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Threshold Analysis of Poverty and Crime Model

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Abstract

The concept of basic reproduction number occupies a fundamental place in epidemic theory, the value of R_0 determine the proportion of the population that becomes infected over course of epidemic. Here we obtained the R_0 using the next generation matrix method, and this was used as a threshold parameter to determined the threshold between crime extinction and outbreak.
We computed an expression for the threshold and sensitivity analysis of the model, this was used to examined, specifically the impact of poverty and longer prison terms and incarceration rate on the prevalence of crime.
It was observed that those parameters that shows a decline in reproduction number are those that when put in place, can actually reduce the outbreak of crime in the population. If government policies can be centered on improving these parameters, the crime rate will reduce in this nation.

Keywords: Crime, Poverty, Reproduction number, Threshold, Sensitivity

1.0 Introduction

Crime is a serious social and political problem. The costs it imposes on victims, the public purse, the economic and society cannot be underestimated.
Violent and non-violent crimes from burglars to terrorism have affected the citizens. The greatest security challenge facing Nigeria now is the imposing impunity of terrorist activities of the Boko Haram, the contemporary Nigeria society is engulfed by this terrible acts and this have seriously caused untold hardship to the populace. Even though the subjects of crime and crime control have been major issues of public debate and despite their regular appearance as one of the nation's most severe problems, significant advances in empirical research related to these issues have been made [1].
Nuno et al. [2] presented a model of a criminal-prone self-protected society; they assumed that a society is criminal-prone if its criminal stationary state is unstable under small perturbations of given socio-economic conditions.
Berenji et al. [3] used an evolutionary game theoretic model to study the hypothetical effect of incarceration and prisoner re-entry policy intervention on re-division. They find that excessively harsh or lenient punishment are both less effective at reducing crime than a policy that optimally dedicates scarce resources to a mix of both punishments and post-punishment intervention program, especially. David et al. [4] presented a series of increasing complex models of crime in a population and studied the relationships among the model parameters, although these abstract models do not generate empirical findings.
Criminology has been seen has a complex mix of economic, social and psychological factors that are difficult to ascertain, insignificant to alert, and highly dependent on individuals. Thus, researchers have been studying criminal activities by analyzing criminal psychology, social environment and economic circumstances [5]. Here, we analyzed crime mathematically taking a population-based approach, similar to some of the systems models and game theoretic models surveyed above, but rooted in models that have been developed mainly to model the spread of diseases. We computed and analyzed similar thresholds, considering the spread of crime and poverty.

2.0 Model Formulation

The model formulation is a five compartmental model which describes the different categories of a certain population. The Poverty (P_1) class, the Non-impoverished class (P_2), is the category of individual who have are rich in the society, the crime class (C), the Jailed class by(J); and the Recovered class (R).

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3.0 Model Assumption

The model assumptions for the P_1P_1CJR model are as follows:

(a) The population has constant size, N i.e. Birth and death occurred in a population at constant rate.
(b) The population is homogeneously mixed, i.e. population interacts in such uniformly random and independent way between time steps
(c) The two interventions considered are the rate of converting those in poverty to non-impoverished and the rate of incarceration.
(d) It is assumed that people moved from jail to crime at a constant rate (γ_2) .
(e) Criminals repent at a constant rate (θ_1)
(f) There exist natural death (μ)
(g) The death rate in the model is not only due to natural death but also induced death.
(h) All parameters are assumed non-negative

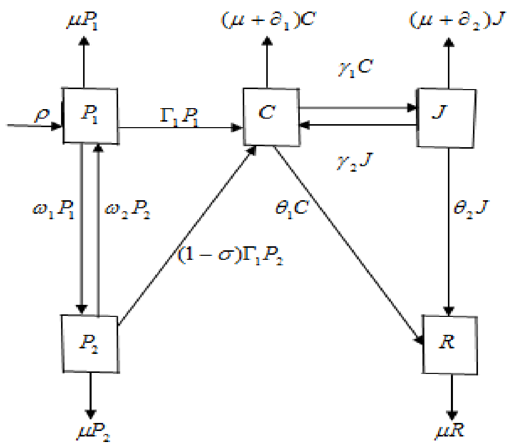

Fig. 1: Flow diagram of the Model.

