EXISTENCE AND STABILITY OF PERIODIC SOLUTIONS FOR A CLASS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, the hypotheses of implicit function theorem and Lyapunov functions are applied to study existence and stability of periodic solutions for a class of second order nonlinear Duffing-type equation. Under appropriate conditions around the origin, a unique periodic solution and asymptotic stability are obtained.

Keywords: Implicit Function theorem, Lyapunov Stability, Cartwright Method, Periodic Solutions, Mathcad Software.

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1. Introduction

The well-known implicit function theorem has been employed by many authors to study existence of solution to non-linear differential equations of various types. Chow and Lasota [1], Chicone [2], Hartman [3] investigated the existence of solution to ordinary differential equations using implicit function theorem. Other researchers [4 - 6] used implicit function theorem to show the existence of periodic solution for non-linear partial differential equations. On the stability of periodic solutions see references [7 - 10]. The Duffing equation (oscillator):

$$\ddot{x} + c\dot{x} + ax + bx^3 = h(t) \qquad (1.1)$$

where a,b,care real constants and h(t)is continuous, has been widely used in physics, economics, engineering, and many other physical phenomena. Given its characteristic of oscillation and chaotic nature, many scientists are inspired by this nonlinear differential equation given its nature to replicate similar dynamics in our natural world. The nonlinear differential equation is used to model damped and driven oscillators [11], prediction of earthquake occurrences [12], modelling of the brain [13], and crash analysis [14].This equation together with Van der Pol's equation have become one of the most common examples of nonlinear oscillation in textbooks and research articles. See for instance [15 - 17] and the references therein. Due to the importance of the Duffing equation in real world problems, the study of existence of solution of the equation has continued to attract the attention of many researchers. [18 - 22] have proposed independently, the existence of periodic solution of Duffing equation of the general form:

$\ddot{x} + c\dot{x} + g(t,x) = p(t)$ (1.2)

where p(t) is continuous and 2π-periodic in $t \in R, g(t,x) = ax + bx^2 + \beta x^3$

and $\beta = 2$ represents a hard spring.

Motivated by the above results and ongoing research in this direction the purpose of this paper is to investigate the existence and stability of periodic solutions for a class of second order nonlinear differential equation of the Duffing's type

$\ddot{x} + c\dot{x} + ax + bx^2 + 2x^3 = h(t)$ (1.3)

where a,b,c are real constants and $h : [0,2\pi] \to R^n$ is continuous with boundary conditions

$x(0) = x(2\pi)$

$\dot{x}(0) = \dot{x}(2\pi)$

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2. Preliminaries

Definition 2.1.(Stability):An equilibrium point x_e of a nonlinear system is said to be stable if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $\bar{x} \in B(x_\varepsilon, \delta) \Longrightarrow \varphi(t, 0, \bar{x}) \in B(x_\varepsilon, \varepsilon) \quad for\ all\ t \geq 0$

Note: The Lyapunov stability of x_e assumes a "simultaneous continuity", more precisely the equicontinuity at x_e of all the functions in the $\{\phi_t : \bar{x} \to \varphi(t, 0, \bar{x})\ for\ t \geq 0$

Definition 2.2.(Asymptotic Stability):The equilibrium point x_e is said to be asymptotically stable, if for all $\varepsilon > 0$, the exists a $\delta > 0$ such that,

(i) $\varphi(t; 0, \bar{x}) \in B(x_e, \varepsilon)$ for all $t \geq 0$

(ii) $\lim_{t\to\infty} \varphi(t; 0, \bar{x}) = x_e$

Definition 2.3. Consider the general non-linear differential equation of the form

$$\dot{x} = f(t, x(t)) where\ f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n\ is\ continuous$$

The function *f* in definition2.3 is said to be *T*-periodic if for every $(t, x) \in \mathbb{R} \times \mathbb{R}^n and\ some\ T > 0, f(t, x) = f(t + T, x) and f(t, x) \neq f(t + T^*, x)\ for\ all\ T^* < T.$

Definition 2.4. A solution *x* of definition 2.3 defined on R such that *x* (*t* + *T*) = *x* (*t*) for all $t \in \mathbb{R}$ is called T- periodic solution or T- periodic harmonic solution.

