Some Properties of a Class of Salagean-Type *P*-valent Functions

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Abstract

In (2008)G$\ddot{u}$ neyintroduced and investigated the subclass $T_p^*(g(z),\alpha,\beta,\gamma,A,B,\lambda)$ consisting of analytic and *p*-valent functions with negative coefficients using a differential operator $D^\lambda = p^\lambda z^p - \sum_{k=1}^{\infty}(p+k)^\lambda a_{p+k} z^{p+k}$ andderived the coefficient inequalities, distortion theorem and extreme points for $T_p^*(g(z),\alpha,\beta,\gamma,A,B,\lambda)$. The purpose of present work is to derive some interesting properties other than those obtained by G$\ddot{u}$ ney for the subclass. In particular, we derive quasi-Hadamard product (quasi-convolution) property, inclusion theorem, radius of close-to-convexity, star-likeness and convexity properties for $T_p^*(g(z),\alpha,\beta,\gamma,A,B,\lambda)$.

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1.0 INTRODUCTION

A function $f(z)$ merophorphic in a domain D is said to be p-valent in D (or multivalent of order p in D) if for each w_0 (infinity included) the equation $f(z) = w_0$ has at most p roots in D (where the roots are counted in accordance with their multiplicity) and if there is some w_1 such that the equation $f(z) = w_1$ has exactly p roots in D [1].

The class of p-valent functions has been widely studied. In fact, Aouf [2] and Hossen [3] studied the subclasses of p-valent functions of order α and type β. In recent years, several interesting classes (and or subclasses) of p-valent analytic functions have been introduced and investigated [4 5 6 7 8 9 10 11 12].

Let A(p) denote the class of functions defined by

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}, \qquad p \in \aleph \tag{1.1}$$

which are analytic and p-valent in the open unit disk $U = \{z \in \mathbf{C}; |z| < 1\}$.

Let T_p^* denote the subclass of $\mathbf{A}(p)$ whose members have the form given by

$$f(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad , \quad a_{p+k} \geq 0,\ p \in \Re. \tag{1.2}$$

It is well known that the Hadamard Product (or Convolution) of two analytic functions, $f(z)$ defined by (1.1) and $g(z)$ given by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k} \quad , \quad b_{p+k} \geq 0,\ p \in \Re. \tag{1.3}$$

is denoted by $(p * g)(z)$ and defined by

$$(p * g)(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k} \quad , \quad p \in \Re.. \tag{1.4}$$

Similarly, the modified Hadamard product of the function $f(z)$ defined by (1.2) and $g(z) \in T_p^*$ given by

$$g(z) = z^p - \sum_{k=1}^{\infty} b_{p+k} z^{p+k} \quad , \quad b_{p+k} \geq 0,\ p \in \Re. \tag{1.5}$$

is denoted by denoted by $(p * g)(z)$ and defined by

$$(p * g)(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k} \quad , \quad p \in \Re.. \tag{1.6}$$

This definition follows from the similar definition by Mugur Acu [13].

In [12], Gü ney introduced the differential operator $D^\lambda f(z)$: $D^\lambda f(z) = D\left(D^{\lambda-1} f(z)\right) = p^\lambda z^p - \sum_{k=1}^{\infty} (p+k)^\lambda a_{p+k} z^{p+k}.$

By making use of this differential operator, hedefined the class $T_p^*(g(z), \alpha, \beta, \gamma, A, B, \lambda)$:

$$T_p^*(g(z), \alpha, \beta, \gamma, A, B, \lambda) := \left\{ f \in T_p^* : \left| \frac{z(D^\lambda f(z))' - pD^\lambda g(z)}{(A-B)(p-\beta)D^\lambda g(z) - B[z(D^\lambda f(z))' - pD^\lambda g(z)]} \right| < \gamma, \quad z \in U, \right\} \tag{1.7}$$

for $g(z) \in T_p^*$, $\quad -1 \leq B < A \leq 1,\ -1 \leq B < 0,\ 0 \leq \alpha < p,\ 0 \leq \beta < p,$ and $0 < \gamma < 1$.

For the class $T_p^*(g(z), \alpha, \beta, \gamma, A, B, \lambda)$, properties such as coefficient inequalities, distortion theorem and extreme points were discussed. In the present work, we derive other properties of the class $T_p^*(g(z), \alpha, \beta, \gamma, A, B, \lambda)$. .

