

On Existence and Uniqueness of Solution to a Special Case of Asset Management

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Abstract

Considering the uncertain differential equation of the form:

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t,$$

we establish the existence and uniqueness of solution for an uncertain optimal control model for management of net risky capital asset under Lipschitz condition and linear growth condition.

Keywords: Uncertainty theory, uncertain process, differential equation.

1.0 Introduction

Liu [1] founded uncertainty theory which is a branch of mathematics for modeling belief degrees. The uncertainty theory has become a tool to research uncertainty in real life problems. Thus, uncertain process is a sequence of uncertain variables indexed by time or space. The study of uncertain process was initialized by Liu [2]. As a duplicate of Brownian motion, Liu [3] established a canonical process that is a Lipschitz continuous uncertain process with stationary and independent increments. Thereafter, uncertain calculus was introduced by Liu [3] to approach differentiation and integration.

Differential equations have been utilized by many fields like engineering, biology, economics and physics but the complexity in real life problems require more precise description. Uncertain differential equation is a differential equation in which one or more terms is a canonical process. In this paper, uncertain differential equations are used to model a special case in asset management where the investor is interested in determining the expected present net asset of his business such that the net worth is maximized. However, it is difficult to solve uncertain differential equation in many cases, thus we discuss how to solve the constraint of the model, which is formed by uncertain differential equations, by proving the existence and uniqueness of the solution for numerical methods solutions to have a theoretical background.

2.0 Preliminaries

Uncertainty theory is a branch of mathematics for modeling belief degrees. This theory is based on some concepts which may be referred to Liu [4]. For easy interpretation, some of the concepts are given.

Let Γ be a nonempty set and L a σ -algebra over Γ such that (Γ, L) be a measurable space. Each element $\Lambda \in L$ is called an event.

Definition 2.1 [1]: A set function M defined on the σ -algebra over L is called an uncertain measure if it satisfies the following axioms:

Axiom 1. (Normality Axiom): $M\{\Gamma\} = 1$ for the universal set Γ .

Axiom 2. (Duality Axiom): $M\{\Lambda\} + M\{\Lambda^c\} = 1$ for any event Λ .

Axiom 3. (Subadditivity Axiom): For every countable sequence of events, $\Lambda_1, \Lambda_2, \dots$, we have

$$M\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} M\{\Lambda_i\} \tag{2.1}$$

Axiom 4. (Product Axiom): Let (Γ_k, L_k, M_k) be uncertainty spaces for $k = 1, 2, \dots$. The product uncertain measure M is an uncertain measure satisfying

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$$M \left\{ \prod_{k=1}^{\infty} \Lambda_k \right\} = \min_{1 \leq k \leq \infty} M_k \{ \Lambda_k \} \tag{2.2}$$

where Λ_k are arbitrarily chosen events from L_k for $k = 1, 2, \dots$, respectively.

Definition 2.2 [3](Liu 2009): An uncertain process C_σ is said to be a canonical Liu process if

- (i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,
- (ii) C_σ has stationary and independent increments,
- (iii) every increment $C_{s+\sigma} - C_s$ is a normal uncertain variable with expected value 0 and variance σ^2 . The uncertainty distribution of C_σ is

$$\Phi_\sigma(x) = \left[1 + \exp\left(\frac{-\pi x}{\sqrt{3}\sigma}\right) \right]^{-1}, \quad x \in \mathfrak{R} \tag{2.3}$$

and the inverse distribution is

$$\Phi_\sigma^{-1}(y) = \frac{\sigma\sqrt{3}}{\pi} \ln \frac{y}{1-y}, \quad y \in \mathfrak{R} \tag{2.4}$$

Definition 2.3[1]: Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} M\{\xi \geq x\} dx - \int_{-\infty}^0 M\{\xi \leq x\} dx \tag{2.5}$$

provided that at least one of the two integrals is finite

Definition 2.4[2]: An uncertain process X_t is said to have independent increments if

$$X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \dots, X_{t_k} - X_{t_{k-1}}$$

are independent uncertain variables where t_1, t_2, \dots, t_k are any times with $t_0 < t_1 < \dots < t_k$

That is, an independent increment process means that its increments are independent uncertain variables whenever the time intervals do not overlap. It is noted that the increments are also independent of the initial state.

Definition 2.5[2]: Suppose C_t is a canonical Liu process, and f and g are two functions. Then

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t \tag{2.6}$$

is called an uncertain differential equation. A solution is a Liu process X_t that satisfies (2.3) and (2.4) identically in t .

