SOME FURTHER PROPERTIES FOR ANALYTIC FUNCTIONS WITH RESPECT TO OTHER POINTS WITH VARYING ARGUMENT

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Abstract

Some further properties for analytic functions with respect to other points with varying argument were discussed. The coefficient bounds, coefficient inequality, majorization, distortion bounds, extreme points and radius of close-to-convexity, starlikeness and convexity for the functions belonging to the class $TU_\gamma S_s^*(\alpha, A, B)$ and $TU_\gamma S_c^*(\alpha, A, B)$ were obtained.

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1. Introduction

Let $\boldsymbol{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the unit disc $E = \{z: |z| < 1\}$. Let S be a subclass of A consisting of analytic and univalent functions. Also, noted that $S^*(\beta)$ and $K(\beta), (0 \leq \beta < 1)$ are the classes of starlike function of order β and convex function of order β which their geometric condition satisfies $Re \frac{zf'(z)}{f(z)} > \beta$ and $Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > \beta$ in E respectively. It is known that $S^*(\beta) \subset S^*(0)$ and $K(\beta) \subset K(0) = K$.

A function $f(z) \in A$ is said to be in $US(\alpha, \beta)$, the class of α-uniformly starlike functions of order $\beta (0 \leq \beta < 1)$ which geometric condition satisfies

$$R\left(\frac{zf'(z)}{f(z)}\right) > \alpha \left|\frac{zf'(z)}{f(z)} - 1\right| + \beta,\ \alpha > 0 \tag{1.2}$$

and a function $f(z) \in A$ is said to be in $UK(\alpha, \beta)$, the class α-uniformly convex functions of order $\beta (0 \leq \beta < 1)$ which geometric condition satisfies

$$R\left(\frac{zf''(z)}{f'(z)} + 1\right) > \alpha \left|\frac{zf''(z)}{f'(z)}\right| + \beta,\ \alpha \geq 0 \tag{1.3}$$

See details in [1,2].

For two functions f and g, analytic in E, we say that the function f is subordinate to g in E, and write

$$f(z) \prec g(z) \tag{1.4}$$

If there exists a Schwarz function, w, which is analytic in E with $w(0) = 0$ and $|w(z)| < 1$, $(z \in E)$, such that $f(z) = g(w(z))$, $(z \in E)$. Furthermore, if the function g is univalent in E, then

$$f(z) \prec g(z),\ (z \in E) \iff f(0) = g(0) \text{ and } f(E) \subset g(E) \tag{1.5}$$

in [3].

Let f and g be analytic in the open unit disc E. We say that f is majorized by g in E see details in [4-6] and write

$$f(z) \ll g(z),\ \ (z \in E), \tag{1.6}$$

if there exists a function $\varphi(z)$ analytic in E such that

$$|\varphi(z)| \leq 1 \ \text{ and } \ f(z) = \varphi(z)g(z) \ \ (z \in E). \tag{1.7}$$

It may be noted here that (1.6) is closely related to the concept of quasi-subordination between analytic functions.

More so, $T_\gamma(\gamma \in R)$ denoted the class of functions $f(z) \in A$ of the form (1.1) for which all of non-vanishing coefficients satisfy the condition

$$\arg(a_n) = \pi + (1 - n)\gamma,\ \ n = 2,3,\cdots \tag{1.8}$$

Putting $\gamma = 0$, the class T_0 of functions with negative coefficient were obtained.
Silverman [7] introduced class T which he defined as

$$T = U_{\gamma \in R} T_\gamma.$$

The class T of functions is called the class of functions with varying argument of coefficients.
Sakaguchi [8] introduced the class $S_s^* \subset S$ consisting of functions given by (1.1) satisfying

$$Re\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0,\ z \in E. \tag{1.9}$$

These functions are called starlike function with respect to symmetric points. Ashwah and Thomas [9] introduced another class of functions namely; the class S_c^* consist of functions with respect to conjugate points which geometric condition satisfying

$$Re\left\{\frac{zf'(z)}{f(z) + \overline{\overline{f(\bar{z})}}}\right\} > 0,\ z \in E \tag{1.10}$$

and their results littered everywhere.

In terms of subordination, Goel and Mehrok [10] introduced a subclass of S_s^* denoted by $S_s^*(A, B)$.
Let $S_s^*(A, B)$ be the class of functions of the form (1.1) and satisfying the condition

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec \frac{1 + Az}{1 + Bz},\ \ -1 \leq B < A \leq 1,\ z \in E. \tag{1.11}$$

Also, let $S_c^*(A, B)$ be the class of functions of the form (1.1) and satisfying the condition

$$\frac{2zf'(z)}{f(z)+\overline{f(\bar{z})}} \prec \frac{1+Az}{1+Bz}, \ -1 \le B < A \le 1, \ z \in E. \tag{1.12}$$

Let $C_s(A,B)$ be the class of functions of the form (1.1) and satisfying the condition

$$\frac{2(zf'(z))'}{(f(z)-f(-z))'} \prec \frac{1+Az}{1+Bz}, \ -1 \le B < A \le 1, \ \ z \in E. \tag{1.13}$$

More so, let $C_c(A,B)$ be the class of functions of the form (1.1) and satisfying the condition

$$\frac{2(zf'(z))'}{(f(z)+\overline{f(\bar{z})})'} \prec \frac{1+Az}{1+Bz}, \ -1 \le B < A \le 1, \ \ z \in E. \tag{1.14}$$

The above classes of functions have been studied by many authors including Selvaraj and Vasaithi [11], Olatunji and Oladipo [12], Keerthi and Chinthamani [13], Janteng and Dahhar [14] and so on. They obtained very interesting results.