In the present work, we derive other properties of the class $T_p^*(g(z), \alpha, \beta, \gamma, A, B, \lambda)$. which are modified Hadamard product property, inclusion theorem, radius of close-to-convexity, star-likeness and convexity properties.

2. PRELIMINARIES

Lemma 2.1[6]: Let the function $g(z)$ be as defined by (1.2). Then $g(z) \in T_p^*(\alpha, \lambda)$ if and only

$$\sum_{k=1}^{\infty} (p+k)^\lambda (p+k-\alpha) b_{p+k} \leq p^\lambda (p-\alpha). \tag{2.1}$$

Now we prove the following Lemma which gives a sufficient condition for functions $f(z)$ defined by (1.2) to be in the class $T_p^*(g(z), \alpha, \beta, \gamma, A, B, \lambda)$. .

Lemma 2.2: Let the function $f(z)$ defined by (1.2) be in the class $T_p^*(g(z), \alpha, \beta, \gamma, A, B, \lambda)$. Then

$$\sum_{k=1}^{\infty} (1-\gamma B)(p+k)^{1+\lambda} a_{p+k} + [(\gamma B - 1)p + (A - B(p-\beta)\gamma)] \frac{p^\lambda (p-\alpha)}{p+k-\alpha} \leq (A-B)\gamma p^\lambda (p-\beta). \tag{2.2}$$

Proof. Since $f \in \mathrm{T}_{\mathrm{p}}^{*}(\mathrm{g(z)}, \alpha, \beta, \gamma, \mathrm{A}, \mathrm{B}, \lambda)$, there exists a function $g(z) \in T_p^*(\alpha, \lambda)$ such that

$$\left| \frac{z(D^{\lambda} f(z))' - pD^{\lambda} g(z)}{(A-B)(p-\beta)D^{\lambda} g(z) - B[z(D^{\lambda} f(z))' - pD^{\lambda} g(z)]} \right| < \gamma, \qquad z \in U.$$

Then, we have

$$\left| \frac{z(D^{\lambda} f(z))' - pD^{\lambda} g(z)}{(A-B)(p-\beta)D^{\lambda} g(z) - B[z(D^{\lambda} f(z))' - pD^{\lambda} g(z)]} \right|$$

$$= \left| \frac{\sum_{k=1}^{\infty} (p+k)^{1+\lambda} a_{p+k} z^{p+k} - \sum_{k=1}^{\infty} p(p+k)^{\lambda} b_{p+k} z^{p+k}}{(A-B)(p-\beta)p^{\lambda} z^{p} + \sum_{k=1}^{\infty} B(p+k)^{1+\lambda} a_{p+k} z^{p+k} - \sum_{k=1}^{\infty} Bp(p+k)^{\lambda} b_{p+k} z^{p+k} - \sum_{k=1}^{\infty} (A-B)(p-\beta)(p+k)^{\lambda} b_{p+k} z^{p+k}} \right| < \gamma$$

Since $\mathrm{Re}(z) \le |z| \ \forall \ z$, we obtain

$$\mathrm{Re}\left\{ \frac{\sum_{k=1}^{\infty} (p+k)^{1+\lambda} a_{p+k} z^{p+k} - \sum_{k=1}^{\infty} p(p+k)^{\lambda} b_{p+k} z^{p+k}}{(A-B)(p-\beta)p^{\lambda} z^{p} + \sum_{k=1}^{\infty} B(p+k)^{1+\lambda} a_{p+k} z^{p+k} - \sum_{k=1}^{\infty} Bp(p+k)^{\lambda} b_{p+k} z^{p+k} - \sum_{k=1}^{\infty} (A-B)(p-\beta)(p+k)^{\lambda} b_{p+k} z^{p+k}} \right\} < \gamma. \tag{2.3}$$

We choose values of z in the real axis so that $\frac{z(D^{\lambda} f(z))'}{D^{\lambda} g(z)}$ is real.

