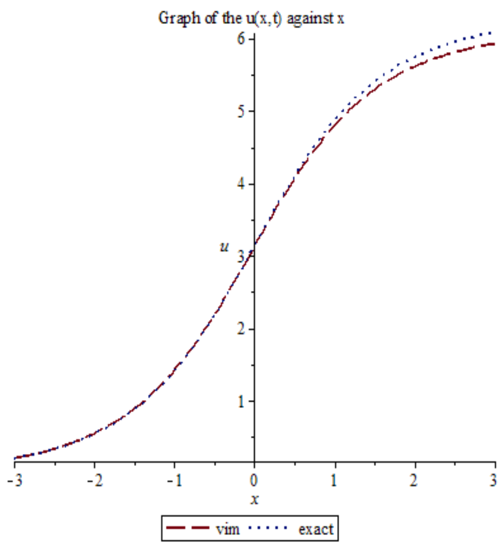
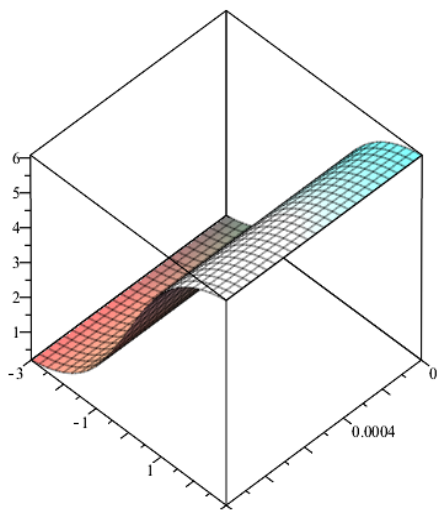




Figure 6a: u(exact) with u(vim.) when t=0.25 Figure 6b: u(exact) with u(vim.) when t=0.5



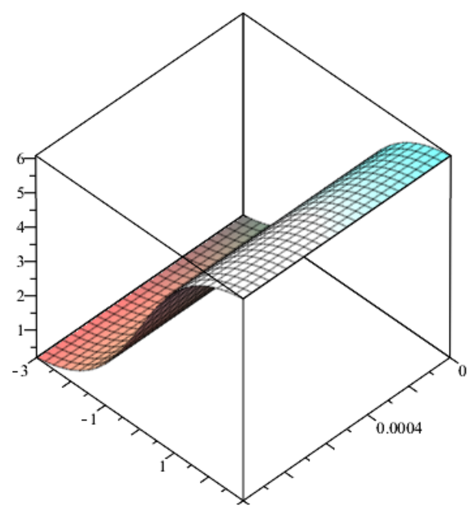




Figure 6c- 3-D graph of 5-iterate of VIM Figure 6d- 3-D graph of exact solution

## 5.0 Conclusion

In this paper, VIM has been successfully used to find the solution of Klein-Gordon and sine-Gordon equations. These equations arise in the propagation of fluxions in Josphson junction between two superconductors, stability of fluid motion, motion of rigid pendular attached to a stretched wire, solid state physics and nonlinear optics. Comparison with the exact solution shows that the method is elegant, effective and a powerful tool for both Klein-Gordon and sine-Gordon equations.

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