For arbitrary fixed real numbers A and $(-1 \le B < A \le 1)$, let $P(A,B)$ denote the class of functions of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k\, z^k, \tag{1.15}$$

which are analytic in E and satisfies the condition

$$p(z) \prec \frac{1+Az}{1+Bz}, \ \ (z \in E). \tag{1.16}$$

The class $P(A,B)$ was studied and introduced by Janowski [15]. We note that a function $f(z) \in P(A,B)$ if and only if

$$\begin{cases} \left|p(z) - \dfrac{1-AB}{1-B^2}\right| < \dfrac{A-B}{1-B^2}, & (-1 \le B < A \le 1, \ z \in E) \\ R\{p(z)\} > \dfrac{1-A}{2}, & (B = -1, z \in E). \end{cases}$$

for details, see [4,6].

In this work, two classes of functions $TU_\gamma S_s^*(\alpha,A,B)$ and $TU_\gamma S_c^*(\alpha,A,B)$ are defined by making use of (1.9) and (1.10). The coefficient bounds , coefficient estimates, distortion bound, extreme points and radii for thetwo classes of functions are obtained by employing Shu-Hai et al.'s [4,6] method.

For the purpose of our results, the following Lemma and definitions shall be necessary.

Lemma 1.1. Let $f(z) \in US_s^*(\alpha,A,B)$. Then

$$f(z) \in US_s^*(\alpha,A,B) \Rightarrow \begin{cases} f(z) \in US_s^*\left(\dfrac{1-A-\alpha(1-B)}{(1-\alpha)(1-B)}\right), & -1 < B < A \le 1, \alpha(1-B) \le 1-A, \\ f(z) \in US_s^*\left(\dfrac{(1-A)-2\alpha}{2(1-\alpha)}\right), & B = -1, 2\alpha \le 1-A. \end{cases} \tag{1.17}$$

Proof. Let $f(z) \in US_s^*(\alpha, A, B$. Then, we obtain

$$\Re\left\{\frac{2zf'(z)}{f(z)-f(-z)}\right\} > \begin{cases} \alpha\Re\left\{\frac{2zf'(z)}{f(z)-f(-z)}\right\} - \alpha + \frac{1-A}{1-B}, & -1 < B < A \leq 1, \alpha(1-B) \leq 1-A; \\ \alpha\Re\left\{\frac{2zf'(z)}{f(z)-f(-z)}\right\} - \alpha + \frac{1-A}{2}, & B = -1, 2\alpha \leq 1-A. \end{cases} \tag{1.18}$$

Or, equivalently,

$$\Re\left\{\frac{2zf'(z)}{f(z)-f(-z)}\right\} > \begin{cases} \frac{1-A-\alpha(1-B)}{(1-\alpha)(1-B)}, & -1 < B < A \leq 1, \alpha(1-B) \leq 1-A; \\ \frac{(1-A)-2\alpha}{2(1-\alpha)}, & B = -1, 2\alpha \leq 1-A. \end{cases} \tag{1.19}$$

If $-1 < B < A \leq 1$ and $\alpha(1-B) \leq 1-A$, then we have

$$0 \leq \frac{(1-A)-\alpha(1-B)}{(1-\alpha)(1-B)} < 1. \tag{1.20}$$

Also, if $B = -1$ and $2\alpha \leq 1-A$, then we obtain

$$0 \leq \frac{(1-A)-2\alpha}{2(1-\alpha)} < 1. \tag{1.21}$$

Thus, we prove Lemma 1.1.

Lemma 1.2. [16] Let $\alpha \geq 0$ and $-1 \leq B < A \leq 1$. If $w(z)$ is an analytic function $w(0) = 1$, then we have

$$w - \alpha(w-1) \prec \frac{1+Az}{1+Bz} \iff w(1-\alpha e^{-i\theta}) + \alpha e^{-i\theta} \prec \frac{1+Az}{1+Bz},\ (\theta \in R). \tag{1.22}$$

Lemma 1.3. [17] Let $\varphi(z)$ be analytic in U satisfying $|\varphi(z)| \leq 1$ for $z \in U$. Then

$$|\varphi'(z)| \leq \frac{1-|\varphi(z)|^2}{1-|z|^2},\ (z \in U). \tag{1.23}$$

Definition 1. Let $f(z) \in TU_\gamma S_s^*(\alpha, A, B)$, then

$$Re\ \frac{2zf'(z)}{f(z)-f(-z)} > \alpha\left|\frac{2zf'(z)}{f(z)-f(-z)} - 1\right| + \beta, \tag{1.24}$$

where $\alpha \geq 0$, $-1 \leq B < A \leq 1, \gamma \in R$ and $0 \leq \beta < 1$.