The model equations

$$\frac{dP_1}{dt} = \varphi + \omega_2 P_2 - (\Gamma_1 + \mu + \omega_1)P_1$$

$$\frac{dP_2}{dt} = \omega_1 P_1 - (\Gamma_2 + \mu + \omega_2)P_2$$

$$\frac{dC}{dt} = \Gamma_1 P_1 + \Gamma_2 P_2 - (\gamma_1 + \theta_1 + \mu + \delta_1)C \quad (1.0) \qquad \frac{dJ}{dt} = \gamma_1 C - (\gamma_2 + \theta_2 + \mu + \partial_2)J$$

$$\frac{dR}{dt} = \theta_1 C + \theta_2 J - \mu R$$

Where

$$\Gamma_1 = \beta_1 C + \beta_2 J$$

$$\Gamma_2 = (1-\sigma)\Gamma_1 = \beta_3 C + \beta_4 J$$

$N = P_1 + P_2 + C + J + R$ is the total size of the population

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Table 1: Definition of Parameters

Symbols	Interpretation
φ	Human Recruitment
ω_1	Movement from impoverished to non-impoverished
ω_2	Movement from non- impoverished to impoverished
δ_1	Crime induced death
δ_2	Induced death at jail
μ	Natural mortality/death rate
θ_1	Recovery from crime
θ_2	Recovery from jail
γ_1	Movement from crime to jail
γ_2	Movement from jail to crime
Γ_1	Rate at which the impoverished commit crime
Γ_2	Rate at which the non-impoverished crime

4.0 The Basic Reproduction Number of the Model

The basic reproduction number R_0 is defined as the effective number of secondary infections (crime) caused by typical criminal individual. It is obtained by taking the largest (dominant)

$$R_0 = \left[\frac{\partial f_i(x_0)}{\partial x_j}\right]\left[\frac{\partial v_i(x_0)}{\partial x_j}\right]^{-1}$$

Where f_i be the rate of appearance of new criminal in the compartments, v_i be the transfer individuals out of the compartments by another means, x_0 be the crime free equilibrium.

$$f_i = \begin{bmatrix} \Gamma_1 P_1 + \Gamma_2 P_2 \\ 0 \end{bmatrix} \Rightarrow F = \begin{bmatrix} \beta_1 P_1 + \beta_3 P_2 & \beta_2 P_1 + \beta_4 P_2 \\ 0 & 0 \end{bmatrix}$$

$$v_i = \begin{bmatrix} k_1 C & -\gamma_2 J \\ -\gamma_1 C & k_2 J \end{bmatrix} \Rightarrow V = \begin{bmatrix} k_1 & -\gamma_2 \\ -\gamma_1 & k_2 \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} \dfrac{k_2}{k_1 k_2 - \gamma_1 \gamma_2} & \dfrac{\gamma_2}{k_1 k_2 - \gamma_1 \gamma_2} \\ \dfrac{\gamma_1}{k_1 k_2 - \gamma_1 \gamma_2} & \dfrac{k_1}{k_1 k_2 - \gamma_1 \gamma_2} \end{bmatrix}$$

$$R_0 = \rho\left(FV^{-1}\right) = \frac{k_2(\beta_1 P_1 + \beta_3 P_2) + \gamma_1(\beta_2 P_1 + \beta_4 P_2)}{k_1 k_2 - \gamma_1 \gamma_2}$$

$$\Rightarrow \frac{k_2 \beta_1 [P_1 + (1-\sigma)P_2] + \gamma_1 \beta_2 [P_1 + (1-\sigma)P_2]}{k_1 k_2 - \gamma_1 \gamma_2}$$

$$\Rightarrow \frac{(k_2 \beta_1 + \gamma_1 \beta_2)[P_1 + (1-\sigma)P_2]}{k_1 k_2 - \gamma_1 \gamma_2} \qquad (1.1)$$