Then upon simplifying the denominator in (2.3) and letting $z \to 1^{-}$ through real values, we obtain

$$\sum_{k=1}^{\infty} (p+k)^{1+\lambda} a_{p+k} - \sum_{k=1}^{\infty} p(p+k)^{\lambda} b_{p+k} \le (A-B)\gamma p^{\lambda}(p-\beta) + \sum_{k=1}^{\infty} B\gamma (p+k)^{1+\lambda} a_{p+k} - \left\{ \sum_{k=1}^{\infty} [\gamma Bp(p+k)^{\lambda} + \gamma(A-B)(p-\beta)(p+k)^{\lambda}] b_{p+k} \right\}$$

i.e. $$\sum_{k=1}^{\infty} (1-\gamma B)(p+k)^{1+\lambda} a_{p+k} + [(\gamma B - 1)p + (A-B)(p-\beta)\gamma](p+k)^{\lambda} b_{p+k} \le (A-B)\gamma p^{\lambda}(p-\beta). \tag{2.4}$$

But from Lemma 2.1, $T_p^*(\alpha, \beta)$ implies

$$\sum_{k=1}^{\infty} (p+k)^{\lambda} b_{p+k} \le \frac{p^{\lambda}(p-\alpha)}{p+k-\alpha} \tag{2.5}$$

So that by (2.4) and (2.5) we have

$$\sum_{k=1}^{\infty} (1-\gamma B)(p+k)^{1+\lambda} a_{p+k} + [(\gamma B - 1)p + (A - B(p-\beta)\gamma)] \frac{p^{\lambda}(p-\alpha)}{p+k-\alpha} \le (A-B)\gamma p^{\lambda}(p-\beta). \tag{2.6}$$

End of proof.

The result is sharp, the extremal function being

$$f(z) = z^{p} - \frac{(A-B)\gamma pk(p-\beta) - p^{1+\lambda}(\gamma B - 1)(p-\alpha)}{(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha)} z^{p+k}, \qquad p, k \in \aleph \tag{2.7}$$

for $k \ge 1$, with respect to

$$g(z) = z^{p} - \frac{p^{\lambda}(p-\alpha)}{(p+k)^{\lambda}(p+k-\alpha)} z^{p+k}.$$

Corollary 2.3. Let the function defined by (1.2) be in the class $T_p^*(g(z), \alpha, \beta, \gamma, A, B, \lambda)$. . Then

$$a_{p+k} \le \frac{(A-B)\gamma pk(p-\beta) + p^{1+\lambda}(1-\gamma B)(p-\alpha)}{(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha)}, \qquad \text{for } p, k \in \aleph.$$

Remark 2. (i) The result obtained in**Lemma 2.2** correct the result obtained by Güney (**Theorem 2.2**of [12]),
(ii) The result obtained in**corollary 2.3** also correct the result obtained by Güney (Corollary **2.3**of[12]).

3.0 Main results

In this section we provide the main results.

Our next result is the Quasi-Hadamard (Modified Hadamard product) for the class $T_p^*(g(z),\alpha,\beta,\gamma,A,B,\lambda)$..

3.1 Modified Hadamard Product

Let the functions $f_i(z)$ $(i=1,2)$ be defined by

$$f_i(z)=z^p-\sum_{k=1}^{\infty}a_{p+k,i}\,z^{p+k},\qquad p\in\aleph. \tag{2.8}$$

Following the definition (1.5), the modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1*f_2)(z)=z^p-\sum_{k=1}^{\infty}a_{p+k,1}a_{p+k,2}\,z^{p+k}.$$

Theorem 3.1. Let the functions $f_i(z)$ $(i=1,2)$ defined by (2.8) be in the class $T_p^*(g(z),\alpha,\beta,\gamma,A,B,\lambda)$.. Then $(f_1*f_2)(z)\in$ $T_p^*(g(z),\delta,\beta,\gamma,A,B,\lambda)$., where

$$\delta=\frac{\begin{array}{c}(A-B)\gamma p^{\lambda}(p-\beta)-(\gamma B-1)p^{\lambda+2}[(1-\gamma B)(p+1)^{1+\lambda}(1+p-\alpha_1)][(1-\gamma B)(p+1)^{1+\lambda}(1+p-\alpha_2)]\\ -(1-\gamma B)(p+1)^{\lambda+2}[(A-B)\gamma p^{\lambda}(p-\beta)-(\gamma B-1)p^{\lambda+1}(p-\alpha_1)][(A-B)\gamma p^{\lambda}(p-\beta)-(\gamma B-1)p^{\lambda+1}(p-\alpha_2)]\end{array}}{\begin{array}{c}(\gamma B-1)(p+1)^{1+\lambda}[(A-B)\gamma p^{\lambda}(p-\beta)-(\gamma-1)p^{1+\lambda}(p-\alpha_1)][(A-B)\gamma p^{\lambda}(p-\beta)-(\gamma B-1)p^{1+\lambda}(p-\alpha_2)]\\ -(\gamma B-1)p^{1+\lambda}[(1-\gamma B)(p+1)^{1+\lambda}(1+p-\alpha_1)][(1-\gamma B)(p+1)^{1+\lambda}(1+p-\alpha_2)]\end{array}} \tag{2.9}$$