2. Main Results

Theorem 2.1. Let $f(z) \in US_s^*(\alpha, A, B)$. Then

$$|a_2| \leq \begin{cases} \frac{A-B}{(1-\alpha)(1-B)}, & -1 < B < A \leq 1, \alpha(1-B) \leq 1-A; \\ \frac{(1+A)}{2(1-\alpha)}, & B = -1, 2\alpha \leq 1-A, \end{cases} \tag{2.1}$$

$$|a_3| \le \begin{cases} \dfrac{A-B}{(1-\alpha)(1-B)}, & -1 < B < A \le 1, \alpha(1-B) \le 1-A; \\ \\ \dfrac{(1+A)}{2(1-\alpha)}, & B = -1, 2\alpha \le 1-A \end{cases} \tag{2.2}$$

and

$$|a_4| \le \begin{cases} \dfrac{(A-B)(1-\alpha)(1-B)+(A-B)^2}{2(1-\alpha)^2(1-B)^2}, & -1 < B < A \le 1, \alpha(1-B) \le 1-A; \\ \\ \dfrac{2(1+A)(1-\alpha)+(1+A)^2}{8(1-\alpha)^2}, & B = -1, 2\alpha \le 1-A. \end{cases} \tag{2.3}$$

Proof. Suppose that $f(z) \in US_S^*(\alpha, A, B)$. Then, by Lemma 1.1 we obtain

$$R\left\{\frac{2zf'(z)}{f(z)-f(-z)}\right\} > \begin{cases} \dfrac{(1-A)-\alpha(1-B)}{(1-\alpha)(1-B)}, & -1 < B < A \le 1, \alpha(1-B) \le 1-A; \\ \\ \dfrac{(1-A)-2\alpha}{2(1-\alpha)}, & B = -1, 2\alpha \le 1-A. \end{cases} \tag{2.4}$$

Let us define the function $p(z)$ by

$$p(z) = \begin{cases} \dfrac{(1-\alpha)\frac{2zf'(z)}{f(z)-f(-z)} - \frac{(1-A)-\alpha(1-B)}{1-B}}{\frac{A-B}{1-B}}, & -1 < B < A \le 1, \alpha(1-B) \le 1-A; \\ \\ \dfrac{(1-\alpha)\frac{2zf'(z)}{f(z)-f(-z)} - \frac{(1-A)-2\alpha}{2}}{\frac{1+A}{2}}, & B = -1, 2\alpha \le 1-A. \end{cases} \tag{2.5}$$

Hence, $p(z)$ is analytic in U with $p(0) = 1$ and $R\, p(z) > 0,\ (z \in U)$. Let

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots. \tag{2.6}$$

So, we obtain

$$\frac{2zf'(z)}{f(z)-f(-z)} = \begin{cases} 1 + \dfrac{A-B}{(1-\alpha)(1-B)}(c_1 z + c_2 z^2 + c_3 z^3 + \cdots), & -1 < B < A \le 1, \alpha(1-B) \le (1-A); \\ \\ 1 + \dfrac{1+A}{2(1-\alpha)}(c_1 z + c_2 z^2 + c_3 z^3 + \cdots), & B = -1, 2\alpha \le 1-A. \end{cases} \tag{2.7}$$

Or equivalently

$$2zf'(z) - [f(z)-f(-z)] = \begin{cases} \dfrac{A-B}{(1-\alpha)(1-B)}[f(z)-f(-z)](c_1 z + c_2 z^2 + c_3 z^3 + \cdots), \\ \qquad -1 < B < A \le 1, \alpha(1-B) \le 1-A; \\ \\ \dfrac{A-B}{2(1-\alpha)}[f(z)-f(-z)](c_1 z + c_2 z^2 + c_3 z^3 + \cdots), \\ \qquad B = -1, 2\alpha \le 1-A. \end{cases} \tag{2.8}$$

which gives

$$4a_2z^2+4a_3z^3+8a_4z^4+\cdots = \begin{cases} \dfrac{A-B}{(1-\alpha)(1-B)}(2z+2a_3z^3+2a_5z^5+\cdots)(c_1z+c_2z^2+c_3z^3+\cdots), \\ \qquad -1<B<A\le 1, \alpha(1-B)\le 1-A; \\ \dfrac{A-B}{2(1-\alpha)}(2z+2a_3z^3+2a_5z^5+\cdots)(c_1z+c_2z^2+c_3z^3+\cdots),\ B=-1, 2\alpha\le 1-A; \end{cases} \tag{2.9}$$

