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Analysis of the Dynamic Buckling of a Model Structure with Quadratic Nonlinearity Struck by a Step Load Superposed on a Quasi-Static Load

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Abstract

In this paper, we aim at the dynamic buckling load of a typical imperfect model elastic structure struck by a step load that is superposed on a pre-static load. We made use of regular perturbation and expanded the variables asymptotically in powers of a small parameter that signifies the amplitude of the imperfection. We also made use of Phase Plane analysis and compared the results. The results show that, in general, pre-loading lowers the dynamic buckling load. With these results we can relate the dynamic buckling load with the corresponding static buckling load such that if one is given the other can be determined without repeating the entire process of evaluation.

Key words: Lindley distribution;Survival function; hazard function; moments; maximumLikelihood; Kolmogorov test Statistic

1.0 Introduction

The stability of elastic structures has become a cause of great concern in recent day because of increasing occurrence of collapse structures such as buildings, bridges, dams and other such life supporting structures. A major concern of most Structural Engineers as well as Applied Mathematicians is the elastic stability of loaded structures. Every Engineering structures buckle when they are loaded beyond the stability limit and the initial imperfection of the system is also of a great consequence. Our study is concerned with the determination of the various buckling loads involve in a pre-statically loaded elastic structure which is suddenly trapped by an explicitly time dependent load. Consequently, we adopt a simple quadratic model structure, from which we are able to determine the classical as well as the static buckling loads of a designed structure with a small imperfection amplitude $\bar{\xi}$.

As mentioned earlier a lot of interest has been generated in recent times on the study of both dynamic and static buckling of engineering structure because of its importance:

Ette [1] determined analytically the dynamic buckling of a finite imperfect simply supported column on mixed quadratic-cubic nonlinear foundations under step loading. The dynamic buckling load was determined nontrivially by means of multiple timing perturbations. The work justifies the claim that it is as important to worry about the load level at which an elastic structure buckles dynamically as well as it is pertinent to worry about the load level at which the same structure buckles statically. Other related investigations, Budiansky and Hutchinson [2,3] have provided criteria and estimates in dynamic stability of elastic structures. Danielson [4] analysed the dynamic buckling loads of imperfection-sensitive structure using perturbation procedures, while Simitses [5-7] also studied the effect of static preloading on dynamic stability of structures and the instability of dynamically loaded structures as well as the dynamic stability of suddenly loaded structures.

Other recent works include McShane et al [8] who investigated the dynamic buckling of inclined structures, while Artem and Aydin [9] worked on exact solution and dynamic buckling analysis of a beam-column loading. Jeong [10] investigated the lateral deflection of rail road track under quasi-static loading, while Chukwuchekwa and Ette [11] analysed asymptotically an improved quadratic model structure subjected to static loading. Similarly, Ette and Osuji [12] investigated the dynamic stability of a viscous damped model structure modulated by a periodic load.

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2.0 Formulation

Budiansky and Hutchinson [1] are known to be the first to investigate the dynamic stability of elastic model structures which include the elastic quadratic model structure. A significant improvement on this model was made by Danielson [4] by incorporating additional mass M on a spring (Fig.1), with a spring constant K. We adopt this model in a simplistic form as in Ette and Osuji [12] and proceed as follows:

2.1 Derivation of Simple Quadratic Model Structure

We now consider a simple quadratic model structure as in (Fig.1) below:

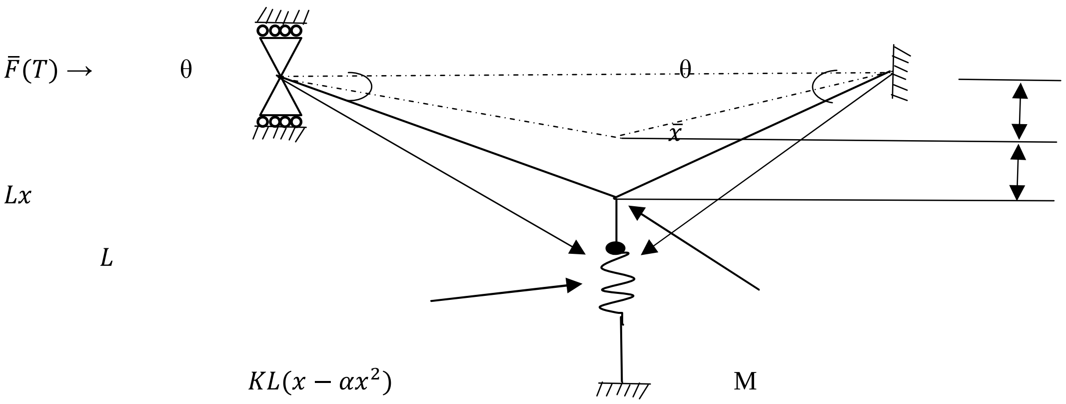

Fig.1: Simple Quadratic- Elastic Model Structure

Let Q be the tension on each arm of the structure and let θ be the angle between the horizontal and each arm of the structure. Let the magnitude of the force applied be $\bar{F}(T)$ and the restoring force per unit length on the spring be $KL(x-\alpha x^2)$, where K,L and α are constants.

Let $\bar{x}$ be the initial deflection of the structure from the horizontal and let x be the additional distance from equilibrium position on the application of the horizontal force $\bar{F}(T)$.

Then for θ small, we have

$$\theta = \frac{\bar{x}}{L} + \frac{x}{L} \qquad (2.1)$$

For equilibrium of forces on the horizontal position we have

$$Q\cos\theta = \bar{F}(T) \qquad (2.2)$$

For equilibrium on the vertical direction, we have

$$mx'' = 2Q\sin\theta - KL(x-\alpha x^2), \qquad (2.3)$$

where $(\ )' \equiv \frac{d}{dT}(\ )$.

If θ is small, we can afford to neglect the nonlinear term, so that

$$\cos\theta \approx 1, \quad \sin \approx \theta \qquad (2.4)$$

Thus, from equation (2.2), we get

$$Q = \bar{F}(T) \qquad (2.5)$$

and from equation (2.3), we get

$$mx'' = 2\bar{F}(T)\left(\frac{\bar{x}}{L} + \frac{x}{L}\right) - KL(x-\alpha x^2).$$

This implies that

$$mx'' + KL\left(1 - \frac{2\bar{F}(T)}{KL^2}\right)x - \alpha KLx^2 = \frac{2\bar{x}}{L}\bar{F}(T) \qquad (2.6)$$

We shall let

$$\xi = \frac{x}{L}, \ \ \bar{\xi} = \frac{\bar{x}}{L}, \ \ t = T\sqrt{\frac{KL}{m}}, \ \ \lambda = \frac{2\bar{F}(0)}{KL^2}, \ \ f(t) = \frac{\bar{F}(t)}{\bar{F}(0)}, \ \ a = \alpha L, \ \bar{F}(0) \neq 0$$

Rewriting equation (2.6) using the non-dimensional parameters, we have

$$\ddot{\xi} + \big(1 - \lambda f(t)\big)\xi - a\xi^2 = \lambda\bar{\xi} f(t) \qquad (2.7)$$

where $(\dot{\ }) \equiv \frac{d}{dt}()$

Now, equation (2.7) is an important equation that characterizes all quadratic structures.

