*Journal of the Nigerian Association of Mathematical Physics*
***Volume35,*(*May, 2016*), pp 439 – 444**
*© J. of NAMP*

# Construction of Point Transformation for the Simple Harmonic Oscillator Equation Using Differential Forms

*[1]Orverem J.M. and [2]Ogundare B.S.*

**[1]Department of Mathematical Sciences, Federal University P.M.B. 5001, Dutsinma, Nigeria.**
**[2]Department of Mathematics,Obafemi Awolowo University, Ile-Ife,Nigeria.**

## *Abstract*

***In this paper, we obtain the point transformation for the simple harmonic oscillator equation of second order ordinary differential equation using the method of differential forms, and hence its solution.***

**Key words:** Point transformation, Differential forms, Simple harmonic oscillator equation, second order differential equations

## 1.0 Introduction

While the introductory problems involving the motion of a particle are often concerned with moving a particle from one place to another, there is an important class of problems where a particle goes through a motion, but at some point in the trajectory the particle returns to the initial position.

While periodic motion is often complex in nature, many problems can be reduced by approximation to a more simple form known as Simple Harmonic Motion (SHM). An example of such an approximation is a simple pendulum where for small oscillations the motion can be approximated to simple harmonic motion. Simple harmonic motion is an oscillation of a particle in a straight line.

Point transformation preserves the integrability of the equation and its Lie symmetry structure [1], and hence the reason for the use of point transformation. The linearizability problem for second order ordinary differential equations using differential forms has been investigated in [2].

In this paper, we construct the point transformation for the simple harmonic oscillator equation and present its solution using the method of differential forms derived in [2].

## 2.0 The Method

Our starting point is a second order ordinary differential equation

$$y'' = f(x, y, y'). \qquad (2.1)$$

We assume a point transformation given by the variables

$$X = F(x, y), \qquad Y = G(x, y), \qquad (2.2)$$

with a requirement that,

$$\frac{d^2Y}{dX^2} = 0. \qquad (2.3)$$

We first construct, using equation (2.2)

$$\frac{dY}{dX} = \frac{G_x + G_y y'}{F_x + F_y y'} \qquad (2.4)$$

where $F_x + F_y y' \neq 0$ and the subscripts $x$ and $y$ denote partial differentiation. The second derivative equation may be written simply in terms of a differential $d\left(\frac{dY}{dX}\right) = 0$ which becomes

$$(F_x + F_y y')(dG_x + y' dG_y + G_y dy') - (G_x + G_y y')(dF_x + y' dF_y + F_y dy') = 0. \qquad (2.5)$$

We can expand (2.5) and write it as

$$T dy' + \rho y'^2 + (\lambda + \delta) y' + \sigma = 0, \qquad (2.6)$$

where

$$T = F_x G_y - F_y G_x, \qquad (2.7)$$

Corresponding author: Orverem J.M., E-mail:orveremjoel@yahoo.com, Tel.: +2347034848671& 8163883787(OBS)

and we have the 1-forms

$$\left.\begin{array}{l}\rho = F_y dG_y - G_y dF_y, \lambda = F_y dG_x - G_y dF_x,\\ \sigma = F_x dG_x - G_x dF_x, \delta = F_x dG_y - G_x dF_y.\end{array}\right\} \quad (2.8)$$

We can rewrite equation (2.6) as

$$dy' = \alpha + \beta y' + \gamma y'^2, \quad (2.9)$$

where

$$\alpha = \frac{-\sigma}{T}, \beta = \frac{-(\lambda+\delta)}{T}, \gamma = \frac{-\rho}{T}. \quad (2.10)$$

For integrability of equation (2.9) we set $ddy' = 0$, that is

$$0 = d\alpha + dy' \wedge \beta + y' d\beta + 2y' dy' \wedge \gamma + y'^2 d\gamma. \quad (2.11)$$

Substituting (2.9) into equation (2.11), we have:

$$0 = d\alpha + (\alpha + \beta y' + \gamma y'^2) \wedge \beta + y' d\beta + 2y'(\alpha + \beta y' + \gamma y'^2) \wedge \gamma + y'^2 d\gamma. \quad (2.12)$$

