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Fifth Order Extended Explicit Runge Kutta Nystrom-Like Method for the Solution of Second Order Ordinary Differential Equations

Aliyu B. Y, Jikantoro Y.D, Aliyu A.I.M and Abubakar A.

Mathematics/Computer Science Department, Faculty of Natural Sciences, Ibrahim Badamasi Babangida University, Lapai, Niger State.

Abstract

Fifth order explicit extended Runge-Kutta Nystrom type method is constructed for numerical integration of special second order system of differential equations of the form $y'' = f(x, y)$. We introduce additional parameters into classical Runge-Kutta Nystrom method in such a way that the first derivative of $f(x, y)$ might be required for the integration. Constant step size code is presented. The method is applied on several model problems in the scientific literature. Results obtained from the numerical experiments suggest the superiority of the new codes over several existing codes of the same algebraic order in the literature.

Keywords: Runge-Kutta Nystrom, Second order equations Stability, Numerical Solution.

1.0 Introduction

In this paper, we present a new Runge-KuttaNyström method that is capable of integrating special second order initial value problems of the form

$$y'' = f(x, y), y(x_0) = y_0, y'(x_0) = y'_0. \qquad (1)$$

The specialty associated with this type of problem is that *f (x, y)* does not depend on y' explicitly. This type of equation is used to model problems arising from different fields of science and engineering, for example, astrophysics, celestial mechanics quantum mechanics, electronics, quantum chemistry etc. One of the ways of solving (1) is by transforming it into a system of first order equations, then apply any suitable method developed for solving first order equations, e.g., RK or linear multi-step method. However, it will be computationally less costly if a method that can directly integrate (1) is developed. This attempt has been made by several authors [1- 8] and this type of method is referred to Runge-KuttaNyström (RKN) method. There are several Runge-KuttaNyström (RKN) methods in the literature. It is worth knowing that RKN method evolved from modification of RK method. As a result, any successful modification done to RK method is capable of producing a modified RKN method [7].

One of the modified RK methods with improved numerical behaviors in the literature today is extended RK method. This type of method was introduced by Goeken and Johnson [9]. Further work on the method can be seen in [10]. In line with this, extended Runge-KuttaNyström methods up to order four was proposed in [11, 12]. We therefore propose fifth order extended Runge-KuttaNyström method of the form

$$y_{n+1} = y_n + hy' + h^2 \sum_{i=1}^{m} (b_i k_{i1} + hc_i k_{i2}) \qquad (2a)$$

$$y'_{n+1} = y'_n + h \sum_{i=1}^{m} ({b_i}' k_{i1} + h{c'}_i k_{i2}) \qquad (2b)$$

$$k_{i1} = f\left(x_n + \hat{c}_i h, y_n + \hat{c}_i h{y'}_n + h^2 \sum_{s=1}^{i-1} a_{is}\, k_{s1} \right), \qquad (2c)$$

$$k_{i2} = G\left(x_n + \hat{c}_i h, y_n + \hat{c}_i h{y'}_n + h^2 \sum_{s=1}^{i-1} a_{is}\, k_{s1} \right). \qquad (2d)$$

Corresponding author:Aliyu B. Y, E-mail:aybadeggi@ibbu.edu.ng, Tel.: +2348036570470

Assume $a_{is},\ b_i,\ b'_i, c_i,\ c'_i\ and\ \hat{c}_i$ are real. $m$might be greater or equal to the number of stages of the method. Suppose $b, b', c, c'\ and\ \hat{c}$are m-dimensional vectors and A an $m \times m$ matrix, where

$b = [b_1, b_2, b_3, \cdots, b_m]^T,$
$b' = [b'_1, b'_2, b'_3, \cdots, b'_m]^T,$
$c = [c_1, c_2, c_3, \cdots, c_m]^T,$
$c' = [c'_1, c'_2, c'_3, \cdots, c'_m]^T,$
$\hat{c} = [\hat{c}_1, \hat{c}_2, \hat{c}_3, \cdots, \hat{c}_m]^T,$

and$A = [a_{is}]$.The coefficients of the method can be summarized as in the tableau below. The rest of the paper is organized as follows: in Section 2, fifth order extended Runge-KuttaNyström method with five functions call per step is derived. Stability analysis of the method is presented in Section 3. In Section 4, numerical results are presented to evaluate the effectiveness of the proposed algorithm. Conclusion is given in Section 5.