The result is sharp for the functions $f_i(z)$ $(i=1,2)$ given by

$$f_i(z)\;=z^p-\frac{(A-B)\gamma p^{\lambda}(p-\beta)-(\gamma B-1)p^{1+\lambda}(p-\alpha_i)}{(1-\gamma B)(p+1)^{1+\lambda}(1+p-\alpha_i)}z^{p+1},\qquad p\in\aleph;\;\; i=1,2. \tag{2.10}$$

Proof. Let $f_i(z)\in$ $T_p^*(g(z),\alpha,\beta,\gamma,A,B,\lambda)$ $(i=1,2)$, we need to find the largest δ such that

$$\sum_{k=1}^{\infty}\frac{(1-\gamma B)(p+k)^{1+\lambda}(p+k-\delta)}{(A-B)\gamma p^{\lambda}k(p-\beta)-(\gamma B-1)p^{1+\lambda}(p-\delta)}a_{p+k,1}a_{p+k,2}\le 1$$

Since $f_i(z)\in$ $T_p^*(g(z),\alpha,\beta,\gamma,A,B,\lambda)$ $(i=1,2)$, in view of (2.2) we have

$$\sum_{k=1}^{\infty}\frac{(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha_i)}{(A-B)\gamma p^{\lambda}k(p-\beta)-(\gamma B-1)p^{1+\lambda}(p-\alpha_i)}a_{p+k,i}\le 1\qquad (i=1,2).$$

Furthermore, by the Cauchy-Schwarz inequalities, we get

$$\sum_{k=1}^{\infty}\frac{\sqrt{[(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha_1][(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha_2]}}{\sqrt{[(A-B)\gamma p^{\lambda}k(p-\beta)-(\gamma B-1)p^{1+\lambda}(p-\alpha_1)][(A-B)\gamma p^{\lambda}k(p-\beta)-(\gamma B-1)p^{1+\lambda}(p-\alpha_2)]}}\sqrt{a_{p+k,1}a_{p+k,2}}\le 1.$$

Thus it is sufficient to show that

$$\frac{(1-\gamma B)(p+k)^{1+\lambda}(p+k-\delta)}{(A-B)\gamma p^{\lambda}k(p-\beta)-(\gamma B-1)p^{1+\lambda}(p-\delta)}a_{p+k,1}a_{p+k,2}$$

$$\le\frac{\sqrt{[(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha_1][(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha_2]}}{\sqrt{[(A-B)\gamma p^{\lambda}k(p-\beta)-(\gamma B-1)p^{1+\lambda}(p-\alpha_1)][(A-B)\gamma p^{\lambda}k(p-\beta)-(\gamma B-1)p^{1+\lambda}(p-\alpha_2)]}}\sqrt{a_{p+k,1}a_{p+k,2}}\qquad (k\in\aleph),$$

Or equivalently,

$$\sqrt{a_{p+k,1}a_{p+k,2}}$$

$$\le\frac{(A-B)\gamma p^{\lambda}k(p-\beta)-(\gamma B-1)p^{1+\lambda}(p-\delta)\sqrt{[(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha_1][(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha_2]}}{(1-\gamma B)(p+k)^{1+\lambda}(p+k-\delta)\sqrt{[(A-B)\gamma p^{\lambda}k(p-\beta)-(\gamma B-1)p^{1+\lambda}(p-\alpha_1)][(A-B)\gamma p^{\lambda}k(p-\beta)-(\gamma B-1)p^{1+\lambda}(p-\alpha_2)]}}\qquad (k\in\aleph).$$