The $y'^3$ term in equation (2.12) vanishes because $\gamma \wedge \gamma = 0$, we expand equation (2.12) and equate the coefficients of the other powers of $y'$ to zero to have:

$$d\alpha = \beta \wedge \alpha, d\beta = 2\gamma \wedge \alpha, dr = \gamma \wedge \beta. \quad (2.13)$$

Now, we go back to equations (2.8) and expand the differentials, to have:

$$\rho = F_y(G_{xy}dx + G_{yy}dy) - G_y(F_{xy}dx + F_{yy}dy),$$
$$\lambda = F_y(G_{xx}dx + G_{xy}dy - G_y(F_{xx}dx + F_{xy}dy),$$
$$\sigma = F_x(G_{xx}dx + G_{xy}dy) - G_x(F_{xx}dx + F_{xy}dy),$$
$$\delta = F_x(G_{xy}dx + G_{yy}dy) - G_x(F_{xy}dx + F_{yy}dy),$$

which can simply be written as

$$\rho = Adx + Bdy, \lambda = Cdx + Ady, \sigma = Ddx + Edy, \delta = Edx + Hdy, \quad (2.14)$$

where

$$A = F_y G_{xy} - G_y F_{xy}, B = F_y G_{yy} - G_y F_{yy}$$
$$C = F_y G_{xx} - G_y F_{xx}, D = F_x G_{xx} - G_x F_{xx}$$
$$E = F_x G_{xy} - G_x F_{xy}, H = F_x G_{yy} - G_x F_{yy}.$$

Thus,

$$\alpha = \frac{-(Ddx+Edy)}{T}, \beta = \frac{-(Cdx+Edx+Ady+Hdy)}{T}, \gamma = \frac{-(Adx+Bdy)}{T}. \quad (2.15)$$

Substituting $\alpha, \beta$ and $\gamma$ into equation (2.9) and dividing by $dx$ to convert the differential forms to functions, we have:

$$y'' + f_0 + f_1 y' + f_2 y'^2 + f_3 y'^3 = 0, \quad (2.16)$$

where the $f_k$ are given by

$$f_0 = \frac{D}{T}, f_1 = \frac{(C+2E)}{T}, f_2 = \frac{(H+2A)}{T}, f_3 = \frac{B}{T}. \quad (2.17)$$

We define $K$ and $L$ as

$$K = \frac{E}{T}, L = \frac{A}{T}, \quad (2.18)$$

and replace $D, C, H$ and $B$ in the 1-forms in equation (2.15) in favour of the $f_k$, $K$ and $L$, obtaining

$$\alpha = -f_0 dx - Kdy, \beta = (K - f_1)dx + (L - f_2)dy, \gamma = -Ldx - f_3 dy. \quad (2.19)$$

We also note that

$$\frac{dT}{T} = (3K - f_1)dx + (f_2 - 3L)dy. \quad (2.20)$$

We see that the 1-forms $\alpha, \beta, \gamma$ in (2.19) and $\frac{dT}{T}$ in equation (2.20) are now expressed in terms of these four known functions $K$ and $L$. The first three of these 1-forms can now be substituted into equation (2.13) on the various functions. If we do that, the first equation for $d\alpha$, gives the equation

$$f_{0y} - K_x = -K(K - f_1) + f_0(L - f_2) \quad (2.21)$$

which is nonlinear in $K$. The other equations give the results:

$$-K_y + f_{1y} + L_x - f_{2x} = 2KL - f_0 f_3 \quad (2.22)$$

and

$$L_y - f_{3x} = -L(L - f_2) + f_3(K - f_1) \quad (2.23)$$

which are also nonlinear. However, we can simplify the situation by defining new variables:

$$T = \frac{1}{W^3},\ E = \frac{U}{W^4},\ A = \frac{V}{W^4}, \quad (2.24)$$

so that from (2.18)

$$K = \frac{U}{W},\ L = \frac{V}{W}, \quad (2.25)$$

and from (2.20)

$$3\frac{dW}{W} = (f_1 - 3K)dx + (3L - f_2)dy. \quad (2.26)$$

We now have this situation. The $dW$ equation (2.26) gives expressions for $W_x$ and $W_y$. The equation (2.21) gives, after substitution for $W_x$, an expression

$$U_x = Wf_{0y} - \frac{2}{3}Uf_1 - Vf_0 + Wf_0f_2 \quad (2.27)$$

which is linear in $U$, $V$ and $W$. The equation (2.23) gives an expression

$$V_y = Wf_{3x} + \frac{2}{3}Vf_2 + Uf_3 - Wf_1f_3 \quad (2.28)$$

which is also linear. The equation (2.22) gives a linear expression.