Solving equations (1.0) and letting ε_0 denote the crime-free equilibrium so that $\varepsilon_0 = (P_1 P_1 C J)$

Thus, $\varepsilon_0 = \left[\frac{(\mu+\omega_2)\varphi}{\mu(\mu+\omega_1+\omega_2)}, \frac{\omega_1\varphi}{\mu(\mu+\omega_1+\omega_2)}, 0, 0 \right]$

$$P_1 + (1-\sigma)P_2 = \left[\frac{\mu+\omega_2+(1-\sigma)\omega_1}{\mu(\mu+\omega_1+\omega_2)} \right]\varphi \qquad (1.2)$$

Inserting (1.2) into (1.1)

$$R_0 = \frac{(k_2\beta_1+\gamma_1\beta_2)(\mu+\omega_2+(1-\sigma)\omega_1)\varphi}{\mu(k_1k_2-\gamma_1\gamma_2)(\mu+\omega_1+\omega_2)} \qquad (1.3)$$

Where $(1-\sigma) > 0$ because $0 \le \sigma \le 1$ and $k_1k_2 > \gamma_1\gamma_2$.

5.0 Threshold Analysis of Parameter

We considered the impact of some major parameters like $\gamma_1, \gamma_2, \beta_1, \beta_2$ etc on the transmission dynamics of crime model. This analysis will afford the opportunity of understanding the impact factor of these parameters on the basic reproduction number of the model.

Theorem: For the reproduction number R_0 in (1.2) for the model equations, the threshold analysis of each parameter has an increasing effect on R_0 if $\frac{\partial R_0}{\partial X_i} > 0$ and a decreasing effect on R_0 if $\frac{\partial R_0}{\partial X_i} < 0$ for each X_i (where X_i are the parameters) [6].

Threshold Analysis of β_1 (Susceptible Crime Contact)

$$R_0 = \frac{(\mu+\omega_2+(1-\sigma)\omega_1)\varphi}{\mu(\mu+\omega_1+\omega_2)} \left[\frac{(k_2\beta_1+\gamma_1\beta_2)}{k_1(\theta_2+\mu+\delta_2)+\gamma_2(\theta_1+\mu+\delta_1)} \right]$$

$$\frac{\partial R_0}{\partial \beta_1} = \frac{\varphi k_2(\mu+\omega_2+(1-\sigma)\omega_1)}{\mu(\mu+\omega_1+\omega_2)} > 0 \qquad (1.4)$$

Fundamentally, positivity of an expression confirms a positive effect in the number of secondary crime rate. Hence, increase in β_1 increases the value of the basic reproduction number and make the equilibrium rate approaches endemic value.

Threshold Analysis of β_2 (Susceptible Jail Contact)

$$R_0 = \frac{(\mu+\omega_2+(1-\sigma)\omega_1)\varphi}{\mu(\mu+\omega_1+\omega_2)} \left[\frac{(k_2\beta_1+\gamma_1\beta_2)}{k_1(\theta_2+\mu+\delta_2)+\gamma_2(\theta_1+\mu+\delta_1)} \right]$$

$$\frac{\partial R_0}{\partial \beta_2} = \frac{\varphi\gamma_1(\mu+\omega_2+(1-\sigma)\omega_1)}{\mu(\mu+\omega_1+\omega_2)} > 0 \qquad (1.5)$$

The parameter β_2 also has an expression that is greater than zero, hence its increase in value give a corresponding increase in the number of secondary crime rate generated within completely susceptible population.