Here, ξ is the displacement, $f(t)$ is the time dependent load, λ is the non-dimensional load parameter, a is a constant, while $\bar{\xi}$ is the imperfection amplitude deemed small relative to unity, i.e. $0 < \bar{\xi} < 1$.

We can solve (2.7) for different forms of $f(t)$ and different initial conditions.

In our study, we shall be interested in determining the dynamic buckling load λ_D of the structure if the structure is preloaded statically and thereafter, trapped by a step load.

2.2 Classical Buckling Load, λ_c

This is defined as the buckling load of the perfect structure if we neglect the non-linear terms and neglect the inertial term $\ddot{\xi}$. It is obtained from the conditions

$$\bar{\xi} = 0, \frac{d\lambda}{d\xi} = 0, a = 0, f(t) \equiv 1$$

Thus, from equation (2.7) we have

$$(1-\lambda)\xi = 0 \qquad (2.8)$$

Here, we assume that λ is continuous function of ξ. Hence differentiating (2.8) by ξ and using $\frac{d\lambda}{d\xi} = 0,$ we get

$$\lambda_c = 1 \qquad (2.9)$$

Therefore, the classical buckling load λ_c takes the value 1.

In most loading histories, the Classical buckling load is the greatest load parameter. Thus, the non-dimensionalized load parameter λ is always less than 1; i.e.

$\frac{\lambda}{\lambda_C} < 1$ or $\lambda < \lambda_C$.

2.3 Static Buckling Load, λ_S of the Model Structure

This is the load that the structure would require to buckle statically.

Here we shall use two methods, namely: (i) the phase plane (exact solution) method and (ii) the perturbation method. The aim is to show that the two methods yield the similar result.

(a): **Static Buckling Load,λ_S by phase plane (exact solution) method:**

Here, we set $\ddot{\xi} = 0$, $f(t) \equiv 1$. Thus, from (2.7), we get

$$(1-\lambda)\xi - a\xi^2 = \lambda\bar{\xi} \qquad (2.11)$$

For buckling, we require that $\frac{d\lambda}{d\xi} = 0$, such that Eqn. (2.11) becomes

$$1 - \lambda_S - \frac{d\lambda}{d\xi}\xi - 2a\xi_S = 0$$

i.e. $1 - \lambda_S = 2a\xi_S$ (2.12)

where ξ_S is the value of ξ at buckling and λ_S is the buckling load. Hence, we have

$$\xi_S = \frac{1}{2a}(1-\lambda_S) \qquad (2.13)$$

If we determine (2.11) at buckling, we get

$$(1-\lambda_S)\xi_S - a{\xi_S}^2 = \lambda_S\bar{\xi}$$

Which yield, from eqn. (2.13), the expression for the static buckling load in terms of $\bar{\xi}$

$$(1-\lambda_S)^2 = 4a\lambda_S\bar{\xi} \qquad (2.14)$$

We, note from (2.14) that if the structure were perfect then $\bar{\xi} = 0$ and $\lambda_S = 1$.

Generally, $\lambda_S < \lambda_{C,}$ i.e. $\frac{\lambda_S}{\lambda_C} < 1$.

The result (2.14) was first obtained by Budiansky and Hutchinson [2] using phase plane analysis.

(b) **Static Buckling Load,λ_S Using Perturbation and Asymptotic Procedure**

To solve equation (2.11) analytically by finding asymptotic value of ξ, we let

$$\xi = \sum_{i=0}^{\infty} \xi_0^{(i)}\bar{\xi}^i, \quad 0 < \bar{\xi} < 1$$

$$= \xi_0^{(1)}\bar{\xi} + \xi_0^{(2)}\bar{\xi}^2 + \xi_0^{(3)}\bar{\xi}^3 + \cdots \qquad (2.15)$$

We substitute (2.15) into (2.11) and get

$$(1-\lambda)\left(\xi_0^{(1)}\bar{\xi} + \xi_0^{(2)}\bar{\xi}^2 + \xi_0^{(3)}\bar{\xi}^3 + \cdots\right)$$

$$-a\left(\xi_0^{(1)}\bar{\xi} + \xi_0^{(2)}\bar{\xi}^2 + \xi_0^{(3)}\bar{\xi}^3 + \cdots\right)^2 = \lambda\bar{\xi} \qquad (2.16)$$

Equating the coefficients of powers of $\bar{\xi}$, we have

$$O(\bar{\xi}): \ (1-\lambda)\xi_0^{(1)} = \lambda \qquad (2.17a)$$

$$O(\bar{\xi}^2): \ (1-\lambda)\xi_0^{(2)} = a\xi_0^{(1)2} \qquad (2.17b)$$

$$O(\bar{\xi}^3): \ (1-\lambda)\xi_0^{(3)} = 2a\xi_0^{(1)}\xi_0^{(2)} \qquad (2.17c)$$

and so on. Thus we have:-

From (2.17a), $\xi_0^{(1)} = \frac{\lambda}{1-\lambda} \equiv B$ (2.18)

From (2.17b), $\xi_0^{(2)} = \frac{a\xi_0^{(1)2}}{1-\lambda} = \frac{aB^2}{1-\lambda}$ (2.19)

From (2.17c), $\xi_0^{(3)} = \frac{2a\xi_0^{(1)}\xi_0^{(2)}}{1-\lambda} = \frac{2a^2B^3}{(1-\lambda)^2}$ (2.20)

and so on. Thus we can write

$$\xi = \xi_0^{(1)}\bar{\xi} + \xi_0^{(2)}\bar{\xi}^2 + \xi_0^{(3)}\bar{\xi}^3 + \cdots$$

$$= B\bar{\xi} + \frac{aB^2}{1-\lambda}\bar{\xi}^2 + \frac{2a^2B^3}{(1-\lambda)^2}\bar{\xi}^3 + \cdots \qquad (2.21)$$

Let $B = c_{1,} \frac{aB^2}{1-\lambda} = c_2$, $\frac{2a^2B^3}{(1-\lambda)^2} = c_3$, and so on. Then, we get

$$\xi = c_1\bar{\xi} + c_2\bar{\xi}^2 + c_3\bar{\xi}^3 + \cdots \qquad (2.22)$$

Since the displacement is unbounded after buckling, we, as in Amazigo and Ette [13], have to reverse the series in (2.22). Thus, we have

$$\bar{\xi} = d_1\xi_0 + d_2\xi_0^2 + d_3\xi_0^3 + \cdots \qquad (2.23)$$

For us to determine $d_1, d_2, d_3, ...$ we substitute for ξ from (2.22) in (2.23) and get

$$\bar{\xi} = d_1\left(c_1\bar{\xi} + c_2\bar{\xi}^2 + c_3\bar{\xi}^3 + \cdots\right) + d_2\left(c_1\bar{\xi} + c_2\bar{\xi}^2 + c_3\bar{\xi}^3 + \cdots\right)^2$$

$$+d_3(c_1\bar{\xi} + c_2\bar{\xi}^2 + c_3\bar{\xi}^3 + \cdots)^3 + \cdots \qquad (2.24)$$