Comparing the coefficients in (2.9), we obtain

$$a_2 = \begin{cases} \dfrac{(A-B)c_1}{2(1-\alpha)(1-B)}, & -1<B<A\le 1, \alpha(1-B)\le 1-A; (13) \\ & (14) \\ \dfrac{(1+A)c_1}{4(1-\alpha)}, & B=-1, 2\alpha\le 1-A, \end{cases} \tag{2.10}$$

$$a_3 = \begin{cases} \dfrac{(A-B)c_2}{2(1-\alpha)(1-B)}, & -1<B<A\le 1, \alpha(1-B)\le 1-A; (16) \\ & (17) \\ \dfrac{(1+A)c_2}{4(1-\alpha)}, & B=-1, 2\alpha\le 1-A \end{cases} \tag{2.11}$$

and

$$a_4 = \begin{cases} \dfrac{(A-B)(c_3+a_3c_1)}{4(1-\alpha)(1-B)}, & -1<B<A\le 1, \alpha(1-B)\le 1-A; \\ \dfrac{(1+A)(c_3+a_3c_1)}{8(1-\alpha)}, & B=-1, 2\alpha\le 1-A. \end{cases} \tag{2.12}$$

This completes the prove of Theorem 2.1. □

Theorem 2.2. Let $f(z)\in US_c^*(\alpha,A,B)$. Then

$$|a_2| \le \begin{cases} \dfrac{(A-B}{(1-\alpha)(1-B)}, & -1<B<A\le 1, \alpha(1-B)\le 1-A; \\ \dfrac{(1+A)}{(1-\alpha)}, & B=-1, 2\alpha\le 1-A, \end{cases} \tag{2.13}$$

$$|a_3| \le \begin{cases} \dfrac{(A-B)(1-\alpha)(1-B)+2(A-B)^2}{(1-\alpha)^2(1-B)^2}, & -1<B<A\le 1, \alpha(1-B)\le 1-A; \\ \dfrac{(1+A)(1-\alpha)+(1+A)^2}{2(1-\alpha)^2}, & B=-1, 2\alpha\le 1-A \end{cases} \tag{2.14}$$

and

$$|a_4| \le \begin{cases} \dfrac{(A-B)\left(2+\frac{4(A-B)}{(1-\alpha)(1-B)}+\frac{2(A-B)(1-\alpha)(1-B)+4(A-B)^2}{(1-\alpha)^2(1-B)^2}\right)}{3(1-\alpha)(1-B)}, & -1<B<A\le 1, \alpha(1-B)\le 1-A; \\ \dfrac{(1+A)\left(1+\frac{1+A}{1-\alpha}+\frac{(1+A)(1-\alpha)+2(1+A)^2}{2(1-\alpha)^2}\right)}{3(1-\alpha)}, & B=-1, 2\alpha\le 1-A. \end{cases} \tag{2.15}$$

Proof. The proof follows from Theorem 2.1. □

Theorem 2.3. Let the function $f\in A$ and suppose that $f\in TU_\gamma S_s^*(\alpha,A,B)$, $0\le\alpha\ne 1$. If $[2zf'(z)]$ is majorized by $[f(z)-f(-z)]$ and $|2zf'(z)|\le |z[f(z)-f(-z)]'|$, then

$$|[2zf'(z)]'| \le |[f(z)-f(-z)]'|,\quad (|z|\le r_o), \tag{2.16}$$

where $r_o = r_o(\alpha, A, B)$ is the smallest positive root of the equation

$$\left[\frac{A-B}{|1-\alpha|}+|B|\right]r^3-[1+2|B|]r^2-\left[\frac{A-B}{|1-\alpha|}+|B|+2\right]r+1=0 \qquad (2.17)$$

$\left(z \in U, -1 \leq B < A \leq 1, 0 \leq \delta \leq r_o, [\frac{A-B}{|1-\alpha|}+|B|]\delta \leq 1\right)$.

Proof. Suppose that $f \in TU_\gamma S_s^*(\alpha, A, B)$. Then by Lemma 1.2, we obtain

$$\frac{2zf'(z)}{f(z)-f(-z)}(1-\alpha e^{-i\theta}) \prec \frac{1+Az}{1+Bz}. \qquad (2.18)$$

Or, equivalently

$$\frac{2zf'(z)}{f(z)-f(-z)} \prec \frac{1+\left(\frac{A-\alpha Be^{-i\theta}}{1-\alpha e^{-i\theta}}\right)z}{1+Bz} \qquad (2.19)$$

which holds true for all $z \in U$.

We find from (2.19) that

$$\frac{2zf'(z)}{f(z)-f(-z)} = \frac{1+\left(\frac{A-\alpha Be^{-i\theta}}{1-\alpha e^{-i\theta}}\right)w(z)}{1+Bw(z)} \qquad (2.20)$$

where $w(z) = c_1z + c_2z^2 + c_3z^3 + \cdots \in W$, W denotes the usual class of the bounded analytic functions in U and satisfies the conditions: $w(0) = 0,\ |w(z)| \leq |z|,\ (z \in U)$.