$\hat{c}^T$	A
	b^T
	c^T
	b'^T
	c'^T

2.0 Fifth Order Extended Runge-Kutta Nystrom Method

The general form of the fifth order method, when we consider m = 5 in (2) is

$$y_{n+1} = y_n + hy' + h^2 \Big[b_1 k_{11} + b_2 k_{21} + b_3 k_{31} + b_4 k_{41} + b_5 k_{51}$$

$$+h(c_1 k_{12} + c_2 k_{22} + c_3 k_{32} + c_4 k_{42} + c_5 k_{52})] \qquad (3a)$$

$$y'_{n+1} = y_n + h\Big[b'_1 k_{11} + b'_2 k_{21} + b'_3 k_{31} + b'_4 k_{41} + b'_5 k_{51}$$

$$+h(c'_1 k_{12} + c'_2 k_{22} + c'_3 k_{32} + c'_4 k_{42} + c'_5 k_{52})] \qquad (3b)$$

$$k_{11} = f(x_n, y_n),$$

$$k_{12} = G(x_n, y_n), \qquad (3c)$$

$$k_{21} = f\big(x_n + \hat{c}_2 h\,, y_n + h\hat{c}_2 y'_n + h^2 a_{21} k_{11}\big),$$

$$k_{22} = G\big(x_n + \hat{c}_2 h\,, y_n + h\hat{c}_2 y'_n + h^2 a_{21} k_{11}\big), \qquad (3d)$$

$$k_{32} = f\Big(x_n + \hat{c}_3 h, \quad y_n + h\hat{c}_3 y'_n + h^2(a_{31} k_{11} + a_{32} k_{21})\Big),$$

$$k_{32} = G\Big(x_n + \hat{c}_3 h,\ y_n + h\hat{c}_3 y'_n + h^2(a_{31} k_{11} + a_{32} k_{21})\Big), \qquad (3e)$$

$$k_{41} = f\Big(x_n + \hat{c}_4 h, \quad y_n + h\hat{c}_4 y'_n + h^2(a_{41} k_{11} + a_{42} k_{21} + a_{43} k_{31})\Big),$$

$$k_{42} = G\Big(x_n + \hat{c}_4 h,\ y_n + h\hat{c}_4 y'_n + h^2(a_{41} k_{11} + a_{42} k_{21} + a_{43} k_{31})\Big), \qquad (3f)$$

$$k_{51} = f\Big(x_n + \hat{c}_5 h, \quad y_n + h\hat{c}_5 y'_n + h^2(a_{51} k_{11} + a_{52} k_{21} + a_{53} k_{31} + a_{54} k_{41})\Big),$$

$$k_{52} = G\Big(x_n + \hat{c}_5 h,\ y_n + h\hat{c}_5 y'_n + h^2(a_{51} k_{11} + a_{52} k_{21} + a_{53} k_{31} + a_{54} k_{41})\Big), \ (3g)$$

We now derive the order condition of the method. The order condition can be derived by employing the Taylor series expansions of (3) and compare the terms of $h, h^2, h^3, h^4, h^5$with that of the exact solution of (1). The following set of equations obtained as a result of the comparison constitutes the order condition of the fifth order.

For y

$$b_1 + b_2 + b_3 + b_4 + b_5 = \tfrac{1}{2}, \qquad (4)$$

$$b_2\hat{c}_2 + b_3\hat{c}_3 + b_4\hat{c}_4 + b_5\hat{c}_5 + c_1 + c_2 + c_3 + c_4 + c_5 = \frac{1}{6}\,, \qquad (5)$$

$$c_1 + c_2 + c_3 + c_4 + c_5 = 0, \qquad (6)$$

$$b_2\hat{c}_2^2 + b_3\hat{c}_3^2 + b_4\hat{c}_4^2 + b_5\hat{c}_5^2 + c_2\hat{c}_2 + c_3\hat{c}_3 + c_4\hat{c}_4 + c_5\hat{c}_5 = \frac{1}{12}, \qquad (7)$$