We now equate the coefficient of $\bar{\xi}^i$ for i = 1, 2, 3, … Thus, for

$$i = 1: \quad 1 = d_1c_1 \Rightarrow d_1 = \frac{1}{c_1} \qquad (2.25a)$$

$$i = 2: \quad 0 = d_1c_2 + d_2{c_1}^2 \Rightarrow d_2 = \frac{-d_1c_2}{{c_1}^2} = \frac{-c_2}{{c_1}^3} \qquad (2.25b)$$

$$i = 3: \quad 0 = d_1c_3 + 2c_1c_2d_2 + d_3{c_1}^3 \Rightarrow d_3 = \frac{d_1c_3 - 2c_1c_2d_2}{{c_1}^3}$$

$$= \frac{-1}{{c_1}^5}(c_1c_3 - 2{c_2}^2) = \frac{2{c_2}^2 - c_1c_3}{{c_1}^5} \qquad (2.25c)$$

We, now apply the condition for buckling is $\frac{d\lambda}{d\xi} = 0$ to eqn. (2.23), noting that $d_i = d_i(\lambda),\ i = 1,2,3, ...$ This implies that

$$\frac{d\bar{\xi}}{d\xi} = \frac{dd_1}{d\lambda}\frac{d\lambda}{d\xi} + d_1 + \frac{dd_2}{d\lambda}\frac{d\lambda}{d\xi} + 2d_2\xi_S + \cdots = 0 \qquad (2.26)$$

$$\Rightarrow d_1 + 2d_2\xi_S = 0$$

where, ξ_S is the value of ξ at static buckling. Hence

$$\xi_S = \left(\frac{-d_1}{2d_2}\right)\Bigg]_{\lambda=\lambda_S} = \frac{-\frac{1}{c_1}}{-2\left(\frac{c_2}{{c_1}^3}\right)}$$

i.e. $\xi_S = \frac{{c_1}^2}{2c_2}$ (2.27)

If we determine (2.23) at buckling, we get

$$\bar{\xi} = d_1\xi_S + d_2\xi_S^2 + d_3\xi_S^3 + \cdots$$

$$= \xi_S(d_1 + d_2\xi_S) = \frac{{c_1}^2}{2c_2}\left[\frac{1}{c_1} - \left(\frac{c_2}{{c_1}^3}\right)\left(\frac{{c_1}^2}{2c_2}\right)\right]$$

i.e. $\bar{\xi} = \frac{c_1}{4c_2}$ (2.28)

On substitution for c_1 and c_2 in (2.28), we get

$$\bar{\xi} = \frac{1}{4}\frac{B(1-\lambda_S)}{aB^2} = \frac{1-\lambda_S}{4aB}$$

$$= \frac{1-\lambda_S}{4a\left(\frac{\lambda_S}{1-\lambda_S}\right)} = \frac{(1-\lambda_S)^2}{4a\lambda_S}$$

i.e. $(1-\lambda_S)^2 = 4a\bar{\xi}\lambda_S$ (2.29)

We observe that equation (2.29) is the same as (2.14). Hence, the result (2.29) re-affirms the result (2.14) which we got by phase plane analysis. This method confirms the authenticity of the perturbation method which we shall henceforth use to solve the remaining problems in this report, bearing in mind that the remaining problems cannot be solved using phase plane analysis.

2.4 Dynamic Buckling Of A Pre-Statically Loaded Elastic Structure Trapped By A Step Load

In this case the structure is first of all loaded statically, i.e. no inertial effect for some time and thereafter, it is trapped by a time dependent step load, i.e. $$f(t)=\begin{cases}1, & t>0\\ 0, & t\le 0\end{cases} \qquad (2.30)$$

Here, two loadings are called to play, i.e. static load and dynamic step load. The step load is superposed on the static load and our aim is to determine the dynamic buckling load of the structure in this case. Consequently, two motions are initiated. These are the displacement ξ_0 due to the static load and the displacement $\xi_m(t)$ due to the strictly dynamic load. We pose that at the superposition of the step load on the erstwhile static load the combined displacement, $z(t)$ is

$$z(t)=\xi_0+\xi_m(t) \qquad (2.31)$$

Thus, turning to (2.11), it is apparent that before the superposition of the step load, the equation satisfied at the strictly static loading is

$$(1-\lambda)\xi_0-a{\xi_0}^2=\lambda\bar{\xi} \qquad (2.32)$$

At the superposition of the step load, where $f(t)=1,\ t>0$, the equation equivalent to (2.7), in this case is

$$\ddot{z}+(1-\lambda)z-az^2=2\lambda\bar{\xi}$$
$$z(0)=0,\ \dot{z}(0)=0 \qquad (2.33)$$

Substituting for $z(t)$ from (2.31) we have

$$\ddot{\xi}_m+(1-\lambda)\xi_0+(1-\lambda)\xi_m-a({\xi_0}^2+2\xi_0\xi_m+{\xi_m}^2)=2\lambda\bar{\xi} \qquad (2.34)$$

Therefore, the motion satisfied by the step load alone is (2.34)-(2.32), which gives

$$\left.\begin{aligned}\ddot{\xi}_m+(1-\lambda)\xi_m-a\left(2\xi_0\xi_m+{\xi_m}^2\right)&=\lambda\bar{\xi}\\ \xi_m(0)=0,\quad \dot{\xi}_m&=0\end{aligned}\right\} \qquad (2.35)$$

Equation (2.35) gives the required equation of motion due to the step load after the imposition of the two loads.

2.4.1 Exact Solution of Associated Strictly Step Loading Problem

Now, from eqn. (2.7) with $f(t)=1,$ we obtain the equations satisfied by the structure if subjected to a step load only as

$$\left.\begin{aligned}\ddot{\xi}+(1-\lambda)\xi-a\xi^2&=\lambda\bar{\xi}\\ \xi(0)=0,\quad \dot{\xi}(0)&=0\end{aligned}\right\} \qquad (2.36)$$

To determine the dynamic buckling load of the structure if loaded by a step load alone, we first multiply (2.36) by $\dot{\xi}$ and obtain

$$\dot{\xi}\ddot{\xi}+(1-\lambda)\xi\dot{\xi}-a\xi^2\dot{\xi}=\lambda\bar{\xi}\dot{\xi} \qquad (2.37)$$

By simple integration with respect to t, we obtain

$$\tfrac{1}{2}\dot{\xi}^2+\tfrac{1}{2}(1-\lambda)\xi^2-\tfrac{a}{3}\xi^3=\lambda\bar{\xi}\xi \qquad (2.39)$$

Since at maximum displacement, $\dot{\xi}=0$, equation (2.39) becomes

$$\tfrac{1}{2}(1-\lambda){\xi_a}^2-\tfrac{a}{3}{\xi_a}^3=\lambda\bar{\xi}\xi_a \qquad (2.40)$$

where ξ_a is the maximum displacement.