Threshold Analysis of γ_1 (Movement from Crime to Jail Class)

$$R_0 = \frac{(\mu+\omega_2+(1-\sigma)\omega_1)\varphi}{\mu(\mu+\omega_1+\omega_2)} \left[\frac{(k_2\beta_1+\gamma_1\beta_2)}{k_1(\theta_2+\mu+\delta_2)+\gamma_2(\theta_1+\mu+\delta_1)} \right]$$

$$\frac{\partial R_0}{\partial \gamma_1} = \frac{\varphi\gamma_1(\mu+\omega_2+(1-\sigma)\omega_1)}{\mu(\mu+\omega_1+\omega_2)(k_1k_2-\gamma_1\gamma_2)^2} \left[-k_2\beta_1(\theta_2+\mu+\delta_2) - \beta_2(\theta_1+\mu+\delta_1) \right]$$

$\frac{\partial R_0}{\partial \gamma_1}$ iff $\beta_1(\theta_2+\mu+\delta_2) \ge \beta_2(\theta_1+\mu+\delta_1)$

$$\frac{\partial R_0}{\partial \gamma_1} \geq 0 \text{ iff } \beta_1(\theta_2+\mu+\delta_2) \leq \beta_2(\theta_1+\mu+\delta_1) \qquad (1.6)$$

If the crime contact rate β_1 and β_2 with the susceptible class are known, and other parameters $(\theta_1, \theta_2, \delta_1, \delta_2, \mu)$ are kept constant, then the effect of γ_1 on the reproduction number can be determined.

Threshold Analysis of ω_1 (Movement Rate from Impoverished Class to Non-Impoverished Class)

$$R_0 = \frac{(\mu+\omega_2+(1-\sigma)\omega_1)\varphi}{\mu(\mu+\omega_1+\omega_2)}\left[\frac{(k_2\beta_1+\gamma_1\beta_2)}{k_1(\theta_2+\mu+\delta_2)+\gamma_2(\theta_1+\mu+\delta_1)}\right]$$

$$\frac{\partial R_0}{\partial \omega_1} = \frac{\varphi(k_2\beta_1+\gamma_1\beta_2)}{\mu(\mu+\omega_1+\omega_2)^2(k_1k_2-\gamma_1\gamma_2)}[(1-\sigma)(\mu+\omega_1+\omega_2)-(\mu+\omega_2+(1-\sigma)\omega_1)]$$

$$= \frac{-\varphi(k_2\beta_1+\gamma_1\beta_2)(\mu+\omega_2)}{\mu(\mu+\omega_1+\omega_2)(k_1k_2-\gamma_1\gamma_2)} < 0 \qquad (1.7)$$

The result obtained depicts a sharp decrease in reproduction number to any increasein ω_1. This agrees with the effect of implementing policies that will move peoplefrom impoverished class to non-impoverished class as any slight increment in thestandard of living of a susceptible population decreases the appealing rate of crimerelated offenses. This explains why, it is very important for government to make policies and provide social interaction with the aim to eliminate poverty in the society, thereby reducing crime rate in the society.

Threshold Analysis of ω_2 (Movement Rate from Non-Impoverished Class to Impoverished Class)

$$\frac{\partial R_0}{\partial \omega_2} = \frac{\varphi(k_2\beta_1+\gamma_1\beta_2)}{\mu(\mu+\omega_1+\omega_2)(k_1k_2-\gamma_1\gamma_2)}[(\mu+\omega_1+\omega_2)-(\mu+\omega_2+(1-\sigma)\omega_1)]$$

$$= \frac{\varphi\sigma\omega_1(k_2\beta_1+\gamma_1\beta_2)}{\mu(\mu+\omega_1+\omega_2)(k_1k_2-\gamma_1\gamma_2)} > 0 \qquad (1.8)$$

If this parameter increases, its effect on the transmission dynamics of crime related activities is a positive one. That is, increment of this parameter value causes the basic reproduction number to increase and tends to the endemic equilibrium state as the value increase.

Threshold Analysis of σ (Effect of financial satisfaction on transmission of dynamics of crime)

$$\frac{\partial R_0}{\partial \sigma} = \frac{-\varphi\omega_1(k_2\beta_1+\gamma_1\beta_2)}{\mu(\mu+\omega_1+\omega_2)(k_1k_2-\gamma_1\gamma_2)} < 0 \qquad (1.9)$$

Increase in this parameter reduces the basic reproduction number and hence serve as an immunity against the rate of contacting crime. This parameter can be likened to the effect of prophylactic vaccine on the transmission dynamics of any infectious disease. Since the efficacy of any vaccine is in reduction of contacting the disease, if σ increases, lesser people will tend to think of crime as a means of livelihood.