$$c_2\hat{c}_2 + c_3\hat{c}_3 + c_4\hat{c}_4 + c_5\hat{c}_5 = 0, \qquad (8)$$

$$\frac{1}{6}b_2\hat{c}_2^3 + \frac{1}{6}b_3\hat{c}_3^3 + \frac{1}{6}b_4\hat{c}_4^3 + \frac{1}{6}b_5\hat{c}_5^3 + \tfrac{1}{2}c_2\hat{c}_2^2 + \frac{1}{2}c_3\hat{c}_3^2 + \frac{1}{2}c_4\hat{c}_4^2 + \frac{1}{2}c_5\hat{c}_5^2 = \frac{1}{120}, \qquad (9)$$

$$b_3 a_{32}\hat{c}_2 + b_4(a_{42}\hat{c}_2 + a_{43}\hat{c}_3) + b_5(a_{52}\hat{c}_2 + a_{53}\hat{c}_3 + a_{54}\hat{c}_4) = \tfrac{1}{120}, \qquad (10)$$

$$c_2\hat{c}_2^2 + c_3\hat{c}_3^2 + c_4\hat{c}_4^2 + c_5\hat{c}_5^2 = 0, \qquad (11)$$

$$\frac{1}{24}b_2\hat{c}_2^4 + \frac{1}{24}b_3\hat{c}_3^4 + \frac{1}{24}b_4\hat{c}_4^4 + \frac{1}{24}b_5\hat{c}_5^4 + \frac{1}{6}c_2\hat{c}_2^3 + \frac{1}{2}c_3\hat{c}_3^3 + \frac{1}{2}c_4\hat{c}_4^3 + \frac{1}{2}c_5\hat{c}_5^3 = \frac{1}{720}, \qquad (12)$$

$$b_3 a_{32}\hat{c}_2 + b_4(a_{42}\hat{c}_2 + a_{43}\hat{c}_3) + b_5(a_{52}\hat{c}_2 + a_{53}\hat{c}_3 + a_{54}\hat{c}_4),$$

$\frac{1}{2}b_3a_{32}\hat{c}_2^2+\frac{1}{2}b_4(a_{42}\hat{c}_2^2+a_{43}\hat{c}_3^2)+\frac{1}{2}b_5(a_{52}\hat{c}_2^2+a_{53}\hat{c}_3^2+a_{54}\hat{c}_4^2)=\frac{1}{144},$ (13)
$c_2\hat{c}_2^3+c_3\hat{c}_3^3+c_4\hat{c}_4^3+c_5\hat{c}_5^3=0,$ (14)
For y'
$b'_1+b'_2+b'_3+b'_4+b'_5=1\,,$ (15)
$b'_2\hat{c}_2+b'_3\hat{c}_3+b'_4\hat{c}_4+b'_5\hat{c}_5+c'_1+c'_2+c'_3+c'_4+c'_5=\frac{1}{2}\,,$ (16)
$c'_1+c'_2+c'_3+c'_4+c'_5=0\,,$ (17)
$\frac{1}{2}b'_2\hat{c}_2^2+\frac{1}{2}b'_3\hat{c}_3^2+\frac{1}{2}b'_4\hat{c}_4^2+\frac{1}{2}b'_5\hat{c}_5^2+c'_2\hat{c}_2+c'_3\hat{c}_3+c'_4\hat{c}_4+c'_5\hat{c}_5=\frac{1}{6},$ (18)
$c'_2\hat{c}_2+c'_3\hat{c}_3+c'_4\hat{c}_4+c'_5\hat{c}_5=0\,,$ (19)
$\frac{1}{6}b'_2\hat{c}_2^3+\frac{1}{6}b'_3\hat{c}_3^3+\frac{1}{6}b'_4\hat{c}_4^3+\frac{1}{6}b'_5\hat{c}_5^3+\frac{1}{2}c'_2\hat{c}_2^2+\frac{1}{2}c'_3\hat{c}_3^2+\frac{1}{2}c'_4\hat{c}_4^2+\frac{1}{2}c'_5\hat{c}_5^2=\frac{1}{24},$ (20)
$b'_3a_{32}\hat{c}_2+b'_4(a_{42}\hat{c}_2+a_{43}\hat{c}_3)+b'_5(a_{52}\hat{c}_2+a_{53}\hat{c}_3+a_{54}\hat{c}_4)=\frac{1}{24},$ (21)
$c'_2\hat{c}_2^2+c'_3\hat{c}_3^2+c'_4\hat{c}_4^2+c'_5\hat{c}_5^2=0,$ (22)
$\frac{1}{24}b'_2\hat{c}_2^4+\frac{1}{24}b'_3\hat{c}_3^4+\frac{1}{24}b'_4\hat{c}_4^4+\frac{1}{24}b'_5\hat{c}_5^4+\frac{1}{6}c'_2\hat{c}_2^3+\frac{1}{2}c'_3\hat{c}_3^3+\frac{1}{2}c'_4\hat{c}_4^3+\frac{1}{2}c'_5\hat{c}_5^3=\frac{1}{120},$ (23)
$b'_3a_{32}\hat{c}_2+b'_4(a_{42}\hat{c}_2+a_{43}\hat{c}_3)+b'_5(a_{52}\hat{c}_2+a_{53}\hat{c}_3+a_{54}\hat{c}_4)$
$+\frac{1}{2}b'_3a_{32}\hat{c}_2^2+\frac{1}{2}b'_4(a_{42}\hat{c}_2^2+a_{43}\hat{c}_3^2)\;+\frac{1}{2}b'_5(a_{52}\hat{c}_2^2+a_{53}\hat{c}_3^2+a_{54}\hat{c}_4^2)=\frac{1}{120},$ (24)
$c'_2\hat{c}_2^3+c'_3\hat{c}_3^3+c'_4\hat{c}_4^3+c'_5\hat{c}_5^3=0,$ (25)
Method. Where $f(x,y)=f,\;\; G(x,y)=f_x+ff_y$ and the row sum condition of RKN method,