Definition 2.6[2]: Let X_t be an uncertain process. Then for each $\gamma \in \Gamma$, the function $X_t(\gamma)$ is called a sample path of X_t

Definition 2.7[4]: An uncertain process X_t is said to be sample-continuous if almost all sample paths are continuous functions with respect to time t .

3.0 Uncertain Optimal Control Problem in Asset Management

Suppose an individual invests his wealth in capital asset of his business A_t from time t_0 to time t_n . He starts with a known net worth X_0 . At time t , he must choose what fraction of his net worth to utilize on capital asset, ψ , what fraction of his net worth is incurred on liability of the business, τ and thus, determine the expected present net asset, E such that the net worth is maximized.

Table 3.1: Definition of Parameters

Parameter	Description
X_t	Net worth at time (state variable) t
K_t	Consumption at time t
A_t	Capital asset at time t
I_t	Investment at time t
T_t	Indirect tax at time t
D_t	Depreciation at time t
Z_t	Net Foreign supply at time t (less home supply from foreign supply)
L_t	Liability at time t
R_t	Net foreign factor revenue generated at time t
b_t	Return on capital asset at time t
τ	Liability ratio (control) $\tau > -1$
σ_r	Diffusion volatility of liability (with variance σ_r^2 per unit time)
ψ	Capital asset ratio (control) $\psi > 0$
σ_b	Diffusion volatility of asset (with variance σ_b^2 per unit time)
α	Capital gain on asset due to inflation
σ_p	Diffusion volatility on asset price (with variance σ_p^2 per unit time)
β	Mean rate of return on asset
ω	Mean interest rate of liability
C_t	Canonical process
μ	Consumption level
f	Investment ratio
j	Tax ratio
g	Depreciation ratio
h	Net foreign asset ratio
η	discount rate
λ	degree of relative risk
U	Utility function
E	Expectation operator

Therefore, a dynamic optimization model of the expected present value of asset over a given the life cycle is herein presented following the study of portfolio selection by Merton [5]. It is assumed that the goal of the asset management is to choose the optimal utilization and asset allocation policies for maximizing a value function which discounts exponentially future uncertain values of Hyperbolic Absolute Risk Aversion (HARA) utility function over a given time horizon with net worth of tangible assets as the state variable.

The risky asset is assumed to earn an uncertain return and an uncertain gain with mean rate of return and capital gain. Furthermore, we express the change in liability as sum of liability service with an assumption of uncertainty, consumptions, investment and net foreign supply, less taxation, depreciation and revenue over a period of time, Latunde and Bamigbola [6]. Thus, we have

$$J(X) = \max_{\psi} E_C \left[\int_{t_0}^{t_n} \frac{1}{\lambda} e^{-\eta t} (\psi X_t)^{\lambda} dt \right] \tag{3.1}$$

subject to

$$dX_t = [(\alpha + \beta)\psi - (\omega(\psi - 1) + \mu + f + h - j - g)]X_t dt + [\psi\sigma_p + \psi\sigma_b - \psi(\sigma_r - 1)]X_t dC_t \quad (3.2)$$

4.0 Existence and Uniqueness of Solution to the Uncertain Optimal Control Problem

Theorem 4.1 [7]: The uncertain differential equation

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t \quad (4.1)$$

has a unique solution if the coefficients $f(t, x)$ and $g(t, x)$ satisfy the linear growth condition

$$|f(t, x)| + |g(t, x)| \leq L(1 + |x|), \quad \forall x \in \mathfrak{R}, t \geq 0 \quad (4.2)$$

and Lipschitz condition

$$|f(t, x) - f(t, y)| + |g(t, x) - g(t, y)| \leq L|x - y|, \quad \forall x, y \in \mathfrak{R}, t \geq 0 \quad (4.3)$$

for some constant L . Moreover, the solution is sample-continuous

We therefore show the existence and uniqueness of solution to the problem considering constraint (3.2).

Theorem 4.2: The constraint (3.2) has a unique solution if the coefficients $[(\alpha + \beta)\psi - (\omega(\psi - 1) + \mu + f + h - j - g)]X_t$ and $[\psi\sigma_p + \psi\sigma_b - \psi(\sigma_r - 1)]X_t$ satisfy the linear growth condition (4.2) and the Lipschitz condition (4.3) for some constant L . Moreover, the solution is sample-continuous.

Proof:

We apply successive approximation method to prove the existence of the solution.