Threshold Analysis of θ_1 (Recovery Rate from Crime)

$$\frac{\partial R_0}{\partial \theta_1} = \frac{-\varphi k_2(k_2\beta_1+\gamma_1\beta_2)(\mu+\omega_2+\omega_1(1-\sigma))}{\mu(\mu+\omega_1+\omega_2)(k_1k_2-\gamma_1\gamma_2)} < 0 \qquad (2.0)$$

Increase in parameter θ_1 causes reduction on the basic reproduction number. Hence, effective rehabilitation of the people into crime can reduce the impact of crime on the susceptible population.

Threshold Analysis of θ_2 (Recovery Rate from Jail)

$$\frac{\partial R_0}{\partial \theta_2} = \frac{-\varphi\gamma_1(k_2\beta_1+\gamma_1\beta_2)(\mu+\omega_2+\omega_1(1-\sigma))}{\mu(\mu+\omega_1+\omega_2)^2(k_1k_2-\gamma_1\gamma_2)} < 0 \qquad (2.1)$$

The effect of capital punishment for every criminal activities serve as a means of rehabilitation for many, hence the basic reproduction number reduces with increase in θ_2. If policy can be implemented such that justice will be done to every criminal activity, then there will be a reduction in the criminal rate.

6.0 Sensitivity Analysis

The reproduction number R_0 was used to perform sensitivity analysis using the parameter values given in table 1. This allows us to measure the relative in a state variable when a model parameter changes.

The sensitivity indices $Q_{x_j}^{R_0} = \frac{\partial R_0}{\partial x_j} \times \frac{x_j}{R_0}$ of the reproduction number R_0 to theParameters x_j .

Table 2: Parameter values used in the simulation for the model equations (1.0)

Parameter	Nominal Value (per year)	Source
β_1	0.01	Hypothetical
β_2	0.004	Hypothetical
ω_1	0.01	Hypothetical
ω_2	0.06	Hypothetical
φ	29	Hypothetical
γ_1	0.003	Hypothetical
γ_2	0.05	Hypothetical
δ_1	0.47	Hypothetical
δ_2	0.03	Hypothetical
μ	0.02	Hypothetical
θ_1	0.1	Hypothetical
θ_2	0.7	Hypothetical

Table 3: Result of the Simulation

Parameter	Sensitivity of R_0
β_1	0.993
β_2	0.0007
ω_1	-0.0056
ω_2	0.003
γ_1	0.0012
γ_2	0.00003
σ	-0.0056
θ_1	-0.0636
θ_2	-0.0003

The normalized forward sensitivity index of a variable u is given $Q_p^u = \frac{\partial u}{\partial p} \times \frac{p}{u}$ [7].

Table 3 shows that the normalized sensitive indices of reproduction number R_0 with respect to parameters. We observed that $\beta_{1,}, \beta_{2,} \gamma_{2,} \omega_2$ will have positive impact on R_0 and therefore increase crime, while $\gamma_{1,} \omega_{1,}, \sigma, \theta_{1,} \theta_2$ will have negative impact on R_0, thus reducing crime burden. For example, 10% increase in β_1, resulting in 9.99% increase in R_0. Finally, parameters $\beta_1, \omega_{1,}, \theta_{1,}$ and σ are most sensitive to R_0, hence we observed significant change in R_0 by small changes in these parameters.