$$\frac{1}{2}\hat{c}_i=\sum a_{is}.$$

$a_{is}$is assumed to hold. A great deal of work is involved in solving the system of twenty two equations of order conditions of the method involving thirty unknown parameters. Due to large number of free parameters the strategy will be to hold some of the free parameters constant so that we can easily choose the remaining parameters based on the strategy proposed in [4], which demands that the following quantities related to the method be minimized to their least values, where p is order of the method,

$$E^{(p+1)}=\left\|\tau^{(p+1)}\right\|_2,\;E'^{(p+1)}=\left\|\tau'^{(p+1)}\right\|_2$$

$\tau^{(p+1)}$and$\tau'^{(p+1)}$ are the principal error terms of the method and $E^{(p+1)}$ and $E'^{(p+1)}$ are the principal error norms of y and y' respectively. We therefore obtain the coefficients of the proposed method in line with the aforementioned strategies as follows, with $E^{[6]}=5.7517\times10^{-4}$ and$E'^{[6]}=2.6506\times10^{-3}$ as the error norms of y and y' respectively.

Table 1. *Coefficients of ERKN5 Method*

$\frac{1}{5}$	$\frac{1}{50}$				
$\frac{1}{2}$	$-\frac{7}{183872}$	$\frac{22991}{183872}$			
$\frac{4}{5}$	$\frac{40299}{359125}$	$\frac{229}{2873}$	$\frac{45996}{359125}$		
1	$-\frac{13223}{258570}$	$\frac{5}{13}$	$\frac{13924}{129288}$	$\frac{1}{17}$	
	$\frac{1}{16}$	$\frac{25}{108}$	$\frac{4}{27}$	$\frac{251}{432}$	0
	00000				
	$\frac{1}{16}$	$\frac{125}{432}$	$\frac{8}{27}$	$\frac{125}{432}$	$\frac{1}{16}$
	00000				

3.0 Stability Analysis

The stability property of a numerical method describes the extent to which the method is consistent with a given problem. In this section we derive stability polynomial of ERKN5 method and present the region of absolute stability, which is the main linear stability property of the method.