Thus, we have

$$(1-\lambda)\xi_a-\tfrac{2a}{3}{\xi_a}^2=2\lambda\bar{\xi} \qquad (2.41)$$

Using the condition for dynamic buckling, i.e. $\frac{d\lambda}{d\xi_a}=0$, we have

$$(1-\lambda_D)-\tfrac{4a}{3}\xi_{aD}=0 \qquad (2.42)$$

where ξ_{aD} is the maximum displacement at buckling. Therefore, we have

$$\xi_{aD}=\tfrac{3}{4a}(1-\lambda_D) \qquad (2.43)$$

Thus, if we determine (2.41) at dynamic buckling using (2.43), we get

$$\xi_{aD}\left[(1-\lambda_D)-\tfrac{2a}{3}\xi_{aD}\right]=2\lambda_D\bar{\xi} \qquad (2.44a)$$

i.e. $$\tfrac{3}{4a}(1-\lambda_D)\left[(1-\lambda_D)-\tfrac{2a}{3}\tfrac{3}{4a}(1-\lambda_D)\right]=2\lambda_D\bar{\xi} \qquad (2.44b)$$

and simplifying, we obtain

$$(1-\lambda_D)^2=\tfrac{16a}{3}\lambda_D\bar{\xi} \qquad (2.45)$$

Dividing the corresponding sides of (2.45) by (2.14), we have

$$\left(\frac{1-\lambda_D}{1-\lambda_S}\right)^2=\frac{4}{3}\left(\frac{\lambda_D}{\lambda_S}\right) \qquad (2.46)$$

We observe that:

(i) Equation (2.46) relates the dynamic buckling λ_D to its static equivalent λ_S. Thus if we are given λ_D we can determine λ_S and vice versa. Moreover, this equation provides an easy means of solving equation (2.36).

(ii) So far, the method of determining equations (2.45) and (2.14) are exact as no approximation was made all through.

We shall in the next session use an approximation method that is asymptotic, to determine the two results. In this case the resultant asymptotic results turn out to be asymptotically exact.

2.4.2 The Governing Equation and Solution of (2.35):

In order that our results might equally analyse the strictly step loading situation and the situation whereby the pre-statically loaded structure is struck by a step load, we multiply the term $(2a\xi_0\xi_m)$ in eqn. (2.35) by a constant a constantβ, to obtain

$$\ddot{\xi}_m + (1-\lambda)\xi_m - a\left(2\beta\xi_0\xi_m + {\xi_m}^2\right) = \lambda\bar{\xi} \tag{2.47a}$$

where β=0 or 1, so that when β=0, we get the equation

$$\ddot{\xi} + (1-\lambda)\xi - a\xi^2 = \lambda\bar{\xi}$$

which characterises the strictly step loading case (for $\xi \equiv \xi_m$), but, when β=1, we get

$$\ddot{\xi}_m + (1-\lambda)\xi_m - a\left(2\xi_0\xi_m + {\xi_m}^2\right) = \lambda\bar{\xi}$$

which characterises the equation of motion for the pre-statically loaded structure. We now solve (2.47a), subject to

$$\xi_m(0) = 0, \quad \dot{\xi}_m(0) = 0 \tag{2.47b}$$

(a) **Phase Plane Solution of Equation (2.47a,b):**

We first multiply eqn. (2.47a) by $\dot{\xi}_m$, and get

$$\dot{\xi}_m\ddot{\xi}_m + (1-\lambda)\xi_m\dot{\xi}_m - a\left(2\xi_0\xi_m\dot{\xi}_m\beta + {\xi_m}^2\dot{\xi}_m\right) = \lambda\bar{\xi}\dot{\xi}_m \tag{2.48}$$

This implies

$$\frac{1}{2}\frac{d}{dt}(\dot{\xi}_m)^2 + \frac{1}{2}(1-\lambda)\frac{d}{dt}(\xi_m^2) - a\left\{2\xi_0\frac{\beta}{2}\frac{d}{dt}(\xi_m^2) + \frac{1}{3}\frac{d}{dt}(\xi_m^3)\right\} = \lambda\bar{\xi}\frac{d}{dt}(\xi_m) \tag{2.49}$$

On integrating (2.49), we get

$$\frac{1}{2}(\dot{\xi}_m)^2 + \frac{1}{2}(1-\lambda)\xi_m^2 - a\left\{\xi_0\beta\xi_m^2 + \frac{1}{3}\xi_m^3\right\} = \lambda\bar{\xi}\xi_m \tag{2.50}$$

At buckling, $\dot{\xi}_m = 0$, therefore, from (2.50), we get

$$\frac{1}{2}(1-\lambda)\xi_{m_c}^2 - a\left\{\beta\xi_0\xi_{m_c}^2 + \frac{1}{3}\xi_{m_c}^3\right\} = \lambda\bar{\xi}\xi_{m_c}$$

i.e. $$(1-\lambda)\xi_{m_c} - 2a\left\{\beta\xi_0\xi_{m_c} + \frac{1}{3}\xi_{m_c}^2\right\} = 2\lambda\bar{\xi} \tag{2.51}$$

where, ξ_{m_c} is the value of ξ_m at maximum displacement.

At buckling, we require

$$\frac{d\lambda}{d\xi_{m_c}} = 0 \tag{2.52}$$

$$\therefore (1-\lambda_D) - 2a\left\{\beta\xi_0 + \frac{2}{3}\xi_{m_c}\right\} = 0$$

i.e. $$\xi_{m_c} = \frac{3}{4a}\left[(1-\lambda_D) - 2a\beta\xi_0\right] \tag{2.53}$$

We now substitute (2.53) in (2.51) and get

$$\xi_{m_c}\left[(1-\lambda_D) - 2a\left(\beta\xi_0 + \frac{1}{3}\xi_{m_c}\right)\right] = 2\lambda_D\bar{\xi} \tag{2.54}$$

This finally yields

$$\left[(1-\lambda_D) - 2a\beta\xi_0\right]^2 = \frac{16a\lambda_D\bar{\xi}}{3} \tag{2.55}$$

Also, using equations (2.55) and (2.29) we have

$$\left[\frac{(1-\lambda_D) - 2a\beta\xi_0}{1-\lambda_S}\right]^2 = \frac{4}{3}\left(\frac{\lambda_D}{\lambda_S}\right) \tag{2.56}$$

(b) **Solution of Equation (2.47a,b) Using Perturbation Method:**

Let

$$\hat{t} = (1-\lambda)^{\frac{1}{2}}\left(1 + \mu_1\bar{\xi} + \mu_2\bar{\xi}^2 + \cdots\right)t \tag{2.57}$$

where, μ_i are constants to be determined. Then

$$\dot{\xi}_m = \frac{d\xi_m}{dt} = (1-\lambda)^{\frac{1}{2}}\left(1 + \mu_1\bar{\xi} + \mu_2\bar{\xi}^2 + \cdots\right)\frac{d\xi_m}{d\hat{t}} \tag{2.58a}$$

And

$$\ddot{\xi}_m = \frac{d^2\xi_m}{dt^2} = (1-\lambda)\left(1 + \mu_1\bar{\xi} + \mu_2\bar{\xi}^2 + \cdots\right)^2\frac{d^2\xi_m}{d\hat{t}^2} \tag{2.58b}$$