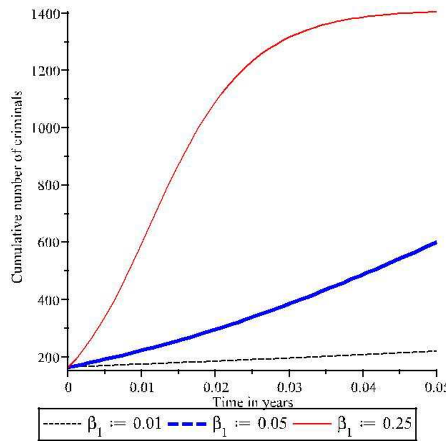

Fig. 2: Effect of $\beta_{1,}$ on Cumulative number of Criminal

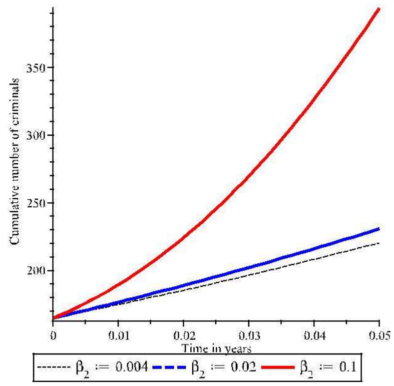

Fig. 3: Effect of $\beta_{2,}$ on Cumulative number of Criminal

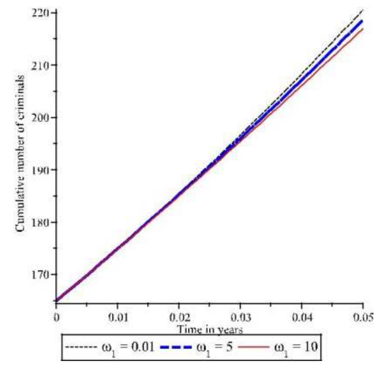

Fig. 4: Effect of ω_1 on Cumulative number of Criminal

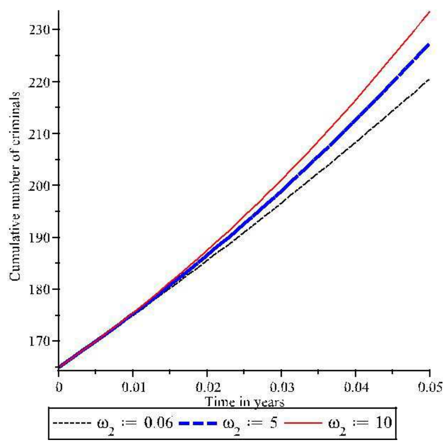

Fig. 5: Effect of $\omega_{2,}$ on Cumulative number of Criminal

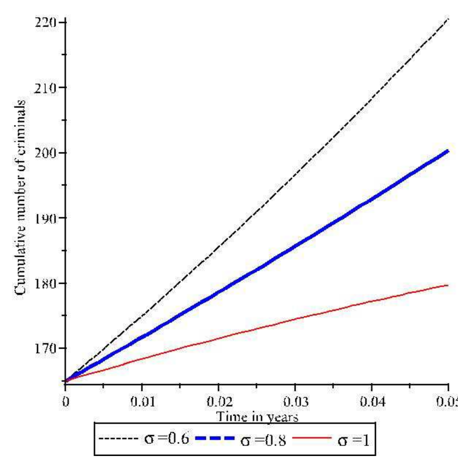

Fig. 6: Effect of σ on Cumulative number of Criminal

7.0 Discussion of Results

A model on crime and poverty was presented and analyzed. We looked intently into the qualitative analysis of the threshold quantity R_0 related to the threshold parameters involved. The parameters that affected the basic reproduction number positively are the parameters that could increase the chance of outbreak of crime in the population. However, those parameters that showed decline in reproduction number are those that when put in place, would reduce the outbreak of crime in the society. Such parameters include β_1, β_2 and ω_1 (movement from impoverished class to non-impoverished class). If government policies can be centered on improving these parameters, the crime rate would reduce in this nation.

8.0 Conclusion

The study underlined the effect of poverty on crime in the society. The model examined and treated the desire to commit crime as an infectious disease motivated by poverty. The basic reproduction number of the model was obtained, used as threshold condition and used to examine how small perturbation on a parameter affects the threshold condition. Sensitivity analysis also done in order to determine the parameters to which our system is most sensitive, which are the contact rate of criminals, rate of movement from the impoverished class to non-impoverished and the rate at which the criminals are released from Jail.

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