To study stability property of ERKN5 method, we apply (2) to homogeneous test

$$y''=-\gamma^2y, \tag{26}$$

equation (26), which gives

$\begin{pmatrix}y_{n+1}\\hy'_{n+1}\end{pmatrix}=R(z)\begin{pmatrix}y_n\\hy'_n\end{pmatrix}$,where $R(z)=\begin{pmatrix}r_{11}(z)&r_{12}(z)\\r_{21}(z)&r_{22}(z)\end{pmatrix}$,

$r_{11}(z) = [1 + zb^T(I - zA)^{-1} + z^2c^T(I + z(I - zA)^{-1})]e,$
$r_{12}(z) = [1 + zb^T(I - zA)^{-1} + z^2c^T(I + z(I - zA)^{-1})]\hat{c},$
$r_{21}(z) = [1 + zb'^T(I - zA)^{-1} + z^2c'^T(I + z(I - zA)^{-1})]e,$
$r_{22}(z) = [1 + zb'^T(I - zA)^{-1} + z^2c'^T(I + z(I - zA)^{-1})]\hat{c},$
$z = -(\gamma h)^2,$

Vectors b, b', c, c' $\hat{c}$ and Matrix A are defined in section 1 and vector $e = [1, 1, \cdots, 1]^T$. $R(z)$ is referred to as stability matrix of the method. If r is the eigenvalue of $R(z)$, then the characteristic equation associated with $R(z)$, is

$$r^2 = T(z)r + D(z) = 0. \qquad (27)$$

And it is the stability polynomial of the method, where $T(z)$ and $D(z)$ are trace and determinant of $R(z)$ respectively.

Definition 1. Interval $(-z\,, 0)$ is referred to interval of absolute stability of the method if for all $z \in (-z\,, 0)$, $|r_1, r_2| < 1$ where r_1, r_2 are the roots of equation (27).

The region bounded by the set of points for which $|r| = 1$ is known as stability region of the method. Figure 1 below shows the region of absolute stability of ERKN5 method.

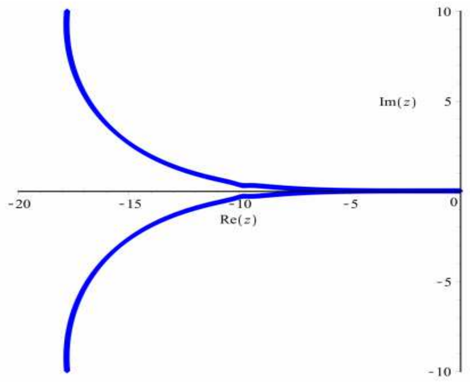

Figure 1. *Stability region of the ERKN5 method.*

4.0 Numerical Results

In this section, we evaluate how effective the proposed method in this paper is by solving some model problems in the literature. The result is compared with results of existing methods of its type in the literature. However, the following acronyms in this paper means:

Max error = $|y(x_{n+1} - y_{n+1})|$: is maximum global error.
ERKN5: New Fifth Order Method obtained in section 2.
RKN5: Fifth Order Method proposed in [4].
MRKN5: Fifth Order Method presented in [13].
BRKN5: Fifth Order Six-Stage Method with first same as last property obtained in [14].
CRK5: Fifth Order RK Method presented in [15].
Problems 1 – 5 below are the test problems in this paper.

Problem 1.

$y'' = -y,\ y(0) = 0;\ y'(0) = 1,$

exact solution: $y(x) = sin(x)$ obtained in [2].

Problem 2.

$y_1'' + y_1 = 0.0\,001cos(x), y_1(0) = 0,\ y_1'(0) = 0.$
$y'' + y = .0\,001sin(),\ y_2(0) = 0, y_2'(0) = 0.9995.$

exact solution: $y_1(x) = cos(x) + 0.0005x\ sin(x),\ y_2(x) = sin(x) - 0.0005x\ cos(x)$ obtained in [1].

Problem 3.

$y'' = -y + x, y(0) = 1, y'(0) = 2,$

exact solution: $y(x) = sin(x) + cos(x) + x$ obtained in [1].

Problem 4.

$y'' = -64y, \qquad y(0) = \frac{1}{4}, y'(0) = -\frac{1}{2}$

exact solution: $y(x) = \frac{\sqrt{17}}{16} sin(8x + t), t = \pi - tan^{-1}(4)$ obtained in [1].