Substituting (2.58b) into (2.47a), using (2.16), we get

$$(1-\lambda)\left(1 + \mu_1\bar{\xi} + \mu_2\bar{\xi}^2 + \cdots\right)^2\frac{d^2\xi_m}{d\hat{t}^2} + (1-\lambda)\xi_m$$
$$-a\left(2\beta\xi_0\xi_m + {\xi_m}^2\right) = \lambda\bar{\xi} \tag{2.59}$$

i.e. $\left(1+\mu_1\bar{\xi}+\mu_2\bar{\xi}^2+\cdots\right)^2\frac{d^2\xi_m}{d\hat{t}^2}+\xi_m$
$-\frac{a}{1-\lambda}\left[2\beta\xi_m\left(\xi_0^{(1)}\bar{\xi}+\xi_0^{(2)}\bar{\xi}^2+\xi_0^{(3)}\bar{\xi}^3+\cdots\right)+{\xi_m}^2\right]=B\bar{\xi}$ (2.60)
where
$B=\frac{\lambda}{1-\lambda}$. (2.61)
For solution, we let
$\xi_m=\sum_{i-1}^{\infty}\xi_m^{(i)}\bar{\xi}^i$ (2.62)
We substitute (2.62) into (2.60) and get

$$\left(1+\mu_1\bar{\xi}+\mu_2\bar{\xi}^2+\cdots\right)^2\frac{d^2}{d\hat{t}^2}\left(\xi_m^{(1)}\bar{\xi}+\xi_m^{(2)}\bar{\xi}^2+\xi_m^{(3)}\bar{\xi}^3+\cdots\right)$$
$$+\left(\xi_m^{(1)}\bar{\xi}+\xi_m^{(2)}\bar{\xi}^2+\xi_m^{(3)}\bar{\xi}^3+\cdots\right)-\frac{a}{1-\lambda}\left[2\beta\left(\xi_m^{(1)}\bar{\xi}+\xi_m^{(2)}\bar{\xi}^2+\xi_m^{(3)}\bar{\xi}^3+\cdots\right)\right.$$
$$\left.\left(\xi_0^{(1)}\bar{\xi}+\xi_0^{(2)}\bar{\xi}^2+\xi_0^{(3)}\bar{\xi}^3+\cdots\right)+\left(\xi_m^{(1)}\bar{\xi}+\xi_m^{(2)}\bar{\xi}^2+\cdots\right)^2\right]=B\bar{\xi} \quad (2.63)$$

$0<\bar{\xi}\ll 1.$
We now equate coefficient of equations of orders of $\bar{\xi}$ to get
$O(\bar{\xi})$: $\frac{d^2\xi_m^{(1)}}{d\hat{t}^2}+\xi_m^{(1)}=B$ (2.64a)
$O(\bar{\xi}^2)$: $\frac{d^2\xi_m^{(2)}}{d\hat{t}^2}+\xi_m^{(2)}=-2\mu_1\frac{d^2\xi_m^{(1)}}{d\hat{t}^2}+\frac{a}{1-\lambda}\left(2\beta\xi_m^{(1)}\xi_0^{(1)}+{\xi_m^{(1)}}^2\right)$ (2.64b)
$O(\bar{\xi}^3)$: $\frac{d^2\xi_m^{(3)}}{d\hat{t}^2}+\xi_m^{(3)}=-2\mu_2\frac{d^2\xi_m^{(1)}}{d\hat{t}^2}-{\mu_1}^2\frac{d^2\xi_m^{(1)}}{d\hat{t}^2}-2\mu_1\frac{d^2\xi_m^{(1)}}{d\hat{t}^2}$
$+\frac{a}{1-\lambda}\left[2\beta\left(\xi_m^{(1)}\xi_0^{(2)}+2\xi_m^{(2)}\xi_0^{(1)}\right)+2\xi_m^{(1)}\xi_0\right]$ (2.64c)
etc.
The corresponding initial conditions are:
$O(\bar{\xi})$: $\xi_m^{(1)}=0,\qquad \frac{d\xi_m^{(1)}(0)}{d\hat{t}}=0$ (2.65a)
$O(\bar{\xi}^2)$: $\xi_m^{(2)}(0)=0,\qquad \frac{d\xi_m^{(2)}(0)}{d\hat{t}}+\mu_1\frac{d\xi_m^{(1)}(0)}{d\hat{t}}=0$ (2.65b)
$O(\bar{\xi}^3)$: $\xi_m^{(3)}(0)=0,\qquad \frac{d\xi_m^{(3)}(0)}{d\hat{t}}+\mu_2\frac{d\xi_m^{(1)}(0)}{d\hat{t}}+\mu_1\frac{d\xi_m^{(2)}(0)}{d\hat{t}}=0$ (2.65c)
The solution of (2.64a) is
$\xi_m^{(1)}=A_1\cos\hat{t}+B_1\sin\hat{t}+B$ (2.66a)
where A_1andB_1 arearbitrary constants.
Using equations (2.65a) on (2.66a), we get

$$A_1=B,\qquad B_1=0$$

$\therefore\xi_m^{(1)}=B(1-\cos\hat{t})$ (2.66b)
We now substitute in equation (3.58) for $\xi_m^{(1)}$ as well as for $\xi_0^{(1)}$ to get
$\frac{d^2\xi_m^{(2)}}{d\hat{t}^2}+\xi_m^{(2)}=-2\mu_1 B\cos\hat{t}+\frac{2\beta aB^2(1-\cos\hat{t})}{1-\lambda}$
$+\frac{aB^2}{1-\lambda}\left(\frac{3}{2}-2\cos\hat{t}+\frac{1}{2}\cos 2\hat{t}\right)$ (2.67)
To ensure a uniformly valid asymptotic solution in $\hat{t}$, we equate to zero in (2.67) the coefficient of$\cos\hat{t}$ to get
$-2\mu_1 B-\frac{2aB^2\beta}{1-\lambda}-\frac{2aB^2}{1-\lambda}=0$ (2.68a)
This gives
$\mu_1=-\left(\frac{a\beta B}{1-\lambda}+\frac{aB}{1-\lambda}\right)$
$=-B\alpha_1$
where
$\alpha_1=\left(\frac{a\beta}{1-\lambda}+\frac{a}{1-\lambda}\right)$ (2.68b)
Substituting for μ_1 and α_1 in eqn. (2.67), we have
$\frac{d^2\xi_m^{(2)}}{d\hat{t}^2}+\xi_m^{(2)}=2aB^2\beta+\frac{3aB^2}{2(1-\lambda)}+\frac{aB^2\cos 2\hat{t}}{2(1-\lambda)}$ (2.69)
The solution of (2.69) together with (2.65b) is

$$\xi_m^{(2)} = A_2\cos\hat{t} + B_2\sin\hat{t} + a\left(\frac{2\beta\mathrm{B}^2}{1-\lambda} + \frac{3\mathrm{B}^2}{2(1-\lambda)}\right) - \frac{\mathrm{aB}^2\cos 2\hat{t}}{6(1-\lambda)} \tag{2.70}$$

Using the initial conditions (2.65b), we get

$$A_2 = -\left(\frac{2\beta\mathrm{B}^2 a}{1-\lambda} + \frac{3\mathrm{aB}^2}{2(1-\lambda)}\right) + \frac{\mathrm{aB}^2}{6(1-\lambda)}$$

$$= -\left(\frac{4\mathrm{aB}^2}{3(1-\lambda)} + \frac{2\beta\mathrm{B}^2 a}{1-\lambda}\right) = -B^2\alpha_2 \tag{6.71a}$$

where

$$\alpha_2 = \left(\frac{4\mathrm{a}}{3(1-\lambda)} + \frac{2\beta a}{1-\lambda}\right) \tag{2.71b}$$

i.e. $A_2 = -B^2\alpha_2, \quad B_2 = 0$ (2.72)