$$V_x - U_y = \frac{U}{3}f_2 + \frac{V}{3}f_1 - Wf_{1y} + Wf_{2x} - 2f_0f_3W. \quad (2.29)$$

The integrability condition on (2.26) gives a linear expression

$$V_x + U_y = \frac{U}{3}f_2 + \frac{V}{3}f_1 + \frac{W}{3}f_{2x} + \frac{W}{3}f_{1y}. \quad (2.30)$$

Equations (2.29) and (2.30) can be solved for $V_x$ and $U_y$. Thus we have expressions for all derivatives of $U$, $V$ and $W$, all of which are linear and homogeneous in the same variables. That is

$$dU = \frac{1}{3}\left(-2Uf_1 - 3Vf_0 + W\left(3f_{0y} + 3f_0f_2\right)\right)dx + \frac{1}{3}\left(-Uf_2 + W\left(2f_{1y} - f_{2x} + 3f_0f_3\right)\right)dy, \quad (2.31)$$

$$dV = \frac{1}{3}\left(Vf_1 + W\left(2f_{2x} - f_{1y} - 3f_0f_3\right)\right)dx + \frac{1}{3}\left(3Uf_3 + 2Vf_2 + W(3f_{3x} - 3f_1f_3)\right)dy, \quad (2.32)$$

$$dW = \frac{1}{3}(-3U + Wf_1)dx + \frac{1}{3}(3V - Wf_2)dy. \quad (2.33)$$

We summarize all these relations in a nice matrix equation

$$dr = Mr, \quad (2.34)$$

where

$$r = \begin{pmatrix} U \\ V \\ W \end{pmatrix} \text{and} M = Pdx + Qdy,$$

$$P = \left(\frac{1}{3}\right)\begin{pmatrix} -2f_1 & -3f_0 & 3f_{0y} + 3f_0f_2 \\ 0 & f_1 & 2f_{2x} - f_{1y} - 3f_0f_3 \\ -3 & 0 & f_1 \end{pmatrix}$$

$$Q = \left(\frac{1}{3}\right)\begin{pmatrix} -f_2 & 0 & 2f_{1y} - f_{2x} + 3f_0f_3 \\ 3f_3 & 2f_2 & 3f_{3x} - 3f_1f_3 \\ 0 & 3 & -f_2 \end{pmatrix}.$$

For integrability of (2.34), $ddr = 0$ giving

$$dM = M \wedge M \quad (2.35)$$

which is not zero since $M$ is a matrix. Substitution for $M$ in terms of $P$ and $Q$ gives the condition

$$Q_x - P_y + QP - PQ = 0. \quad (2.36)$$

This matrix condition in (2.36) reduces to two equations:

$$f_{0yy} + f_0\left(f_{2y} - 2f_{3x}\right) + f_2f_{0y} - f_3f_{0x} + \left(\frac{1}{3}\right)\left(f_{2xx} - 2f_{xy} + f_1f_{2x} - 2f_1f_{1y}\right) \quad (2.37)$$

and

$$f_{3xx} + f_3\left(2f_{0y} - f_{1x}\right) + f_0f_{3y} - f_1f_{3x} + \left(\frac{1}{3}\right)\left(f_{1yy} - 2f_{2xy} + 2f_2f_{2x} - f_2f_{1y}\right) = 0. \quad (2.38)$$

To summarize, we note that the original differential equation is cubic in $y'$, with the coefficients satisfying equations (2.37)and (2.38).

