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Multiple Soliton Solutions of the Boussinesq Equation by the Simplified Hirota's Method

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Abstract

Due to the difficulties in finding the bilinear form for some nonlinear partial differential equations (NLPDE$_s$), in Hirota's method, a simplified version of the method which does not require the bilinear form of the (NLPDE) will be used to obtain the multiple soliton solutions of Boussinesq Equation.

1.0 Introduction

Hirota developed a bilinear method for finding the solution of non linear evolution equations. The method has been one of the most successful methods for constructing exact solutions of various NLPDE from soliton theory [1]. It also allows the testing of whether or not a given PDE satisfies the necessary requirements to admit solitary wave and soliton solutions. The drawback of the conventional Hirota's method is that the bilinear form of the NLPDE must be known in advance [2]. In other words, the technique can only be applied to equations that can be written in a bilinear form; however, contrary to the conventional Hirota's method the simplified version is more efficient in the sense that it does not require the knowledge of the bilinear form in order to find the soliton solutions. We consider the Boussinesq equation

$$u_{tt} - u_{xx} - 3(u^2)_{xx} - u_{xxxx} = 0$$

Equation (1) has been solved using methods such as Hirota bilinear transform [3-5], Wronskian technique [6], inverse scattering transform and etc. In this paper we are going to use simplified version of the Hirota's bilinear method to get the soliton solutions of equation (1).

The paper is organize as follows in section two we are going to give a brief on the methodology of the propose method, while in section three we give the implementation of the method to solve equation (1) and finally the paper is going to be concluded in section four.

2.0 Methodology:

Using a dependent variable transformation of the form

$$u = 2(\ln f)_{xx}$$

(2)

Equation (1) can be written in the form

$$f(f_{tt} - f_{xx} - f_{xxxx}) - f_t^2 + f_x^2 + 4f_x f_{xxx} - 3f_{xx}^2 = 0$$

(3)

Note that the transformation (2) follows from truncated Painlneve expansion [7,8]

In order for the simplified version of the Hirota's technique be applicable to equations that are difficult to write in bilinear form, Hirota's bilinear operators will not be introduced. Instead we write equation (3) as [9-10]

$$fL(f)+N(f)=0$$

(4)

where L denotes the linear differential operator defined by

$$L\cdot=\frac{\partial^2\cdot}{\partial t^2}-\frac{\partial^2\cdot}{\partial x^2}-\frac{\partial^4\cdot}{\partial x^4}$$

(5)

and the nonlinear differential operator denoted by N and is defined by

$$N(f,f)=-\frac{\partial^2 f}{\partial t^2}+\frac{\partial^2 f}{\partial x^2}+4\frac{\partial f}{\partial x}\frac{\partial^3 f}{\partial x^3}-3\left(\frac{\partial^2 f}{\partial x^2}\right)^2$$

(6)

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3.0 Implementation

Suppose we want to find f such that

$$f(t,x)=1+\sum_{i=1}^{\infty}\varepsilon^n f^{(n)}(t,x)$$

(7)

where $f^{(n)}$ (n=1,2,..) are to be determined and for simplicity we take ε =1 substituting (7) in to (4) yield a polynomial in powers of ε. We equate to zero terms of like powers of ε to obtain a hierarchy of equations for $f^{(n)}$ as

$$\varepsilon^1:L(f^{(1)})=0$$

(8)

$$\varepsilon^2:L(f^{(2)})=-f^{(1)}L(f^{(1)})-N(f^{(1)},f^{(1)})=0$$

(8a)

$$\varepsilon^3:L(f^{(3)})=-f^{(1)}L(f^{(2)})-N(f^{(1)},f^{(2)})-N(f^{(2)},f^{(1)})$$

(9)

$$\vdots$$

$$\varepsilon^n:L(f^{(n)})=-\sum_{j=1}^{n-1}[N(f^{(j)},f^{(n-j)})+f^{(j)}L(f^{(n-j)})]$$

(10)

Solving the family of equation (10) simultaneously will determine the function $f^{(n)}$ (n=1,2,…). If there exist an integer m≤n such that $f^{(n)}$ = 0 the series (7) truncates at order m-1 to become

$$f(t,x)=1+\sum_{i=1}^{m-1}\varepsilon^n f^{(n)}(t,x)$$

(11)

Note that the first term of (8) is linear and hence admit exponential solutions.