Thus, we may rewrite equation (2.70) as

$$\xi_m^{(2)} = A_2\cos\hat{t} + \alpha_3 - \alpha_4\cos 2\hat{t} \tag{2.73a}$$

where

$$\alpha_3 = \frac{2\beta\mathrm{B}^2 a}{1-\lambda} + \frac{3\mathrm{aB}^2}{2(1-\lambda)} \tag{2.73b}$$

$$\alpha_4 = \frac{\mathrm{aB}^2}{6(1-\lambda)} \tag{2.73c}$$

In the next substitution into (2.64c), we shall need the following simplifications:

$$\xi_m^{(1)}\xi_0^{(2)} = \frac{\mathrm{aB}^3}{1-\lambda}(1 - cos\hat{t})$$

$$\xi_m^{(2)}\xi_0^{(1)} = B(A_2 cos\hat{t} + \alpha_3 - \alpha_4 cos2\hat{t})$$

$$\xi_m^{(1)}\xi_m^{(2)} = B(1 - cos\hat{t})[A_2 cos\hat{t} + \alpha_3 - \alpha_4 cos2\hat{t}]$$

$$= B[A_2 cos\hat{t} + \alpha_3 - \alpha_4 cos2\hat{t} - \frac{A_2}{2}(1 + cos2\hat{t})$$

$$-\alpha_3 cos\hat{t} - \frac{\alpha_4}{2}(cos3\hat{t} + cos\hat{t})]$$

$$= B[\left(\alpha_3 - \frac{A_2}{2}\right) + \left(A_2 - \alpha_3 - \frac{\alpha_4}{2}\right) cos\hat{t}$$

$$+\left(\alpha_4 + \frac{A_2}{2}\right) cos2\hat{t} - \frac{\alpha_4}{2} cos3\hat{t}]$$

Thus, substituting these equations in (2.64c), we get

$$\frac{d^2}{d\hat{t}^2}\xi_m^{(3)} + \xi_m^{(3)} = -2\mu_2 B cos\hat{t} - \mu_1{}^2 B cos\hat{t} - 2\mu_1(-A_2 cos\hat{t} + \alpha_4 cos2\hat{t})$$

$$+\frac{a}{1-\lambda}[2\beta\left\{\frac{\mathrm{aB}^3(1 - cos\hat{t})}{1-\lambda} + B(A_2 cos\hat{t} + \alpha_3 - \alpha_4 cos2\hat{t})\right\}$$

$$+2B\left\{\left(\alpha_3 - \frac{A_2}{2}\right) + \left(A_2 - \alpha_3 - \frac{\alpha_4}{2}\right) cos\hat{t}\right.$$

$$\left.-\left(\alpha_4 + \frac{A_2}{2}\right) cos2\hat{t} - \frac{\alpha_4}{2} cos3\hat{t}\right\}] \tag{2.74}$$

To ensure a uniformly valid solution in $\hat{t}$, we equate to zero (in equation (2.74)) the coefficient of $cos\hat{t}$ and obtain

$$-2\mu_2 B - \mu_1{}^2 B + 2\mu_1 A_2 + \frac{2a}{1-\lambda}\left[\left(\frac{-\mathrm{aB}^3}{1-\lambda} + A_2 B\right)\beta\right.$$

$$\left.+ 2B\left(A_2 - \alpha_3 - \frac{\alpha_4}{2}\right)\right] = 0 \tag{2.75}$$

This gives

$$\mu_2 = \frac{1}{2B}\left[2\mu_1 A_2 - \mu_1{}^2 B + \frac{2a}{1-\lambda}\left\{\beta\left(A_2 B - \frac{\mathrm{aB}^3}{1-\lambda} + 2B\left\{A_2 - \alpha_3 - \frac{\alpha_4}{2}\right\}\right)\right\}\right] \tag{2.76}$$

If we substitute for A_2 in equation (2.76) from equation (2.72), we get

$\mu_2 = \frac{1}{2B}\left[-2\mu_1 B^2\alpha_2 - \mu_1{}^2 B + \frac{2a}{1-\lambda}\left\{-\beta\left(B^3\alpha_2 + \frac{\mathrm{aB}^3}{1-\lambda}\right) - 2B\left(B^2\alpha_2 + \alpha_3 + \frac{\alpha_4}{2}\right)\right\}\right]$ That is

$$\mu_2 = \left[-2\mu_1 B\alpha_2 - \frac{\mu_1{}^2}{2} + \left(\frac{a}{1-\lambda}\right)\left\{-\beta\left(B^2\alpha_2 + \frac{\mathrm{aB}^2}{1-\lambda}\right) - \left(B^2\alpha_2 + \alpha_3 + \frac{\alpha_4}{2}\right)\right\}\right] \tag{2.73}$$

We can now rewrite (2.74) as

$$\frac{d^2\xi_m^{(3)}}{d\hat{t}^2} + \xi_m^{(3)} = \alpha_5 + \alpha_6 cos2\hat{t} + \alpha_7 cos3\hat{t} \tag{2.75a}$$

where

$$\alpha_5 = \frac{a}{1-\lambda}\left[\beta\left(B\alpha_2 + \frac{2\mathrm{aB}^3}{1-\lambda}\right) + 2B\left(\alpha_3 - \frac{A_2}{2}\right)\right] \tag{2.75b}$$

$\alpha_6 = -\left[2\alpha_4\mu_1 + \frac{3aB\beta\alpha_4}{1-\lambda} + \frac{2aB}{1-\lambda}\left(\alpha_4 + \frac{A_2}{2}\right)\right]$ (2.75c)

$\alpha_5 = \frac{-a\alpha_4 B}{1-\lambda}$ (2.75d)

We now solve equations (2.75a) using (2.65c) and obtain

$\xi_m^{(3)} = A_3 cos\hat{t} + B_3 sin\hat{t} + \alpha_5 - \frac{\alpha_6 cos2\hat{t}}{3} - \frac{\alpha_7 cos3\hat{t}}{8}$ (2.76a) $A_3 = \frac{\alpha_6}{3} - \alpha_5 + \alpha_7, \quad B_3 = 0$ (2.76b)

So far, we have

$\xi_m = \xi_m^{(1)}\bar{\xi} + \xi_m^{(2)}\bar{\xi}^2 + \xi_m^{(3)}\bar{\xi}^3 + \cdots$ (2.77)

Next, we determine the maximum displacement, ξ of ξ_m, and the condition for this is

$\frac{d}{dt}\xi_m = (1 - \lambda)^{1/2}(1 + \mu_1\bar{\xi} + \mu_2\bar{\xi}^2 + \ldots)\frac{d}{d\hat{t}}\xi_m = 0$ (2.78a)

i.e. $\frac{d}{d\hat{t}}\xi_m^{(1)}\bar{\xi} + \frac{d}{d\hat{t}}\xi_m^{(2)}\bar{\xi}^2 + \frac{d}{d\hat{t}}\xi_m^{(3)}\bar{\xi}^3 + \cdots = 0$ (2.78b)

It easily follows, from performing (2.78a,b) that

$sin\hat{t}_a = 0$ (2.79)

where $\hat{t}_a$ is the value of $\hat{t}$ at maximum displacement.