Now, we shall construct the point transformations proper. We will need $U$, $V$ and $W$ therefore we need to solve equations (2.34). Once the equations are solved, we construct $K$ and $L$ from equation (2.25).

In order to find the $F(x,y)$ and $G(x,y)$ for which we are seeking, we revert to equations (2.8) and solve for $dF_x$, $dF_y$, $dG_x$ and $dG_x$. Solution for $dF_x$ and $dF_y$ gives

$$dF_x = \frac{\left(F_y\sigma - F_x\lambda\right)}{T}, \qquad dF_y = \frac{\left(F_y\delta - F_x\rho\right)}{T}.$$

Solution for $dG_x$ and $dG_y$, shows that they satisfy the same equation, so we will write only equations for the derivatives of $F$. We note that

$$\delta + \lambda = -T\beta \text{ and } \delta - \lambda = dT,$$

So we can solve these equations for $\delta$ and $\lambda$. We can also substitute for $\sigma$ and $\rho$ in terms of $\alpha$ and $\gamma$. We get finally

$$dF_x = -F_y\alpha + F_x\frac{\left(\beta + \frac{dT}{T}\right)}{2}, \qquad dF_y = F_x\gamma + F_y\frac{\left(-\beta + \frac{dT}{T}\right)}{2}.$$

We substitute for $\alpha, \beta, \gamma$ and $\frac{dT}{T}$ from equations (2.19)and (2.20) respectively in terms of the expressions obtained above, with the $f_k$, $K$ and $L$.

We now have two equations which can be expressed in matrix form as follows;

$$dR = ZR,\ dS = ZS \tag{2.39}$$

where

$$Z = \begin{pmatrix} (2K - f_1)dx - Ldy & f_0dx + Kdy \\ -Ldx - f_3dy & Kdx + (f_2 - 2L)dy \end{pmatrix},$$

$$R = \begin{pmatrix} F_x \\ F_y \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} G_x \\ G_y \end{pmatrix}.$$

This linear equation set can be solved for $R$. There will be two independent solutions, which can be taken as $R$ and $S$ as seen in equation (2.39). Integrability is guaranteed by setting $ddR = 0$. Finally, one can solve

$$dF = (dx \quad dy)R,\ dG = (dx \quad dy)S \tag{2.40}$$

for $F$ and $G$.

We can summarize the procedure as follows:

1. Make sure that the original differential equation is a cubic in $y'$ as in equation (2.16)
2. Test the coefficients $f_k$ to see whether they satisfy equations (2.37) and (2.38).
3. Construct the $3 \times 3$ matrix $M$ and solve equation (2.34) (linear) for the three components of $r$ − a special solution is usually sufficient and construct $K$ and $L$.
4. Construct the $2 \times 2$ matrix $Z$ and solve equation (2.39) (linear) for $R$ or $S$.
5. Solve equation (2.40); the two independent solutions may be taken as $F$ and $G$.

## 3.0 Construction of the Point Transformation

The simple harmonic oscillator equation is given as:

$$y'' + y = 0 \tag{3.1}$$

Comparing equation (3.1) with equation(2.16), we see that: $f_0 = y, f_1 = f_2 = f_3 = 0,$ and the coefficients satisfy the conditions stated in equations (2.37)and (2.38).

We now construct a $3 \times 3$ matrix $M$ given as

$$M = Pdx + Qdy$$

where

$$Pdx = \begin{pmatrix} 0 & -ydx & dx \\ 0 & 0 & 0 \\ -dx & 0 & 0 \end{pmatrix}, Qdy = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & dy & 0 \end{pmatrix},$$

so that

$$M = \begin{pmatrix} 0 & -ydx & dx \\ 0 & 0 & 0 \\ -dx & dy & 0 \end{pmatrix}.$$

Now, we solve equation (2.34) to have:

$$dr = \begin{pmatrix} -yVdx + Wdx \\ 0 \\ -Udx + Vdy \end{pmatrix}.$$

The equation for $V$ is $dV = 0$, so that we may take $V = 0$. If we do that, we then have

$dU = Wdx$ and $dW = -Udx$ or we simply write $\frac{dU}{dx} = W$ and $\frac{dW}{dx} = -U$.