From (2.20), it is seen that

$$|f(z)-f(-z)| \leq \frac{1+|B||z|}{1-\left(\frac{A-B}{|1-\alpha|}+|B|\right)|z|}|2zf'(z)|. \qquad (2.21)$$

Next, since $2zf'(z)$ is majorized by $(f(z)-f(-z))$ in U, from (1.7), we have

$$2zf'(z) = \varphi(z)[f(z)-f(-z)] \qquad (2.22)$$

Differentiating (2.22) with respect to z and multiplying by z, we get

$$[2zf'(z)]' = \varphi'(z)[f(z)-f(-z)] + \varphi(z)[f(z)-f(-z)]' \qquad (2.23)$$

Thus, by Lemma 1.3, (2.21) and (2.22), we obtain

$$|[2zf'(z)]'| \leq \left[|\varphi(z)| + \frac{1-|\varphi(z)|^2}{1-|z|^2}.\frac{(1+|B||z|)|z|}{\left[1-\left(\frac{A-B}{|1-\alpha|}+|B|\right)|z|\right]}\right]|[f(z)-f(-z)]'| \qquad (2.24)$$

which upon setting $|z| = r$ and $|\varphi(z)| = \rho,\ (0 \leq \rho \leq 1)$ leads us to the inequality:

$$|[2zf'(z)]'| \ \leq \left[\frac{\Psi(\rho)}{(1-r^2)\left[1-\left(\frac{A-B}{|1-\alpha|}+|B|\right)r\right]}\right]|[f(z)-f(-z)]'| \qquad (2.25)$$

where

$$\Psi(\rho) = -r(1+|B|r)\rho^2 + (1-r^2)\left[1-\left(\frac{A-B}{|1-\alpha|}+|B|\right)r\right]\rho + r(1+|B|r) \qquad (2.26)$$

takes its maximum value at $\rho = 1$ with $r_o = r_o(\alpha, A, B)$ is the smallest positive root of (2.17). Furthermore, if $0 \le \delta \le r_o(\alpha, A, B)$, then the function $\Psi(\rho)$ defined by

$$\Psi(\delta) = -\delta(1+|B|\delta)\rho^2 + (1-\delta^2)[1 - \left(\frac{A-B}{|1-\alpha|} + |B|\right)\delta]\rho + (1+|B|\delta)\delta \tag{2.27}$$

is an increasing function on interval $0 \le \rho \le 1$, so that

$$\Psi(\rho) \le \Psi(1) = (1-\delta^2)\left[1 - \left(\frac{A-B}{|1-\alpha|} + |B|\right)\delta\right], \;\; (0 \le \rho \le 1,\; 0 \le \delta \le r_o(\alpha, A, B)) \tag{2.28}$$

Hence, upon setting $\rho = 1$ in (2.27), we conclude that (2.16) of Theorem 2.3 holds true for $|z| \le r_o = r_o(\alpha, A, B)$, which completes the proof of Theorem 3. □

Theorem 2.4. Let the function $f(z) \in A$ and suppose that $f \in TU_\gamma S_c^*(\alpha, A, B)$, $(0 \le \alpha \ne 1)$. If $[2zf'(z)]$ is majorized by $[f(z) + \overline{f(\bar{z})}]$ and $|2zf'(z)| \le |z[f(z) + \overline{f(\bar{z})}]'|$, then

$$|[2zf'(z)]'| \le |[f(z) + \overline{f(\bar{z})}]'|, \;\; (|z| \le r_o), \tag{2.29}$$

where $r_o = r_o(\alpha, A, B)$ is the smallest positive root of the equation

$$\left[\frac{A-B}{|1-\alpha|} + |B|\right] r^3 - [1 + 2|B|]r^2 - \left[\frac{A-B}{|1-\alpha|} + |B| + 2\right] r + 1 = 0, \tag{2.30}$$

$$\left(z \in U, -1 \le B < A \le 1,\; 0 \le \delta \le r_o, \left[\frac{A-B}{|1-\alpha|} + |B|\right]\delta \le 1\right).$$

Proof.
The proof follows from Theorem 2.3. □

Theorem 2.5. If $f(z) \in A$ satisfies

$$\sum_{k=2}^{\infty} [\,|2k - [1-(-1)^k]|(1+\alpha+\alpha|B|) + |2kB - [1-(-1)^k]A|]|a_k| \le A - B \tag{2.31}$$

for some $\alpha \ge 0$, $-1 \le B < A \le 1$, then $f \in TU_\gamma S_s^*(\alpha, A, B)$.