One soliton

In order to obtain the one solitonwe let $f^{(1)} = e^{q_1}$

With $q_1 = e^{k_1 x + w_1 t + d_1}$

The computation of first of (8) yields the following dispersion term

$$p(k_i, w_i) = 0$$

Where, $p(k_i, w_i) = w_i^2 - k_i^2 - k_i^4 \qquad i = 1, 2, ..., N$

(12)

Computing the right hand side of (8a) gives

$$- N(f^{(1)}, f^{(1)}) = 0$$

(13)

This means that $f^{(2)}$ will be equals to zero, it is easy to check that for n≥2 $f^{(2)}$ =0 the series (7) truncates and yields

$$f(x,t) = 1 + e^{q_1}$$

(14)

Therefore the one soliton solution of equation (1) is generated from equation (2) as:

$$u = 2[\ln(1 + e^{\theta_1})]_{xx}$$

(15)

Two soliton solutions

To get the two- soliton solution we use

$$f^{(1)} = e^{q_1 + q_2}$$

(16)

The dispersion terms are obtained from the computation of $(o)e^1$ as

$$w_1 = \sqrt{k_1^2 + k_1^4} \qquad and \quad w_2 = \sqrt{k_2^2 + k_2^4}$$

Next we compute the right hand side of equation (8a) as

$$- N(f^{(1)}, f^{(1)}) = \underset{1£ i< j£ N}{\text{å}} 2k_i k_j[\sqrt{(1 + k_i^2)(1 + k_j^2)} - 1 - 2(k_i^2 + k_j^2) + 3k_i k_j]f_i.f_i$$

(17)

this means $f^{(2)}$ will take the form

$$\underset{1\pounds\, i< j\pounds\, N}{\mathring{a}} a_{ij} f_i f_j$$

(18)

Substituting (19) in to (8a) we compute the left hand side of the (8a) as

$$Lf^{(2)} = \underset{1\pounds\, i< j\pounds\, N}{\mathring{a}} p(k_i + k_j, w_i + w_j) a_{ij} f_i f_j$$

(19)

$$= \underset{1\pounds\, i< j\pounds\, N}{\mathring{a}} 2k_i k_j [\sqrt{(1+ k_i^2)(1+ k_j^2)} - 1- 2(k_i^2 + k_j^2) - 3k_i^2 k_j^2] a_{ij} f_i f_j$$

(20)

Equating (17) and (21) gives a_{ij} as

$$a_{ij} = \frac{\sqrt{(1+ k_i^2)(1+ k_j^2)} - 1- 2(k_j^2 + k_i^2) + 3k_i^2 k_j^2}{\sqrt{(1+ k_i^2)(1+ k_j^2)} - 1- 2(k_j^2 + k_i^2) - 3k_i^2 k_j^2}, \quad 1\pounds\ i< j\pounds\ N$$

(21)

Consequently we obtain the explicit value of $f^{(2)}$ as

$$f^{(2)} = \underset{1\pounds\, i< j\pounds\, N}{\mathring{a}} a_{ij} e^{(k_i + k_j)x + (w_i + w_j)t + d_i + d_j}$$

(22)

It is easy to check that $f^{(n)}=0$ for $n\geq 3$

Hence $f = 1+ e^{q_1} + e^{q_2} + a_{12} e^{q_1 + q_2}$

(23)

Therefore, the two soliton solution is given by substituting equation (24) in to equation (2) as

$$u = 2[\ln(1+e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2})]_{xx}$$

Three soliton solutions

Proceeding in a similar fashion with equation (9) leads to the explicit form of $f^{(3)}$ i.e. for N=3 we will have

$$f^{(3)} = a_{123} e^{q_1 + q_2 + q_3}$$

$$= a_{123} e^{(k_1 + k_2 + k_3)x + (w_1 + w_2 + w_3)t + d_1 + d_2 + d_3}$$

Where $a_{123} = a_{12}a_{13}a_{23}$

For N = 3, we calculate $f^{(n)}$ for n>3 by using the right hand side of (9) to find $f^{(n)} = 0$, for n > 4
Therefore, the expansion of (7) truncates and yields

$$f = 1 + e^{q_1} + e^{q_2} + e^{q_3} + a_{12}e^{q_1+q_2} + a_{13}e^{q_1+q_3} + a_{23}e^{q_2+q_3} + a_{123}e^{q_1+q_2+q_3}$$

(24)

with e =1 and upon substituting (24) in to (2) the well-known three soliton solutions of the Boussinesq equation follows.

$$u = 2[\ln(1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{23}e^{\theta_2+\theta_3} + a_{123}e^{\theta_1+\theta_2+\theta_3})]_{xx}$$

4.0 Conclusion:

Using the simplified version of the Hirota's method the three-soliton solution for the Boussinesq equation is obtained. The method is more efficient in the sense that it does not require the knowledge of the bilinear form in order to find the multi-soliton solution of the NLPDE.The N-soliton solution for N>3 can also be constructed in a similar manner. However, the calculation becomes very lengthy, and more elegant to find. The form of the N-soliton solution may be found using the Wronskian technique.

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