Definition 2.5. Let *E*, *F* be Banach spaces, *U* an open subset of *E* and let $x_0 \in U.\ Let\ f: U \to F$ be a mapping of *U* into *F*. *f* is said to be Frechet differentiable at x_0 if there exists a continuous linear mapping; $L: E \to F$ such that

$$\lim_{x \to x_0} \frac{||f(x) - f(x_0) - L(x - x_0)||}{||x - x_0||} = 0$$

Definition 2.6. Let *E*, *F*, *G* be Banach spaces, let $U = u_1 \times u_2 \in E \times F$ be open set and $f: U \to F$ be a mapping of *U* into *G*, with $(x_0, y_0) \in U$. f is said to be Frechet differentiable with respect to the first variable *x* at (x_0, y_0) if the following conditions hold

(i)There exists a continuous linear mapping $L_1: E \to G$ such that $\forall \varepsilon > 0\ \exists \delta > 0\ such\ that\ \forall\ x, \bar{x} \in u_1\ then\ \left||x - x_0|\right| \leq \delta,\ \left||\bar{x} - x_0|\right| \leq \delta\ and\ \forall y \in u_2 such that \left||y - y_0|\right| \leq \delta$, it follows that

(ii)$\left|\left|f(x, y) - \left||f(\bar{x}, y) - L_1(x - \bar{x})|\right| \leq \varepsilon\right|\right| x - \bar{x}||$

(iii)The mapping$y \to f(x_0, y)\ is\ \ continuous\ at\ y_0$.

Proposition 2.7. Then condition (i) of the definition 2.6 is satisfied if the partial Frechet derivative $f_1^1(x_0, y_0)$ exists for (x, y) in a neighbourhood of (x_0, y_0) and if the mapping $(x, y) \to f_1^1(x_0, y_0)$ is continuous at (x_0, y_0).

Proposition 2.8. If f is Frechet differentiable with respect to the first variable at (x_0, y_0), it is Frechet differential with respect to this variable at (x_0, y_0) with the same L_1. Moreover, this is unique. L_1 is called the strong partial Frechet derivative with respect to the first variable at (x_0, y_0) and denoted by$f_1^1(x_0, y_0)$.

Definition 2.9.(Contraction map):Let Mbe a subset of a Banach space E. A function $f: M \to E$ is called a contraction mapping if there exists a constant $k, 0 \le k < 1$ such that

$$\| f(x) - f(y) \| \ \le \ k \| x - y \| \quad \ for\ all\ x, y \in M$$

Lemma 2.10. (The Banach fixed point theorem):Let E be a Banach space and $f: E \to E$ be a contractionmapping, then f has a unique fixed point in E, i.e. there exists a unique$x \in E$ *such that* $f(x) = x$

Lemma 2.11.(The implicit function theorem):Let E, F,G be Banach spaces and let $U_1 \subset E, U_2 \subset F.$ *Set* $U = U_1 \times U_2.\ For\ arbitrary\ (x_0, y_0) \in U, let\ f: U \to G be\ a$ mapping satisfying the following conditions

$$(i)\ \ f(x_0, y_0) = 0$$

(ii)f is Frechet differentiable with respect to the first variable at(x_0, y_0).

$(iii) f_1'(x_0, y_0): E \to G$*is a linear homeomorphism.*

Then there exists a neighborhood$V_1 \times V_2 \subset U_1 \times U_2\ of\ (x_0, y_0)\ a$nd a unique mapping $\zeta: V_1 \to V_2$such that for each$y \in V_2$the equation $f(x, y) = 0$ has in V_1, the unique solution $f(\zeta\ (y), y) = 0$*:* Moreover, $\zeta$is continuous at y_0.

3.Main Results

We present in this section, the main results of this paper

Theorem 3.1 Let $C_{2\pi}^2$ = {x: [0,T] → R^n : x is a class of C^2} and equipped with the usual uniform norm $||x||_\infty$= max $\{|x(t)|, |\dot{x}(t), |\ddot{x}(t)|\} C = \{x: [0, T] \to \mathbb{R}: x\ is\ continous\}\ with\ the\ usual\ norm,\ J = [0, T], J_{2\pi} = [0,2\pi]$.