By noting that,

$$\sqrt{a_{p+k,1}a_{p+k,2}}\le\frac{\sqrt{[(A-B)\gamma p^{\lambda}k(p-\beta)-(\gamma B-1)p^{1+\lambda}(p-\alpha_1)][(A-B)\gamma p^{\lambda}k(p-\beta)-(\gamma B-1)p^{1+\lambda}(p-\alpha_2)]}}{[(1-\gamma B)(p+k)^{1+\lambda}(1+p-\alpha_1)][(1-\gamma B)(p+k)^{1+\lambda}(1+p-\alpha_2)]}\qquad (k\in\aleph)$$

Consequently, we need only to prove that

$$\frac{(A-B)\gamma p^{\lambda}k(p-\beta)-(\gamma B-1)p^{1+\lambda}(p-\delta)\sqrt{[(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha_1][(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha_2]}}{(1-\gamma B)(p+k)^{1+\lambda}(p+k-\delta)\sqrt{[(A-B)\gamma p^{\lambda}k(p-\beta)-(\gamma B-1)p^{1+\lambda}(p-\alpha_1)][(A-B)\gamma p^{\lambda}k(p-\beta)-(\gamma B-1)p^{1+\lambda}(p-\alpha_2)]}}$$

$$\leq \frac{(A-B)\gamma p^{\lambda}k(p-\beta)-(\gamma B-1)p^{1+\lambda}(p-\delta)}{(1-\gamma B)(p+k)^{1+\lambda}(p+k-\delta)} \qquad (k \in \aleph),$$

which is equivalent to

$$\begin{aligned}&(1-\gamma B)(p+k)^{1+\lambda}(p+k-\delta)[(A-B)\gamma p^{\lambda}k(p-\beta)-(\gamma B-1)p^{1+\lambda}(p-\alpha_1)][(A-B)\gamma p^{\lambda}k(p-\beta)-(\gamma B-1)p^{1+\lambda}(p-\alpha_2)]\\ &\leq (A-B)\gamma p^{\lambda}k(p-\beta)-(\gamma B-1)p^{1+\lambda}(p-\delta)[(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha_1)][(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha_2)]\end{aligned}$$

Such that

$$\begin{aligned}&\delta\{(1-\gamma B)\gamma p^{\lambda}k(p+k)^{1+\lambda}[(A-B)\gamma p^{\lambda}k(p-\beta)-(\gamma B-1)p^{1+\lambda}(p-\alpha_1)][(A-B)\gamma p^{\lambda}k(p-\beta)-(\gamma B-1)p^{1+\lambda}(p-\alpha_2)]\\ &\qquad -(1-\gamma B)p^{1+\lambda}[(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha_1)][(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha_2)]\}\\ &\leq (A-B)\gamma p^{\lambda}k(p-\beta)-(\gamma B-1)p^{\lambda+2}[(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha_1)][(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha_2)]\\ &-(1-\gamma B)(p+k)^{\lambda+2}[(A-B)\gamma p^{\lambda}k(p-\beta)-(\gamma B-1)p^{1+\lambda}(p-\alpha_1)][(A-B)\gamma p^{\lambda}k(p-\beta)-(\gamma B-1)p^{1+\lambda}(p-\alpha_2)]\end{aligned}$$

i.e.

$$\delta=\frac{\begin{array}{c}(A-B)\gamma p^{\lambda}k(p-\beta)-(\gamma B-1)p^{\lambda+2}[(1-\gamma B)(p+k)^{1+\lambda}(k+p-\alpha_1)][(1-\gamma B)(p+k)^{1+\lambda}(k+p-\alpha_2)]\\ -(1-\gamma B)(p+1)^{\lambda+2}[(A-B)\gamma p^{\lambda}k(p-\beta)-(\gamma B-1)p^{\lambda+1}(p-\alpha_1)][(A-B)\gamma p^{\lambda}k(p-\beta)-(\gamma B-1)p^{\lambda+1}(p-\alpha_2)]\end{array}}{\begin{array}{c}(\gamma B-1)(p+k)^{1+\lambda}[(A-B)\gamma p^{\lambda}k(p-\beta)-(\gamma-1)p^{1+\lambda}(p-\alpha_1)][(A-B)\gamma p^{\lambda}(p-\beta)-(\gamma B-1)p^{1+\lambda}(p-\alpha_2)]\\ -(\gamma B-1)p^{1+\lambda}[(1-\gamma B)(p+k)^{1+\lambda}(k+p-\alpha_1)][(1-\gamma B)(p+k)^{1+\lambda}(k+p-\alpha_2)]\end{array}}=E(k) \quad (k\in\aleph) \qquad (2.11)$$