This yields the special solution $U = \sin x$ and $W = \cos x$.

Defining $K = \frac{U}{W}$ and $L = \frac{V}{W}$, we see that $K = \tan x$ and $L = 0$.

We now construct the $2 \times 2$ matrix $Z$ to be

$$Z = \begin{pmatrix} 2\tan x\, dx & ydx + \tan x\, dy \\ 0 & \tan y \end{pmatrix}.$$

Setting

$$R = \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} F_x \\ F_y \end{pmatrix}, dR = ZR$$

From equation (2.39)becomes

$$dR = \begin{pmatrix} 2b\tan x\,dx + c(ydx + \tan x\,dy) \\ c\tan x\,dx \end{pmatrix}.$$

We see from the above that,

$$dc = c\tan x\,dx. \qquad (3.2)$$

We integrate equation (3.2) to obtain

$$c = k\sec x \qquad (3.3)$$

where$k$ is a constant.

We also note from the matrix$dR$that

$$db = 2b\tan x\,dx + c(ydx + \tan x\,dy). \qquad (3.4)$$

Substituting equation (3.3) into equation (3.4) and simplifying, we have

$$db = (2b\tan x + ky\sec x)dx + k\sec x\tan x\,dy,$$

so that

$$b_x = 2b\tan x + ky\sec x \qquad (3.5)$$

and

$$b_y = k\sec x\tan x. \qquad (3.6)$$

Differentiating equation (3.3) with respect to$x$, we obtain

$$c_x = k\sec x\tan x$$

which implies that$c_x = b_y$.

Integrating equation (3.6) we have

$$b = ky\sec x\tan x + g(x). \qquad (3.7)$$

Differentiating equation (3.7) with respect to$x$, we have

$$b_x = ky\sec x + 2ky\,tan^2 x\sec x + g'(x). \qquad (3.8)$$

Equating equations (3.5) and (3.5) and simplifying, we have:

$$g' - 2\tan x\,g = 0. \qquad (3.9)$$

Using the integrating factor to solve for$g$ in the above, we have

$$g = m\,sec^2 x. \qquad (3.10)$$

Substituting equation(3.10) into equation (3.7), we have

$$b = ky\sec x\tan x + m\,sec^2 x, \qquad (3.11)$$

where$m$is another constant.

In summary,$b = F_x$is as obtained in equation (3.11) and$c = F_y$ is obtained in equation (3.3).

Considering$F_y = k\sec x$, on integration, we obtain

$$F = ky\sec x + h(x). \qquad (3.12)$$

Next, we differentiate the above with respect to $x$ to have

$$F_x = ky\tan x\sec x + h'(x). \qquad (3.13)$$

Equating (3.11) and (3.13) and simplifying, we obtain

$$h'(x) = m\,sec^2 x. \qquad (3.14)$$

Integrating the above, we have

$$h(x) = m\tan x. \qquad (3.15)$$

Substituting equation (3.15) into equation (3.12), we have

$$F = ky\sec x + m\tan x. \qquad (3.16)$$

From equation (3.16), we take the coefficients of the constants$k$and $m$ as the point transformation and have that,

$X = F(x, y) = \tan x,\ \ Y = G(x, y) = y\sec x$

as state in [3].

Considering a transformation$Y = aX + b$, stated in [2], where$a$ and$b$ are known constants; the solution of the simple harmonic oscillator equation can be readily obtained as

$y\sec x = a\tan x + b$ or $y = a\sin x + b\cos x$.

## 4.0 References

[1] Grissom, C., Thompson, G. and Wilkens, G., 1989. Linearization of Second Order Ordinary Differential Equations via Cartan's Equivalence Method. J. Diff. Eqns 77, 1-15.

[2] Harrison, B.K., 2002. An Old Problem Newly treated with Differential Forms: When and How can the equation $y'' = f(x, y, y')$ Be Lineanized? Proceedings of the Institute of Mathematics of NAS of Ukraine 43, 27-35.

[3] Mahomed, F.M., 2007. Symmetry group Classification of ordinary differential equations: survey of some results. Math. Appl. Sci. Wiley InterScience 30, 1995-2012.