Then, equation (1.2) *is equivalent to*

$f(x,h) = 0 \; in \; C^2_{2\pi} \times C$ (3.1)

Where $f: C^2_{2\pi} \times C \to C$ *is defined by*

$f(x,h) = \ddot{x} + c\dot{x} + ax + bx^2 + 2x^3 - h$ (3.2)

Proof:We first remark that with the norm defined above,$C^2_{2\pi}$is a Banach space. The strategy for the proof involves application of the implicit function theorem to the function $f$defined in equation (3.1).We split the proof into steps

Step 1: $f(0,0) = 0$.This follows trivially from the definition of *f:* Hence

$f(0,h) = -h$ (3.3)

Step 2: *f* is Frechet differentiable with respect to *x* at (0*;* 0). Observe that

$|f(x,h) - f(\bar{x},h) - f_x(0,0)(x,\bar{x})|$

$$= \ddot{x} + c\dot{x} + ax + bx^2 + 2x^3 - h - (\ddot{\bar{x}} + \dot{\overline{cx}} + a\bar{x} + b\bar{x}^2 + 2\bar{x}^3 - h)|$$

$$= |(\ddot{x} - \ddot{\bar{x}}) + c(\dot{x} - \dot{\bar{x}}) + a(x - \bar{x}) + b(x^2 - \bar{x}^2) + 2(x^3 - \bar{x}^3)|$$

$$\leq |x - \bar{x}|.\,|b(x + \bar{x}) + 2(x^2 + x\bar{x} + \bar{x}^2)|$$

$$\leq |x - \bar{x}|b\big(|x| + |\overline{x}|\big) + 2|(x^2 + x\bar{x} + \bar{x}^2)|$$

$$\leq |x - \bar{x}|b\big(|x| + |\overline{x}|\big) + 2(|x|^2 + |x\bar{x}| + |\bar{x}|^2)|$$

$$\leq |x - \bar{x}|(2bp + 6p^2), with \; |x| \leq p, |\overline{x}| \leq p$$

Consequently, $|f(x,h) - f(\bar{x},h) - f_x(0,0)(x,\bar{x})| \to 0, as \; |x - \bar{x}| \to 0$ (3.4)

Combining (3.3) and (3.4), we obtain that *f* is Frechet differentiable with respect to

thefirst variable at (0,0).

Step 3: $f_x(0,0): C^2_{2\pi} \to C$ defined by $z \to \ddot{z} + c\dot{z} + az$ is a linear homeomorphism.The mapping $f_x(0,0): z \to \ddot{z} + c\dot{z} + az$ is linear and continuous and hence bounded. It is also an onto mapping. Linear homeomorphism would have been established if the mapping is shown to be one to one. This is equivalent to requiring that

$$\ddot{z} + c\dot{z} + az = 0 \qquad (3.5)$$

With

$$z(0) = z(2\pi) \ \ and \ \ \dot{z}(0) = \dot{z}(2\pi) \qquad (3.6)$$

be non-critical.

It suffices to place appropriate conditions on the constants *a*, *c* such that equation

(3.5) is solvable. The auxiliary equation of (3.5) is $\lambda^2 + c\lambda + a = 0$.

Case I:

If $c = 0, and\ a = k^2\ where\ k\ is\ a\ natural\ number, then\ \lambda = \pm ik\ and$

$$z(t) = c_1 coskt + c_2 sinkt \qquad (3.7)$$

for arbitrary constants $c_1\ and\ c_2$. Clearly

$$z(0) = z(2\pi)\ and\ \ \dot{z}(0) = \dot{z}(2\pi) \qquad (3.8)$$

and the solution is non-trivial.

Case II:

If $c = 0\ and\ a \neq k^2, then\ condition$

$$\dot{z}(0) = \dot{z}(2\pi) \qquad (3.9)$$

is satisfied only by the trivial solution *z* = 0.

Case III:

If $c \neq 0\ and\ a \neq 0$,only the trivial solution exists.