Since $E(k)$ is an increasing function of k $(k \in \aleph)$, letting $k=1,$ we obtain

$$\delta=\frac{\begin{array}{c}(A-B)\gamma p^{\lambda}(p-\beta)-(\gamma B-1)p^{\lambda+2}[(1-\gamma B)(p+1)^{1+\lambda}(1+p-\alpha_1)][(1-\gamma B)(p+1)^{1+\lambda}(1+p-\alpha_2)]\\ -(1-\gamma B)(p+1)^{\lambda+2}[(A-B)\gamma p^{\lambda}(p-\beta)-(\gamma B-1)p^{\lambda+1}(p-\alpha_1)][(A-B)\gamma p^{\lambda}(p-\beta)-(\gamma B-1)p^{\lambda+1}(p-\alpha_2)]\end{array}}{\begin{array}{c}(\gamma B-1)(p+1)^{1+\lambda}[(A-B)\gamma p^{\lambda}(p-\beta)-(\gamma-1)p^{1+\lambda}(p-\alpha_1)][(A-B)\gamma p^{\lambda}(p-\beta)-(\gamma B-1)p^{1+\lambda}(p-\alpha_2)]\\ -(\gamma B-1)p^{1+\lambda}[(1-\gamma B)(p+1)^{1+\lambda}(1+p-\alpha_1)][(1-\gamma B)(p+1)^{1+\lambda}(1+p-\alpha_2)]\end{array}}.$$

which is (2.9) that is required.

Corollary 3.2. For $f_i(z)$ $(i=1,2)$ as in Theorem 3.1, we have

$$h(z)=z^p-\sum_{k=1}^{\infty}\sqrt{a_{p+k,1}\,a_{p+k,2}}\;z^{p+k}$$

belongs to the class $T_p^*(g(z),\alpha,\beta,\gamma,A,B,\lambda)$ $(i=1,2)$. The result is sharp with the function given by (2.10).

Proof. The result follows from the Cauchy-Schwarz Inequality.

3.2 INCLUSION PROPERTIES

Theorem 3.2. Let the functions $f_i(z)$ $(i=1,2)$ defined by (2.8) be in the class $T_p^*(g(z),\alpha,\beta,\gamma,A,B,\lambda)$ $(i=1,2)$. Then the function

$F(h)=z^p-\sum_{k=1}^{\infty}\left(a_{p+k,1}^2+a_{p+k}^2\right)z^{p+k}$ belongs to $T_p^*(g(z),\delta,\beta,\gamma,A,B,\lambda)$ $(i=1,2)$, where

$$\delta=\frac{(1-\gamma B)(p+1)^2(p+1-\alpha)^2[(A-B)\gamma p(p-\alpha)+(1-\gamma B)p^{1+\lambda}]-2(p+1)[(A-B)(p-\alpha)\gamma p+p^{1+\lambda}(1-\gamma B)(p-\alpha)]^2}{p^{1+\lambda}(1-\gamma B)^2(1+p-\alpha)^2-2[(A-B)\gamma p(p-\alpha)+p^{1+\lambda}(1-\gamma B)(p-\alpha)]^2}$$

The result is sharp for the function $f_i(z)$ $(i=1,2)$ defined by (2.10).

Proof. By virtue of Lemma 2.2, we obtain

$$\sum_{k=1}^{\infty}\left\{\frac{(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha)}{(A-B)(p-\alpha)\gamma pk-p^{1+\lambda}(\gamma B-1)(p-\alpha)}\right\}^2 a_{p+k,1}^2 \leq \left\{\sum_{k=1}^{\infty}\frac{(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha)}{(A-B)(p-\alpha)\gamma pk-p^{1+\lambda}(\gamma B-1)(p-\alpha)}a_{p+k,1}\right\}^2 \leq 1 \qquad (2.12)$$

and

$$\sum_{k=1}^{\infty}\left\{\frac{(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha)}{(A-B)(p-\alpha)\gamma pk-p^{1+\lambda}(\gamma B-1)(p-\alpha)}\right\}^2 a_{p+k,2}^2 \leq \left\{\sum_{k=1}^{\infty}\frac{(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha)}{(A-B)(p-\alpha)\gamma pk-p^{1+\lambda}(\gamma B-1)(p-\alpha)}a_{p+k,2}\right\}^2 \leq 1. \qquad (2.13)$$