Proof. Suppose that (2.31) is true for $\alpha \ge 0$, $-1 \le B < A \le 1$. For $f(z) \in A$, let us define the function $p(z)$ by

$$p(z) = \frac{2zf'(z)}{f(z) - f(-z)} - \alpha\left|\frac{2zf'(z)}{f(z) - f(-z)} - 1\right|. \tag{2.32}$$

It suffices to show that

$$\left|\frac{p(z) - 1}{A - Bp(z)}\right| < 1, \;\; (z \in U). \tag{2.33}$$

It is noted that

$$\left|\frac{p(z)-1}{A-Bp(z)}\right| = \left|\frac{2zf'(z) - \alpha e^{i\theta}|2zf'(z) - [f(z)-f(-z)]| - [f(z)-f(-z)]}{A[f(z)-f(-z)] - B[2zf'(z) - \alpha e^{i\theta}|2zf'(z) - [f(z)-f(-z)]|]}\right| \tag{2.34}$$

$$= \left|\frac{[2zf'(z) - [f(z)-f(-z)]] - \alpha e^{i\theta}|2zf'(z) - [f(z)-f(-z)]|}{(A-B)[f(z)-f(-z)] - B[[2zf'(z) - [f(z)-f(-z)]] - \alpha e^{i\theta}|2zf'(z) - [f(z)-f(-z)]|]}\right|$$

$$= \left|\frac{\sum_{k=2}^{\infty}[\,2k - (1-(-1)^k)]a_k z^{k-1} - \alpha e^{i\theta}\left|\sum_{k=2}^{\infty}[\,2k - (1-(-1)^k)]a_k z^{k-1}\right|}{(A-B) - \sum_{k=2}^{\infty}[\,2kB - (1-(-1)^k)A]a_k z^{k-1} - \alpha B e^{i\theta}|[2k - (1-(-1)^k)]a_k z^{k-1}|}\right| \tag{2.35}$$

$$\leq \frac{\sum_{k=2}^{\infty}|2k-(1-(-1)^k)||a_k||z|^{k-1}+\alpha|e^{i\theta}|\sum_{k=2}^{\infty}|2k-(1-(-1)^k)||a_k||z|^{k-1}}{(A-B)-\sum_{k=2}^{\infty}|2kB-(1-(-1)^k)A||a_k||z|^{k-1}-\alpha|B||e^{i\theta}|\sum_{k=2}^{\infty}|2k-(1-(-1)^k)||a_k||z|^{k-1}}$$
$$\leq \frac{\sum_{k=2}^{\infty}|2k-(1-(-1)^k)||a_k|+\alpha|\sum_{k=2}^{\infty}|2k-(1-(-1)^k)||a_k|}{(A-B)-\sum_{k=2}^{\infty}|2kB-(1-(-1)^k)A||a_k|-\alpha|B|\sum_{k=2}^{\infty}|2k-(1-(-1)^k)||a_k|}. \tag{2.36}$$

The last expression is bounded above by 1, if

$$\sum_{k=2}^{\infty}|2k-(1-(-1)^k)||a_k|+\alpha|\sum_{k=2}^{\infty}|2k-(1-(-1)^k)||a_k| \leq (A-B)-\sum_{k=2}^{\infty}|2kB-(1-(-1)^k)A||a_k|-\alpha|B|\sum_{k=2}^{\infty}|2k-(1-(-1)^k)||a_k| \tag{2.37}$$

which is equivalent to our condition (2.31). This completes the proof of the theorem. □

Theorem 2.6. If $f(z) \in A$ satisfies

$$\sum_{k=2}^{\infty}[\,|2k-2|(1+\alpha+\alpha|B|)+|2kB-2A|]|a_k| \leq A-B \tag{2.38}$$

for some $\alpha \geq 0,\ -1 \leq B < A \leq 1$, then $f \in TU_\gamma S_c^*(\alpha, A, B)$.

Proof. It follows from Theorem 2.5. □

Theorem 2.7. Let the function $f(z)$ defined by (1.1) satisfy (1.8). We define

$$f_1(z)=z,\ \ f_k(z) = z-\frac{A-B}{(1+\alpha+\alpha|B|)|2k-[1-(-1)^k]|+|2kB-[1-(-1)^k]A|}e^{i(1-k)\gamma}z^k,\ \ (k=2,3,\cdots) \tag{2.39}$$

Then, $f(z) \in TU_\gamma S_s^*(\alpha, A, B)$ if and only if it can be expressed in the form

$$f(z)=\sum_{k=1}^{\infty}\lambda_k f_k(z) \tag{2.40}$$

where $\lambda_k > 0$ and $\sum_{k=1}^{\infty}\lambda_k = 1$.