From equation (2.79) we have

$\hat{t}_a = n\pi, \quad n = 0, \pm 1, \pm 2, \pm 3 \ldots$ (2.80)

Since we need the least non trivial value of $\hat{t}_a$, we set $n = +1$, so that

$\hat{t}_a = \pi$ (2.81)

The maximum displacement ξ_{m_a} of ξ_m is obtained by substituting into equation (2.76) and simplifying thereafter. Thus, we get

$\xi_{m_a} = \xi_{m_a}^{(1)}\bar{\xi} + \xi_{m_a}^{(2)}\bar{\xi}^2 + \xi_{m_a}^{(3)}\bar{\xi}^3 + \cdots$ (2.82)

Now, we simplify the terms and obtain the following:

$\xi_{m_a}^{(1)} = 2B$ (2.83)

$$\xi_{m_a}^{(2)} = -A_2 + \alpha_3 - \alpha_4$$

$$= B^2\alpha_2 + \frac{2a\beta B^2}{1-\lambda} + \frac{3aB^2}{2(1-\lambda)} - \frac{aB^2}{6(1-\lambda)}$$

$$= B^2\left(\frac{4a}{3(1-\lambda)} + \frac{3\beta a}{1-\lambda}\right)_2 + \frac{2B^2\beta a}{1-\lambda} + \frac{3aB^2}{2(1-\lambda)} - \frac{aB^2}{6(1-\lambda)}$$

$= \frac{4a\beta B^2}{1-\lambda} + \frac{8aB^2}{3(1-\lambda)}$ (2.84)

$$\xi_{m_a}^{(3)} = -A_3 + \alpha_5 - \frac{\alpha_6}{3} + \alpha_7$$

$$= -\frac{\alpha_6}{3} + \alpha_5 - \alpha_7 + \alpha_5 - \frac{\alpha_6}{3} + \alpha_7$$

$$= 2(\alpha_5 - \frac{\alpha_6}{3})$$

$$= 2\Big[\frac{a}{1-\lambda}\Big\{\beta\Big(B\alpha_2 + \frac{2aB^3}{1-\lambda}\Big) + 2B\Big(\alpha_3 - \frac{A_2}{2}\Big)\Big\}$$
$$+ \frac{1}{3}\Big\{2\alpha_4\mu_1 + \frac{3aB\beta\alpha_4}{1-\lambda} + \frac{2aB}{1-\lambda}\Big(\alpha_4 + \frac{A_2}{2}\Big)\Big\}\Big]$$
$$= 2\Big[\frac{a}{1-\lambda}\Big\{\beta\Big(2\beta B^3 + \frac{2aB^3}{1-\lambda} + \frac{3aB^3}{3(1-\lambda)}\Big) + 2B\Big(2\beta B^2 + \frac{3aB^2}{1-\lambda}$$
$$+ \frac{B}{2}\Big\{\frac{4a}{3(1-\lambda)} + 2\beta\Big\}\Big)\Big\} + \frac{1}{3}\Big\{\frac{-aB^3}{6(1-\lambda)}\Big(\frac{a}{1-\lambda} + \beta\Big) + \frac{2aB\beta}{1-\lambda}\Big(\frac{aB^2}{6(1-\lambda)}\Big)$$
$$+ \frac{2aB}{1-\lambda}\Big\{\frac{aB^2}{6(1-\lambda)} - \frac{B^2}{2}\Big(\frac{4a}{3(1-\lambda)} + 2\beta\Big)\Big\}\Big\}\Big] \qquad (2.85a)$$

Hence, we can write

$\xi_{m_a}^{(3)} = \frac{4a\alpha_8 B^3}{(1-\lambda)^2}$ (2.85b)

Where

$$\alpha_8 = \Big[\Big\{\beta\Big(4\beta + \frac{7a}{4(1-\lambda)}\Big) + \frac{11a}{3(1-\lambda)}\Big\}\frac{1}{3}\Big\{\frac{-a}{12} + \frac{\beta}{12}(a - (1-\lambda))\Big\}$$

$$+\left\{\beta\left(4\beta(1-\lambda)+\frac{7a}{4}\right)+\frac{11a}{3}\right\}\Big] \qquad (2.85c)$$

To determine the dynamic buckling load λ_D, we follows the same process as equations (2.23) to (2.30). Thus, from (2.77), we let

$$\bar{\xi}=e_1\xi_{m_a}+e_2\xi_{m_a}^2+e_3\xi_{m_a}^3+\cdots \qquad (2.86)$$

$$\therefore e_1=\frac{1}{\xi_m^{(1)}}, \qquad (2.87a)$$

$$e_2=\frac{\xi_m^{(2)}}{\left(\xi_m^{(1)}\right)^2}, \qquad (2.87b)$$

$$e_3=\frac{2\left(\xi_m^{(2)}\right)^2-\xi_m^{(1)}\xi_m^{(3)}}{\left(\xi_m^{(1)}\right)^2} \qquad (2.87c)$$

Now, we may limit (2.86) to the first two terms on the right hand side. Thus, we have

$$\bar{\xi}=\frac{e_1}{4e_2}=\frac{\xi_{m_a}^{(1)}}{4\xi_{m_a}^{(2)}} \qquad (2.88)$$

$$=\frac{2B}{4\left[\frac{8aB^2}{3(1-\lambda)}+\frac{4a\beta B^2}{1-\lambda}\right]}\Bigg|_{\lambda=\lambda_D}=\frac{3B(1-\lambda_D)}{16B^2\left[1+\frac{3\beta(1-\lambda_D)}{2}\right]},$$

$$=\frac{3(1-\lambda_D)^2}{16a\lambda_D\left[1+\frac{3\beta}{2}\right]}$$

where $B=\frac{\lambda}{1-\lambda}$.

$$\therefore(1-\lambda_D)^2=\frac{16a\bar{\xi}\lambda_D}{3}\left[1+\frac{3\beta}{2}\right] \qquad (2.89)$$

We notice that when $\beta=0$, the result (2.89) reduces to (2.45) as expected.

If we divide equivalent sides of (2.89) and (2.14), we get

$$\left(\frac{1-\lambda_D}{1-\lambda_S}\right)^2=\frac{4}{3}\left(\frac{\lambda_D}{\lambda_S}\right)\left[1+\frac{3\beta}{2}\right] \qquad (2.90)$$

Hence, it is possible to determine λ_D from the knowledge of λ_S, and vice versa, without the labour of repeating the whole process, once one of them is known.

3.0 Analysis of Results and Discussion

In our investigation, we have analyse the dynamic buckling of a model structure with quadratic nonlinearity struck by a step load superposed on a quasi-static load, using both perturbation and non-perturbation procedures. The results for both methods of solution give similar results and in some cases with insignificant difference.

The use of the perturbation method is possible because of the existence of small mathematically non-dimensional parameters which allow the adoption of asymptotic series expansions of the variable in the mathematical formulations.

With the results obtained, we are also able to analytically relate the dynamic buckling load with its static buckling load equivalent such that if either of them is given the other can be determined without the labour of repeating the process all over.

Following the approach adopted in this investigation, we now present the analysis of the results obtained and the derived relationship between the dynamic buckling loads and the corresponding static buckling load equivalent. With these, we have also established the effect of the pre- static loading on dynamic buckling of the model structure as this is achieved by setting the constant β =0 which characterises the strictly step loading case and β =1 when we have superposed loading. This effect is also depicted in graphically as shown below.

The dynamic buckling load λ_D of the quadratic model structure, obtained from perturbation process is

$$(1-\lambda_D)^2=\frac{16a\bar{\xi}\lambda_D}{3}\left(1+\frac{3\beta}{2}\right) \qquad (3.1)$$

whereas, the equivalent static buckling load λ_S is obtained as

$$(1-\lambda_s)^2=4a\bar{\xi}\lambda_S \qquad (3.2)$$

From the two results above, we get

$$\left(\frac{1-\lambda_D}{1-\lambda_S}\right)^2=\frac{4}{3}\left(\frac{\lambda_D}{\lambda_S}\right)\left(1+\frac{3\beta}{2}\right) \qquad (3.3)$$

The following graphs illustrate the effect the pre-load on the dynamic buckling load λ_D, of the simple quadratic model structure, using the perturbation method . β=0 characterises the strictly step loading case and β=1 when we have a superposed loading.