Problem 5.

$$y'' - v^2 y + (v^2 - 1)\sin(x), \qquad y(0) = 1, \qquad y'(0) = \frac{1}{2}$$

exact solution: $y(x) = cos(vx) + sin(vx) + sin(x)$ obtained in [1].
The problems are solved using ,$h = 0.1, 0.05, 0.01, 0.005$ and 0.001.
Tables 2-6 show the numerical accuracy of the new ERKN5 and the existing methods for the constant step-sizes. The accuracy is measured in terms of maximum global error taken over the whole interval considered for the approximation. From the tables, we observe that the new method proposed in this paper has smaller maximum global errors when compared to the existing methods considered in the paper. This implies that the new method converges faster than the existing methods. The derivation as well as the implementation algorithm of the method is coded using computer algebra package (MAPLE).

Table 2. *Comparison of maximum global error of fifth order RKN methods for Problem 1.*

h	Max error				
	ERKN5	RKN5	MRKN5	BRKN5	CRK5
0.1	3.5713×10^{-10}	2.5993×10^{-08}	1.4197×10^{-10}	5.4466×10^{-09}	1.1507×10^{-07}
0.05	5.7465×10^{-12}	8.205×10^{-10}	2.2148×10^{-12}	1.7182×10^{-10}	3.5905×10^{-09}
0.01	6.3711×10^{-16}	2.6478×10^{-13}	1.4148×10^{-16}	5.5377×10^{-14}	1.1474×10^{-12}
0.005	1.7387×10^{-17}	8.2825×10^{-15}	3.2227×10^{-17}	1.7321×10^{-15}	3.5850×10^{-14}
0.001	5.3296×10^{-21}	2.6525×10^{-18}	8.6544×10^{-20}	5.5464×10^{-19}	1.1720×10^{-17}

Table 3. *Comparison of maximum global error of fifth order RKN methods for Problem 2.*

h	Max error				
	ERKN5	RKN5	MRKN5	BRKN5	CRK5
0.1	1.4796×10^{-10}	2.7548×10^{-08}	1.0324×10^{-08}	5.7140×10^{-09}	1.1492×10^{-07}
0.05	2.5017×10^{-12}	8.5919×10^{-10}	3.2232×10^{-10}	1.7852×10^{-10}	3.5856×10^{-09}
0.01	3.7551×10^{-16}	2.7453×10^{-13}	1.0311×10^{-13}	5.7142×10^{-14}	1.1458×10^{-12}
0.005	1.0328×10^{-17}	8.5774×10^{-15}	3.2220×10^{-15}	1.7857×10^{-15}	3.5799×10^{-14}
0.001	3.1056×10^{-21}	2.7444×10^{-18}	1.1105×10^{-18}	5.7143×10^{-19}	1.1660×10^{-17}

Table 4.*Comparison of maximum global error of fifth order RKN methods for Problem 3.*

h	Max error				
	ERKN5	RKN5	MRKN5	BRKN5	CRK5
0.1	4.8780×10^{-10}	4.1167×10^{-08}	1.0483×10^{-08}	8.2850×10^{-09}	1.6936×10^{-07}
0.05	7.8363×10^{-12}	1.2852×10^{-09}	3.2504×10^{-10}	2.5714×10^{-10}	5.2878×10^{-09}
0.01	9.9000×10^{-16}	4.1081×10^{-13}	1.0341×10^{-13}	8.2254×10^{-14}	1.6900×10^{-12}
0.005	2.7931×10^{-17}	1.2836×10^{-14}	3.2291×10^{-15}	2.5703×10^{-15}	5.2823×10^{-14}
0.001	8.4028×10^{-21}	4.1071×10^{-18}	8.2245×10^{-18}	8.2245×10^{-19}	3.2000×10^{-17}

Table 5. *Comparison of maximum global error of fifth order RKN methods for Problem 4*

h	Max error				
	ERKN5	RKN5	MRKN5	BRKN5	CRK5
0.1	1.8895×10^{-04}	1.8810×10^{-03}	8.9670×10^{-05}	3.9273×10^{-04}	6.4756×10^{-02}
0.05	3.1076×10^{-06}	6.0740×10^{-05}	2.6600×10^{-06}	1.2720×10^{-05}	2.0261×10^{-03}
0.01	2.0461×10^{-10}	1.9616×10^{-08}	8.4439×10^{-10}	4.1100×10^{-09}	6.5043×10^{-07}
0.005	3.3721×10^{-12}	6.1297×10^{-10}	6.407×10^{-11}	1.2844×10^{-10}	2.0324×10^{-08}
0.001	4.4579×10^{-16}	8.9612×10^{-13}	8.4576×10^{-15}	4.1102×10^{-14}	6.5025×10^{-12}