Proof.
Suppose that

$$f(z)=\sum_{k=1}^{\infty}\lambda_k f_k(z) = z-\sum_{k=1}^{\infty}\lambda_k\frac{A-B}{(1+\alpha+\alpha|B|)|2k-[1-(-1)^k]|+|2kB-[1-(-1)^k]A|}e^{i(1-k)\gamma}z^k. \tag{2.41}$$

Then

$$\sum_{k=1}^{\infty}[\,(1+\alpha+\alpha|B|)|2k-[1-(-1)^k]|+|2kB-[1-(-1)^k]A|] \times|\lambda_k\frac{A-B}{(1+\alpha+\alpha|B|)|2k-[1-(-1)^k]|+|2kB-[1-(-1)^k]A|}e^{i(1-k)\gamma}| \tag{2.42}$$
$$=(A-B)\sum_{k=2}^{\infty}\lambda_k \tag{2.43}$$
$$=(A-B)(1-\lambda_1) \tag{2.44}$$
$$< A-B \tag{2.45}$$

Conversely, suppose that $f(z) \in TU_\gamma S_s^*(\alpha, A, B)$. Since

$$|a_k| \le \frac{A-B}{(1+\alpha+\alpha|B|)|2k-[1-(-1)^k]|+|2kB-[1-(-1)^k]A|}, \quad (k=2,3,\cdots), \tag{2.46}$$

we may set

$$\lambda_k = \frac{(1+\alpha+\alpha|B|)|2k-[1-(-1)^k]|+|2kB-[1-(-1)^k]A|}{(A-B)|e^{i(1-k)\gamma}|}|a_k|, \quad (k=2,3,\cdots) \tag{2.47}$$

and

$$\lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k. \tag{2.48}$$

Then,

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z). \tag{2.49}$$

Theorem 2.8. Let the function $f(z)$ defined by (1.1) satisfy (1.8). We define

$$f_1(z) = z, \quad f_k(z) = z - \frac{A-B}{(1+\alpha+\alpha|B|)|2k-[1-(-1)^k]|+|2kB-[1-(-1)^k]A|} e^{i(1-k)\gamma} z^k, \quad (k=2,3,\cdots) \tag{2.50}$$

then, $f(z) \in TU_\gamma S_c^*(\alpha, A, B)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z) \tag{2.51}$$

where $\lambda_k > 0$ and $\sum_{k=1}^{\infty} \lambda_k = 1$.

Proof. The proof follows from Theorem 2.7. □

Theorem 2.9. Let the function $f(z)$ defined by (2) be in the class $TUS_s^*(\alpha, A, B)$. Then, we have

1. The function $f(z)$ is close-to-convex of μ, $(0 \le \mu < 1)$ in $|z| < r_1$ where

$$r_1 = \inf_{k \ge 2} \left\{ \frac{(1-\mu)[(1+\alpha+\alpha|B|)|2k-[1-(-1)^k]|+|2kB-[1-(-1)^k]A|]}{k(A-B)} \right\}^{\frac{1}{k-1}}. \tag{2.52}$$

2. The function $f(z)$ is starlike of ψ $(0 \le \psi < 1)$ in $|z| < r_2$, where

$$r_2 = \inf_{k \ge 2} \left\{ \frac{(1-\psi)[(1+\alpha+\alpha|B|)|2k-[1-(-1)^k]|+|2kB-[1-(-1)^k]A|]}{(k-\psi)(A-B)} \right\}^{\frac{1}{k-1}}. \tag{2.53}$$

3. The function $f(z)$ is convex of ϵ $(0 \le \epsilon < 1)$ in $|z| < r_3$ where

$$r_3 = \inf_{k \ge 2} \left\{ \frac{(1-\epsilon)[(1+\alpha+\alpha|B|)|2k-[1-(-1)^k]|+|2kB-[1-(-1)^k]A|]}{k(k-\epsilon)(A-B)} \right\}^{\frac{1}{k-1}}. \tag{2.54}$$

Proof.

1. We have to show that $|f'(z) - 1| < 1 - \mu$ for $|z| < r_1$. We have

$$|f'(z)-1| \le \sum_{k=2}^{\infty} k\,|a_k||z|^{k-1}. \tag{2.55}$$

Thus, $|f'(z)-1| < 1-\mu$ if

$$\sum_{k=2}^{\infty} \frac{k}{1-\mu}|a_k||z|^{k-1} \le 1. \tag{2.56}$$

By Theorem 2.5, we have

$$\sum_{k=2}^{\infty} \frac{[(1+\alpha+\alpha|B|)|2k-[1-(-1)^k]|+|2kB-[1-(-1)^k]A|]}{A-B}|a_k| \le 1. \tag{2.57}$$

Hence, (2.57) will be true if

$$\frac{k|z|^{k-1}}{1-\mu} \le \frac{[(1+\alpha+\alpha|B|)|2k-[1-(-1)^k]|+|2kB-[1-(-1)^k]A|]}{A-B}. \tag{2.58}$$

Or, if

$$|z| \le \left\{\frac{(1-\mu)[(1+\alpha+\alpha|B|)|2k-[1-(-1)^k]|+|2kB-[1-(-1)^k]A|]}{k(A-B)}\right\}^{\frac{1}{k-1}},\ (k \ge 2) \tag{2.59}$$

which follows from (2.52). Similarly, we can prove (2.53) and (2.54). The proof of Theorem 2.9 is complete.