It follows from (2.12) and (2.13) that

$$\sum_{k=1}^{\infty}\frac{1}{2}\left\{\frac{(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha)}{(A-B)(p-\alpha)\gamma pk-p^{1+\lambda}(\gamma B-1)(p-\alpha)}\right\}^2(a_{p+k,1}^2+a_{p+k,2}^2)\le 1.$$

Therefore, we need to find the largest δ such that

$$\frac{(1-\gamma B)(p+k)^{1+\lambda}(p+k-\delta)}{(A-B)(p-\alpha)\gamma pk-p^{1+\lambda}(\gamma B-1)(p-\delta)}\le\frac{1}{2}\left\{\frac{(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha)}{(A-B)(p-\alpha)\gamma pk-p^{1+\lambda}(\gamma B-1)(p-\alpha)}\right\}^2, \qquad k\ge 1 \tag{2.14}$$

A simple computation yields that

$$\delta\le\frac{(1-\gamma B)(p+k)^2(p+k-\alpha)^2[(A-B)\gamma pk(p-\alpha)+(1-\gamma B)p^{1+\lambda}]-2(p+k)[(A-B)(p-\alpha)\gamma pk+p^{1+\lambda}(1-\gamma B)(p-\alpha)]^2}{p^{1+\lambda}(1-\gamma B)^2(p+k)^2(p+k-\alpha)^2-2[(A-B)(p-\alpha)\gamma pk+p^{1+\lambda}(1-\gamma B)(p-\alpha)]^2}$$

$= G(k),\ k\in\aleph$

Since $G(k)$, is an increasing function of $k(k\ge 1)$ for the parameters involve and $p\in\aleph.$ letting $k=1,$ we have

$$\delta=\frac{(1-\gamma B)(p+1)^2(p+1-\alpha)^2[(A-B)\gamma p(p-\alpha)+(1-\gamma B)p^{1+\lambda}]-2(p+1)[(A-B)(p-\alpha)\gamma p+p^{1+\lambda}(1-\gamma B)(p-\alpha)]^2}{p^{1+\lambda}(1-\gamma B)^2(1+p-\alpha)^2-2[(A-B)\gamma p(p-\alpha)+p^{1+\lambda}(1-\gamma B)(p-\alpha)]^2},$$

which completes the proof of the theorem.

3.3 RADII OF CLOSE-TO-CONVEX, STARLIKENESS AND CONVEXITY

Theorem 3.3. Let the function $f(z)$ defined by (1.2) be in the class $T_p^*\big(g(z),\alpha,\beta,\gamma,A,B,\lambda\big)$ $(i=1,2)$.
Then $f(z)$ is P-*valently* close-to-convex of order $\rho(0\le\rho<p)$ in $|z|<r_1$ where

$$r_1=\inf_{k\in\aleph}\left\{\frac{(p-\delta)(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha)}{(p+k)[(A-B)\gamma pk(p-\beta)+p^{1+\lambda}(1-\gamma B)(p-\alpha)]}\right\}^{\frac{1}{k}}.$$

The result is sharp with the extremal function given by (2.10).

Proof. It suffices to show that

$\left|\dfrac{f'(z)}{z^{p-1}}-p\right|\le p-\rho, \quad |z|<r_1$. In deed we have

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|\le\sum_{k=1}^{\infty}(p+k)a_{p+k}|z|^k \tag{2.15}$$

Hence (2.15) is true if

$\displaystyle\sum_{k=1}^{\infty}(p+k)a_{p+k}|z|^k \le p-\rho$ or

$$\sum_{k=1}^{\infty}\frac{(p+k)}{(p-\rho)}a_{p+k}|z|^k \le 1 \tag{2.16}$$

By the Theorem, (2.15) is true if

$$\frac{(p-\delta)}{p-\rho}|z|^k\le\frac{(p-\delta)(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha)}{(p+k)[(A-B)\gamma pk(p-\beta)+p^{1+\lambda}(1-\gamma B)(p-\alpha)]}, \qquad k\ge 1 \tag{2.17}$$