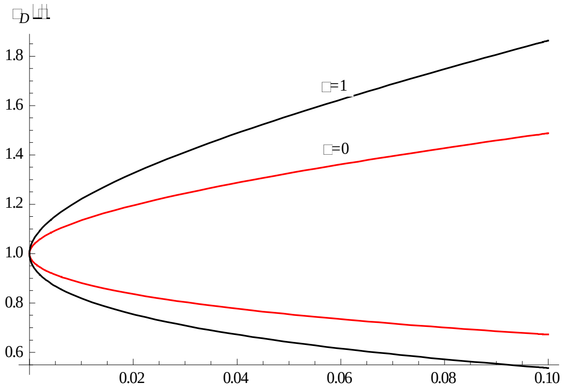

Fig.2: The dynamic buckling load λ_D vs $\bar{\xi}$, with β=0 and β=1, for simple quadratic model Structure using perturbation method

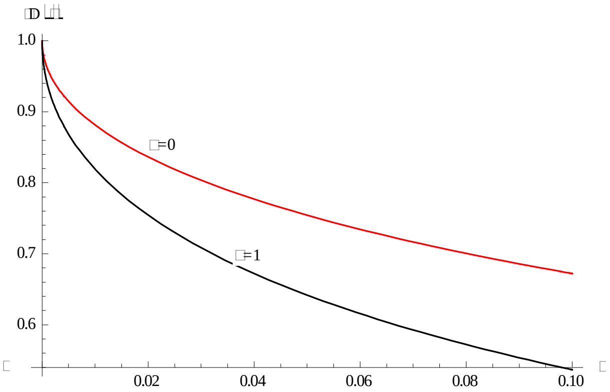

Fig.3: The dynamic buckling load λ_D vs $\bar{\xi}$, $0 < \lambda_D < 1$with β=0 and β=1, for the model structure using perturbationmethod (eqn. 3.1)

REMARK

Fig. (2, 3) show the plot of λ_D against $\bar{\xi}$. We note the parabolic nature of the curves for both β=1 and β=0. We equally note that for $\lambda_D > 1$, the values of λ_D for β=1 are higher than the corresponding values for β=0, while for $0 < \lambda_D < 1$, the values of λ_D for β=1 are less than those for β=0. Mathematically Fig.(2) is valid, but on physical consideration, we may ignore (Fig. 2) and adopt (Fig. 3) because we have by non-dimensionalization, admitted that $0 < \lambda < 1$, which implies $0 < \lambda_D < 1$. We may regard the values of λ_D in the interval, $0 < \lambda_D < 1$ as the lower bounds of the two possible cases, i.e. the cases where $0 < \lambda_D < 1$ and $\lambda_D > 1$.

The phase plane solution (exact solution) of the quadratic model structure superposed on a pre-load is

$$[(1-\lambda_D) - 2a\beta\xi_0]^2 = \frac{16a\bar{\xi}\lambda_D}{3} \tag{3.4}$$

while, the equivalent result without a pre-load is

$$(1-\lambda_D)^2 = \frac{16a\bar{\xi}\lambda_D}{3} \tag{3.5}$$

Using (3.4) and (3.2), we get, in case of pre-load

$$\left[\frac{(1-\lambda_D)-2a\beta\xi_0}{1-\lambda_S}\right]^2 = \frac{4}{3}\left(\frac{\lambda_D}{\lambda_S}\right). \tag{3.6}$$

In the case of no pre-load, using (3.5) and (3.2), we have

$$\left(\frac{1-\lambda_D}{1-\lambda_S}\right)^2 = \frac{4}{3}\left(\frac{\lambda_D}{\lambda_S}\right). \tag{3.7}$$

In particular, we observe from (3.4) that if the structure were perfect, i.e. $\bar{\xi} = 0$, we get

$$\lambda_D = (1 - 2a\beta\xi_0) \tag{3.8}$$

This means that the dynamic buckling load of the structure (in the case where the structure is perfect) and in the pre-loaded case is lower or higher than that of non-pre-loaded case according as to whether the imperfection-sensitivity parameter, a, is positive or negative for the perfect structure. Again, we observe the effect of the pre-static load here, in the sense that while the usual dynamic buckling load for the perfect structure is $\lambda_D = 1$, we now observe the case where the dynamic buckling load λ_D of the perfect structure can be lower than 1.

The following graphs show the dynamic buckling load λ_D against $\bar{\xi}$ when we use the perturbation method (Fig.4) and the phase plane analysis (Fig.5).

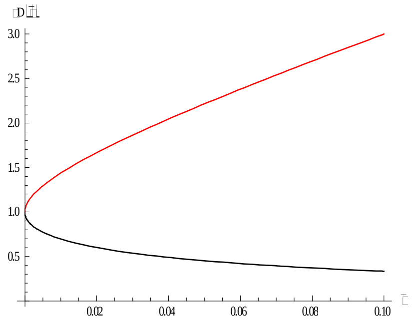

Fig.4: The dynamic buckling load λ_D vs $\bar{\xi}$, of a simple quadratic structure using Perturbation and asymptotic procedure (eqn. 3.1)

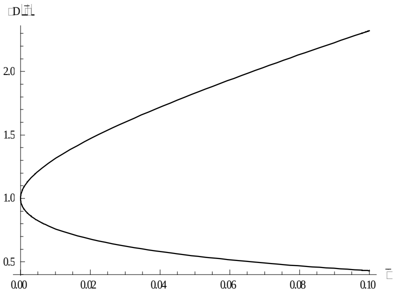

Fig.5: The dynamic buckling load λ_D vs $\bar{\xi}$, of a simple quadratic structure using Phase plane (exact) solution method (eqn. 3.4)

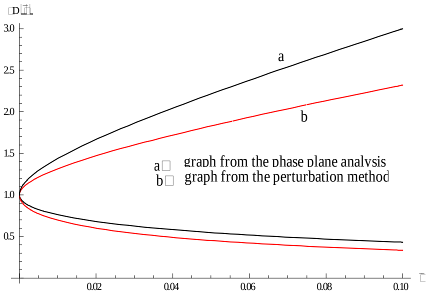

Fig.6: Dynamic buckling load λ_D of the simple quadratic structure for both perturbation and Phase plane (exact) methods.

REMARK:
We observed that without a pre-load (i.e. if $\beta = 0$) the two solution methods give exactly the same result but with a pre-load, the solution for equation (2.47a) using the phase plane analysis is as in equation (2.55), though it differs from equation (2.89) which is obtained from the perturbation method. We, however, note that (2.89) is asymptotic and if $\bar{\xi}$ is small, then the disparity is insignificantly very small.

4.0 Conclusion

With these results we have able to establish analytically the static buckling load of a model structure with quadratic nonlinearity and the dynamic buckling load induced when the pre-static structure is struck by a step load superposed on it.
We then derived a very useful formula which enable us to determine the dynamic buckling load from the knowledge of the static buckling load and vice versa. The results also agree with some empirical estimates and is depicted in the graph above.

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