Thus with any of these conditions imposed, one deduces the one to oneness of $f_x^{|}(0,0)$.Hence by Lemma 2.11 $[f_x^{|}(0,0)]^{-1}$ exists as a bounded linear operator. Linear homeomorphism of $f_x^{|}(0,0)$ follows. Existence of a unique solution is now assured by the implicit function theorem

4. Stability Analysis

We adapt the method of construction of Lyapunov function from Cartwright [23] and extend it to the second order non-linear differential equation of the Duffing type

$$\ddot{x} + c\dot{x} + ax + bx^2 + 2x^3 = p(t) \qquad (4.1)$$

Transforming (4.1) into a system form gives

$$\dot{x} = y$$

$$\dot{y} = -cy - ax - h(x) \qquad (4.2)$$

Where $h(x) = \ bx^2 + 2x^3$ and $p(t) = 0$

Writing equation (4.2) incompact form, we have

$$\dot{X} = Ax \qquad (4.3)$$

Where $A = \begin{bmatrix} 0 & 1 \\ -a - h(x) & -c \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \end{bmatrix}$ (4.4)

The method discussed here is based on the fact that the matrix A defined in equation (4.4) has all its eigenvalues with negative real parts. Then from the general theory which corresponds to any positive quadratic form $U(x)$, there exists another positive definite quadratic form $V(x)$ such that

$$\dot{V} = -U \qquad (4.5)$$

We choose the most general quadratic form of order two and pick the coefficient in the quadratic form to satisfy equation (4.5) along the solution paths of equation (4.2)

Let V be defined by $2V = Ax^2 + By^2 + 2Kxy$ (4.6)

$$\dot{V} = Ax\dot{x} + By\dot{y} + K(\dot{x}y + \dot{y}x)$$

$$= Axy + B\big(-cy - ax - h(x)\big) + Ky^2 + Kx(-cy - ax - h(x))$$

$$= Axy - Bcy^2 - Byax - Byh(x) + Ky^2 - Kcxy - Kax^2 - Kxh(x)$$

Simplifying the coefficients we have

$$\dot{V} = (A - Ba - Kc)xy + (K - Bc)y^2 - (Ka)x^2 - (Kx + By)h(x) \qquad (4.7)$$

Table 1.1: A table showing Terms and Coefficients of the equation (4.7)

Terms	Coefficient
xy	$A - B - Kc$
y^2	$K - Bc$
x^2	$-Ka$
$h(x)$	$-(Kx + By)$

To make $\dot{V}$ negative definite, we equate the coefficient of mixed variable to zero and the coefficients of x^2 and y^2 to any positive constant (say δ) i.e.

$$A - Ba - Kc = 0 \qquad \text{(i)}$$

$$K - Bc = \delta \qquad \text{(ii)}$$

$$-Ka = \delta \qquad \text{(iii)}$$

$$-(Kx + By) = 0 \qquad \text{(iv)}$$

From equation (iii) above

$$-Ka = \delta$$

$$K = -\frac{\delta}{a}$$

Then substituting the value of K into equation (ii) we obtain

$$-\frac{\delta}{a} - Bc = \delta$$

$$-Bc = \delta + \frac{\delta}{a}$$

$$B = -\frac{\delta(a+1)}{ca}$$

Substituting for K and B in (i) we have

$$A = Ba + Kc$$

$$= -\frac{\delta(a+1)}{c} - \frac{\delta c}{a}$$

which by further simplification gives that

$$A = -\frac{\delta}{ca}\left[(a+1)a + c^2\right]$$

Substituting for the values of the constant A, B, K in equation (4.6) gives

$$2V = -\frac{\delta}{ca}\left[(a+1)a + c^2\right]x^2 - \frac{\delta}{ca}[a+1]y^2 - 2\frac{\delta}{a}xy \qquad (4.8)$$

$$V = -\frac{\delta}{2ca}\left[\left((a+1)a + c^2\right)x^2 + (a+1)y^2 + 2cxy\right]$$

For V to be positive definite we choose $\frac{\delta}{ca} = -1$

Hence $V(x) = \frac{1}{2}\left[\left((a+1)a + c^2\right)x^2 + (a+1)y^2 + 2cxy\right] > 0$ is the required Lyapunov function.

Using equation (4.7), the equilibrium point is asymptotically stable since $\dot{V} < 0$

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