Solving (2.17) for $|z|$, we get the desired result. That is,

$$r_1=\inf_{k\in\aleph}\left\{\frac{(p-\delta)(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha)}{(p+k)[(A-B)\gamma pk(p-\beta)+p^{1+\lambda}(1-\gamma B)(p-\alpha)]}\right\}^{\frac{1}{k}}.$$

Theorem 3.4. Let the function $f(z)$ defined by (1.2) be in the class $T_p^*\big(g(z),\alpha,\beta,\gamma,A,B,\lambda\big)$ $(i=1,2)$.
Then $f(z)$ is P-*valently* starlike of order $\rho(0\le\rho<p)$ in $|z|<r_2$ where

$$r_2=\inf_{k\in\aleph}\left\{\frac{(p-\delta)(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha)}{(p+k-\rho)[(A-B)\gamma pk(p-\beta)+p^{1+\lambda}(1-\gamma B)(p-\alpha)]}\right\}^{\frac{1}{k}}.$$

The result is sharp with the extremal function given by (2.10).

Theorem 3.5. Let the function $f(z)$ defined by (1.2) be in the class $T_p^*(g(z),\alpha,\beta,\gamma,A,B,\lambda)$ $(i=1,2)$.
Then $f(z)$ is P-*valently* convex of order $\rho(0 \le \rho < p)$ in $|z| < r_3$ where

$$r_3 = \inf_{k\in\aleph}\left\{\frac{p(p-\rho)(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha)}{(p+k)(p+k-\rho)[(A-B)\gamma pk(p-\beta)+p^{1+\lambda}(1-\gamma B)(p-\alpha)]}\right\}^{\frac{1}{k}}.$$

The result is sharp with the extremal function given by (2.10).

References

[1] Goodman A. W., *Univalent Function,* Mariner Publishing Company, Inc. USA, 1983.

[2] Aouf, M.K., “A generalization of multivalent functions with negative coefficients,” J. *Korean Math. Soc.* 25(1), 53-66, 1988.

[3] Hossen, H. M., “Quasi-Hadamard product of certain p-valent functions, *Demonstration Math.* 33(2), 277-281, 2000.

[4] Khairnar, S. M. and More, N. H., “On a subclass of certain p-valent starlike functions with negative coefficients,” *General Mathematics,* vol. 18, No. 9, 95-119, 2010.

[5] Sharma, P. and Srivastava, P., “Some properties of a class of p-valent analytic functions associated with convolution,” *European Journal of Pure and Applied Mathematics,* vol. 3, No. 6, 1093-1112, 2010.

[6] Amsheri, S. M., Zharkova, V., “Subclasses of p-valent starlike functions defined by using certain fractional derivative operator,” *International Journal of Mathematical Science Education,* vol. 4, No. 1, 17-32, 2011.

[7] Amsheri, S. M., Zharkova, V., “Subclasses of p-valent functions defined by convolution involving certain fractional derivative operator,” *Int. j. Contemp. Math. Sciences,* vol. 6, No. 32, 1545-1567, 2011.

[8] Pathak, M. and Sharma P., “A family of p-valent functions defined by a fractional calculus operator,” *Int. J. of Pure and Applied Mathematics* vol. 87, No. 1, 129-143, 2013.

[9] Jeyaraman, M. P., Suresh, T. K. and Reddy, K E., “Some subclasses of p-valent Analytic Functions,” *Global Journal. Of Science Frontier and Decision* Sciences, vol. 13, Issue 3 Version 1.0, 2013.

[10] Arif, M., Sokól, J. and Ayaz, M., “Coefficient Inequalities for a of p-valent Analytiv functions,” *The Scientific World Journal*, vol. 2014, Article ID801751, 5 pages.

[11] Noor, K. I. and Khan, N., “Some Classes of p-valent Analytic functions Associated with Hypergeometric Functions,” Filomat 29: 5(2015), 1031-1038, DOI 10.2298/FIL 1505031N. Avaialble at http://www.pmf.ni.ac.rs/filomat.

[12] Güney, H. Ö. “A new Class of Salagean-Type P-valent functions, “Advances *in inequalities for Series*, 121-129, 2008.

[13] Acu, M. “On Some Analytic Functions with negative coefficients,” *General Mathematics* Vol. 15, Nr. 2-3, 190-200. 2007.

[14] Salagean, G.S, *Subclasses of Univalent Functions:* Lecture Notes in Mathematics, Springer-Verlag Publisher, Heidelberg and Newyork, 1983, 362-372.