Table 6.*Comparison of maximum global error of fifth order RKN methods for Problem 5.*

h	Max error				
	ERKN5	RKN5	MRKN5	BRKN5	CRK5
0.1	7.0513×10^{-08}	2.5600×10^{-06}	3.7691×10^{-07}	5.2303×10^{-07}	2.2150×10^{-05}
0.05	1.1473×10^{-09}	7.9901×10^{-08}	1.1564×10^{-08}	1.6326×10^{-08}	6.8998×10^{-07}
0.01	1.0327×10^{-13}	2.5545×10^{-11}	3.6499×10^{-12}	5.2257×10^{-12}	2.2010×10^{-10}
0.005	2.3774×10^{-15}	7.9817×10^{-13}	1.1387×10^{-13}	1.6329×10^{-13}	6.8751×10^{-12}
0.001	6.0630×10^{-19}	2.5538×10^{-16}	3.6318×10^{-17}	5.2252×10^{-17}	2.2011×10^{-15}

5.0 Conclusion

Extended explicit Runge-KuttaNyström method of order five with five stages is derived in this paper. The method is applied on some test problems and the results are presented. From the numerical results we can conclude that the new method is more promising than the existing methods of the same algebraic order considered in the paper.

6.0 References

[1] S. Z. Ahmad, F. Ismail, N. Senu and M. Suleiman, Semi implicit hybrid methods with higher order dispersion for solving oscillatory problems, Abstr. Appl. Anal. (2013), doi:10.1155/2013/136961, pp.10.

[2] R. A. Al-Khasawneh, F. Ismail and M. Suleiman, Embedded diagonally implicit Runge-Kutta-Nyström pair for solving special second-order IVPs, App. Math. Comp. 4(3) (2007), 1803-1814.

[3] J.R. Dormand, M. E. A. El-Mikkawy and P. J. Prince, Families of Runge-KuttaNyström formulae, IMA J. Numer. Anal. 7(2) (1987), 235-250.

[4] J.R. Dormand, Numerical Methods for Differential Equations: A Computational Approach, CRC Press, New York, 1996.

[5] N. Senu, M. Suleiman and F. Ismail, An embedded explicit Runge-Kutta-Nyströmmethod for solving oscillatory problems, Phys. Scr. (2009), 015005-015013.

[6] N. Senu, M. Suleiman, F. Ismail and M. Othman, A pair Runge-Kutta-Nyström method for periodic initial value problems, Sains. Malay. 4(3) (2010), 639-646.

[7] F. Rabiei, F. Ismail, S. Norazak and S. Seddighi, Accelerated Runge-KuttaNyströmmethod for solving autonomous second-order ordinary differential equations $y¢¢ = f(x)$, World App. Sci. Jourl. (2012), 1549-1555.

[8] P. J. van der Houwen and B. P. Sommeijer, Explicit Runge-Kutta(-Nyström) methodswith reduced phase errors for computing oscillating solutions, SIAM J. Numer. Anal. 24(3) (1987), 595-617.

[10] Xinyuan Wu and Jianlin Xia, Extended Runge-Kutta-like formulae, Appl. Numer. Math. 56(12) (2006), 1584-1605.

[11] Y. D. Jikantoro, Y. B. Aliyu and A. I. Ma'ali, Extended Runge-KuttaNyström methods for the solution of special second order differential equations, Jewel Journal of Scientific Research (Submited)

[12] D. J. Yusuf and Y. B. Aliyu, Third order extended Runge-KuttaNyström methods for the solution of special second order differential equations, Jewel Journal of Scientific Research (JJSR) (2015), 158-164.

[13] M. Mohamed, S. Norazak, M. Suleiman and F. Ismail, Fifth order explicit Runge-KuttaNyström methods for oscillatory problems, World App. Sci. Jourl. 17 (2012), 16-20.

[14] D. G. Bettis, Runge-KuttaNyström algorithm, Celest. Mech. (1973), 229-233.

[15] J. C. Butcher, Numerical Methods for Ordinary Differential Equations, John Wiley & Sons, Ltd., Chichester, 2003.