Let $X_t^0 = X_0$ and

$$\begin{aligned} X_t^{(n)} &= X_0 + \int_{t_0}^t [(\alpha + \beta)\psi - (\omega(\psi - 1) + \mu + f + h - j - g)]X_t^{(n-1)} dt \\ &+ \int_{t_0}^t ([\psi\sigma_p + \psi\sigma_b - \psi(\sigma_r - 1)]X_t^{(n-1)}) dC_t \end{aligned} \quad (4.4)$$

for each $n = 1, 2, \dots$

For each $\gamma \in \Gamma$, we define

$$S_t^n(\gamma) = \max_{t_0 \leq t \leq t_n} |X_t^{(n+1)}(\gamma) - X_t^{(n)}(\gamma)| \quad (4.5)$$

for each $n = 0, 1, 2, \dots$

When $n = 0$, from equation (4.5), we have

$$S_t^0(\gamma) = \max_{t_0 \leq t \leq t_n} |X_t^{(1)}(\gamma) - X_t^{(0)}(\gamma)| \quad (4.6)$$

And when $n = 1$, from equation (4.4), we

$$\begin{aligned} X_t^{(1)} - X_0 &= \int_{t_0}^t [(\alpha + \beta)\psi - (\omega(\psi - 1) + \mu + f + h - j - g)]X_t^{(0)} dt \\ &+ \int_{t_0}^t ([\psi\sigma_p + \psi\sigma_b - \psi(\sigma_r - 1)]X_t^{(0)}) dC_t \end{aligned} \quad (4.7)$$

Following the linear growth condition and Lipschitz condition, we substitute equation (4.7) into equation (4.6) to have

$$\begin{aligned} S_t^0(\gamma) &= \max_{t_0 \leq t \leq t_n} \left| \int_{t_0}^t [(\alpha + \beta)\psi - (\omega(\psi - 1) + \mu + f + h - j - g)]X_t^{(0)}(\gamma) dt \right. \\ &+ \left. \int_{t_0}^t ([\psi\sigma_p + \psi\sigma_b - \psi(\sigma_r - 1)]X_t^{(0)}(\gamma)) dC_t(\gamma) \right| \\ &\leq \int_{t_0}^t [(\alpha + \beta)\psi - (\omega(\psi - 1) + \mu + f + h - j - g)]X_t^{(0)}(\gamma) dt \\ &+ k(\gamma) \int_{t_0}^t ([\psi\sigma_p + \psi\sigma_b - \psi(\sigma_r - 1)]X_t^{(0)}(\gamma)) dt \\ &\leq (1 + |X_0|)L(1 + k(\gamma))t \end{aligned}$$

where $k(\gamma)$ represents the Lipschitz constant of the sample path $C_t(\gamma)$.

This implies that, for each $\gamma \in \Gamma$, the sample paths $X_t(\gamma)$ converges uniformly on the time interval $t_0 < t < t_n$. Thus, there exist a solution of the form

$$X_t = X_0 + \int_{t_0}^{t_n} [(\alpha + \beta)\psi - (\omega(\psi - 1) + \mu + f + h - j - g)]X_t dt + \int_{t_0}^{t_n} ([\psi\sigma_p + \psi\sigma_b - \psi(\sigma_r - 1)]X_t) dC_t$$

We now prove that the solution is unique.

Suppose X_t and X_t^* are two distinct solutions of the constraint (3.2) and $t_n < t < t_0$ then following the linear growth and Lipschitz conditions, we have

$$|X_{t_n}(\gamma) - X_{t_n}^*(\gamma)| \leq L(1 + k(\gamma)) \int_{t_0}^{t_n} |X_t(\gamma) - X_t^*(\gamma)| dt$$

Using the Gronwall inequality, we get

$$|X_{t_n}(\gamma) - X_{t_n}^*(\gamma)| \leq 0 \cdot \exp(L(1 + k(\gamma))t) = 0$$

. Thus, $X_{t_n}(\gamma) = X_{t_n}^*(\gamma)$ which proves the uniqueness of the solution.

However, for each $\gamma \in \Gamma$, we have

$$|X_{t_n}(\gamma) - X_t(\gamma)| = \left| \int_t^{t_n} [(\alpha + \beta)\psi - (\omega(\psi - 1) + \mu + f + h - j - g)]X_t(\gamma) dt + \int_t^{t_n} ([\psi\sigma_p + \psi\sigma_b - \psi(\sigma_r - 1)]X_t(\gamma)) dC_t(\gamma) \right| \rightarrow 0$$

as $t \rightarrow t_n$. Hence, the solution is sample-continuous which completes the proof.

5.0 Conclusion

Uncertain differential equation which is a relevant tool in modeling was used to model a special case in asset management where the investor is interested in determining the expected present net asset such that the net worth is maximized.

Thus, the existence and uniqueness of solution to the asset management problem was stated and proved with respect to Lipschitz condition and linear growth condition. In a subsequent research work, the numerical solution to the system shall be discussed.

6.0 References

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