□

Theorem 2.10. Let the function $f(z)$ defined by (2) be in the class $TUS_c^*(\alpha, A, B)$. Then, we have

1. The function $f(z)$ is close-to-convex of μ, $(0 \le \mu < 1)$ in $|z| < r_1$ where

$$r_1 = \inf_{k\ge 2}\left\{\frac{(1-\mu)[(1+\alpha+\alpha|B|)|2k-2|+|2kB-2A|]}{k(A-B)}\right\}^{\frac{1}{k-1}}. \tag{2.60}$$

2. The function $f(z)$ is starlike of ψ $(0 \le \psi < 1)$ in $|z| < r_2$, where

$$r_2 = \inf_{k\ge 2}\left\{\frac{(1-\psi)[(1+\alpha+\alpha|B|)|2k-2|+|2kB-2A|]}{(k-\psi)(A-B)}\right\}^{\frac{1}{k-1}}. \tag{2.61}$$

3. The function $f(z)$ is convex of ϵ $(0 \le \epsilon < 1)$ in $|z| < r_3$ where

$$r_3 = \inf_{k\ge 2}\left\{\frac{(1-\epsilon)[(1+\alpha+\alpha|B|)|2k-2|+|2kB-2A|]}{k(k-\epsilon)(A-B)}\right\}^{\frac{1}{k-1}}. \tag{2.62}$$

Proof. The proof follows from Theorem 2.9. □

References

[1] S. Shams, S.R. Kulkarni and J.M. Jahangiri, *Classes of Uniformly Starlike and Convex Functions*, Int. J. Math. Math. Sci., 55(2004), 2959–2961.

http:dx.doi.org/10.1155/s0161171204402014.
[2] S. Shams and S.R. Kulkarni, *On a Class of Univalent Functions Defined by Ruscheweyh Derivatives*, Kyungpook Math. J., 43(2003), 579-585.
[3] S.S. Miller and P.T. Mocanus, *Differential Subordinations: Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied mathematics, Vol. 255, Marcel Dekker, Inc, New York, 2000.
[4] Shu-Hai Li, Huo. Tang, Li-Na. Ma and En Ao, *Some Further Properties for Analytic Functions with Varying Argument Defined by Hadamard Products*, Int. Mathematical Forum, Vol. 10, 2015, No. 2, 75–93. http://dx.doi.org/10.12988/imf.2015.412205.
[5] T.H. Macgregor, *Majorization by Univalent Functions*, Duke Math. J., 34(1967), 95–102. http://dx.doi.org/10.1215/s0012-7094-67-03411-4.
[6] Shu-Hai Li, Huo. Tang. and Jing-Yu. Jang, *A New Class of Analytic Functions Defined by Convolution with Varying Argument*, Tamkang J. Math., 44(1) (2013), 31–39. http://dx.doi.org/10.5556/j.tkjm.44.2013.944.
[7] H. Silverman, *Univalent Functions with varying Arguments*, Houston J. Math., 7(2) (1981), 283–287.
[8] K. Sakaguchi, *On a Certain Univalent Mapping*, J. Math. Soc. Japan, 11, (1959), 72–75.
[9] R.M. El-Ashwah and D.K. Thomas, *Some Subclasses of Close-To-Convex Functions*, J. Ramanujan Math. Soc., 2 (1987), 86–100.
[10] R.M. Goel and B.C. Mehrok, *A Subclass of Starlike Functions With Respect To Symmetric Points*, Tamkang. J. Math., 13(1), (1982), 11–24.
[11] C. Selvaraj and N. Vasanti, *Subclass of Analytic Functions With Respect To Symmetric and Conjugate Points*, Tamkang Jour. of Mathematics, 42, (2011), 87–94.
[12] S.O. Olatunji and A.T. Oladipo, *On a New Subfamilies of Analytic and Univalent Functions with Negative Coefficient With respect To Other points*, Bulletin of Mathematical Analysis and Application, Volume 3 Issue 2 (2011), 159–166.
[13] B.S. Keerthi and S. Chinthamani, *Subclasses of Analytic Functions With Respect To Symmetric and Conjugate Points*, Theoretical Mathematics and Applications, Vol. 3, No. 3, 2013, 1–11.
[14] A. Janteng and S.A.F.M. Dahhar, *A Subclass of Starlike Functions With Respect To Conjugate Points*, Int. Mathematical Forum, 4, (2009), 1373–1377.
[15] W. Janowski, *Some Extremal Problem for Certain Families of Analytic Functions*, Ann. Polon. math., 28(1973), 648–658.
[16] Shu-Hai Li, Huo Tang and En Ao, *Majorization Properties for Certain New Classes of Analytic Functions using the Salagean Operator*, Journal of Inequalities and Applications, 2013, 2013: 86. http://dx.doi.org/10.1186/1029-242x-2013-86.
[17] Z. Nehari, *Conformal Mapping*, McGraw-Hill Book Company, New York, Toronto and London, 1952.